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Remarks on solutions to a generalization of the radical functional equations

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Abstract. During the 16th International Conference on Functional Equations and Inequalities a talk was given concerning the stability of the so-called radical functional equation $f(\sqrt{x^2+y^2}\,) = f(x) + f(y)$. The second author's question about the general solution of the equation itself was answered later by the first one. Contrary to some assertions in the literature the general solution is not an arbitrary quadratic function, but of the form $x \mapsto a(x^2)$ with additive a. Here we present far reaching generalizations of this result.

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1. Introduction and preliminaries

Let S be a nonempty set, $(Y, \star)$ and $(W, *)$ be groupoids (i.e., Y and W are nonempty sets endowed with binary operations $\star : Y^2 \to Y$ and $* : W^2 \to W$), $\Pi : S \to Y$, and $P_0 := \Pi(S)$. Let $p : P_0 \to S$ be a selection of Π, i.e.,

$$\Pi(p(x)) = x, \qquad x \in P_0.$$

We consider some versions of the conditional functional equation

$$f(p(\Pi(x) \star \Pi(y))) = f(x) * f(y), \qquad x, y \in S, \Pi(x) \star \Pi(y) \in P_0, \tag{1.1}$$

for functions $f : S \to W$.

Clearly, if $S = \mathbb{R}$ (the set of reals), $(Y, \star)$ is the additive group of real numbers, $n \in \mathbb{N}$ (positive integers), $\Pi(x) := x^n$ for $x \in S$ and $p(u) := \sqrt[n]{u}$ for $u \in P_0$, then (1.1) takes the form

$$f\left(\sqrt[n]{x^n + y^n}\,\right) = f(x) * f(y).$$

Particular cases of that equation have been considered in [2,3,7,11,12,15,20] (see also [19, p. 196]), and some descriptions of solutions to them have been

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proposed (not always correct). Moreover, the solutions and stability of the equation

$$f\big(\sqrt{ax^2+by^2}\,\big)=af(x)+bf(y) \tag{1.2}$$

have been considered in [16], for functions f mapping $\mathbb{R}$ into a real linear space X, with real $a,b>0$ such that $a+b\neq 1$. The authors have proved that every such solution to (1.2) must be a quadratic function [i.e., a solution to the quadratic functional Eq. (4.22)]. We present far reaching generalizations of all those results mentioned above (see [5] for some recent related results).

Another case of (1.1) is the Pythagorean mean functional equation

$$f\big(\sqrt{x^2+y^2}\,\big)=\frac{f(x)f(y)}{f(x)+f(y)}, \tag{1.3}$$

considered in [18] for $f:(0,\infty)\to\mathbb{R}$. Certainly, we have somehow to exclude in (1.3) the cases when $f(x)+f(y)=0$ (which was not explicitly done by the authors in [18]).

Below we provide two more natural simple examples of (1.1) for real functions. The first one, for $S\subset\mathbb{R}$, is the equation

$$f(\lfloor x\rfloor+\lfloor y\rfloor)=f(x)*f(y),\qquad x,y\in S,\lfloor x\rfloor+\lfloor y\rfloor\in\Pi(S), \tag{1.4}$$

where $\Pi:S\to\mathbb{Z}$ is the floor function, i.e., $\Pi(x):=\lfloor x\rfloor$ for $x\in S$ ($\lfloor x\rfloor$ denotes the largest integer not greater than a real number x), S is such that $\Pi(S)\subset S$ and $p(n)=n$ for $n\in\Pi(S)$.

The second one, also for $S\subset\mathbb{R}$, has the form

$$f(\{x\}+\{y\})=f(x)*f(y),\qquad x,y\in S,\{x\}+\{y\}\in\Pi(S), \tag{1.5}$$

where $\Pi:S\to[0,1)$ is given by $\Pi(x)=\{x\}:=x-\lfloor x\rfloor$ for $x\in S$, S is such that $\Pi(S)\subset S$ and $p(x)=x$ for $x\in\Pi(S)$.

At the end of the paper we give three corollaries with some results on solutions to functional Eqs. (1.3)–(1.5).

Let us recall that $(W,*)$ is right cancellative provided

$$x*z\neq y*z,\qquad x,y,z\in W,x\neq y.$$

Analogously, $(W,*)$ is left cancellative if

$$z*x\neq z*y,\qquad x,y,z\in W,x\neq y.$$

Next, we say that $e\in W$ is a left (right) neutral element in $(W,*)$ if $e*v=v$ ($v*e=v$, resp.) for each $v\in W$. We write

$$N_l:=\{e\in W:\ e\text{ is a left neutral element in }(W,*)\},$$
$$N_r:=\{e\in W:\ e\text{ is a right neutral element in }(W,*)\}.$$

We say that $\zeta \in W$ is a left (right) zero element in $(W, *)$ if $\zeta * v = \zeta$ ($v * \zeta = \zeta$, resp.) for each $v \in W$. We write

$$Z_l := \{\zeta \in W : \ \zeta \text{ is a left zero element in } (W, *)\},$$
$$Z_r := \{\zeta \in W : \ \zeta \text{ is a right zero element in } (W, *)\}.$$

Let us also recall that $u \in W$ is an idempotent element if $u * u = u$. Write

$$I := \{\iota \in W : \ \iota \text{ is an idempotent element in } (W, *)\}.$$

Clearly, $N_l \cup N_r \cup Z_l \cup Z_r \subset I$.

2. Main results

To simplify some reasonings, we assume in the sequel that W has at least two elements (the case where W has only one element is trivial). Moreover, given nonempty $D \subset Y$ and $x \in Y$, we write

$$x \star D := \{x \star d : d \in D\}, \qquad D \star x := \{d \star x : d \in D\}.$$

We have the following.

Theorem 2.1. *Let $P \subset P_0$ be nonempty. Assume that one of the following four conditions is valid.*

(i) $(W, *)$ *is right cancellative and*

$$P \cap (x \star P) \neq \emptyset, \qquad x \in P_0. \tag{2.1}$$

(ii) $(W, *)$ *is left cancellative and*

$$P \cap (P \star x) \neq \emptyset, \qquad x \in P_0. \tag{2.2}$$

(iii) $P = P_0$, $(Y, \star)$ *has a left neutral element* $e \in P_0$, $I \subset N_l \cup N_r \cup Z_l$ *and*

$$w * w \notin Z_l, \qquad w \in W \backslash Z_l. \tag{2.3}$$

(iv) $P = P_0$, $(Y, \star)$ *has a right neutral element* $e \in P_0$, $I \subset N_l \cup N_r \cup Z_r$ *and*

$$w * w \notin Z_r, \qquad w \in W \backslash Z_r. \tag{2.4}$$

Then $f : S \to W$ satisfies the conditional equation

$$f(p(\Pi(s) \star \Pi(t))) = f(s) * f(t), \qquad s, t \in S, \Pi(s) \star \Pi(t) \in P, \tag{2.5}$$

if and only if there exists a solution $A : P_0 \to W$ of the conditional equation

$$A(u \star v) = A(u) * A(v), \qquad u, v \in P_0, u \star v \in P, \tag{2.6}$$

such that $f = A \circ \Pi$. Moreover, such A is unique and $A = f \circ p$.

Proof. Assume that f fulfils (2.5). Let $A = f \circ p$. We show that (2.6) holds. To this end take $u, v \in P_0$ with $u \star v \in P$. Then

$$\begin{aligned} A(u \star v) &= f(p(u \star v)) = f(p(\Pi(p(u)) \star \Pi(p(v)))) \\ &= f(p(u)) * f(p(v)) = A(u) * A(v). \end{aligned}$$

Suppose first that $(W, *)$ is right cancellative and (2.1) is fulfilled. Fix $s, t \in S$ with $\Pi(s) = \Pi(t)$. Then, by (2.1), there is $u \in P$ with $\Pi(s) \star u \in P$. Clearly, $u = \Pi(r)$ for some $r \in S$. Hence, by (2.5),

$$\begin{aligned} f(s) * f(r) &= f(p(\Pi(s) \star \Pi(r))) = f(p(\Pi(t) \star \Pi(r))) \\ &= f(t) * f(r) \end{aligned}$$

and consequently $f(s) = f(t)$. In this way we have shown that

$$f(s) = f(t), \qquad s, t \in S, \Pi(s) = \Pi(t), \tag{2.7}$$

whence

$$f(s) = f(p(\Pi(s))) = A(\Pi(s)), \qquad s \in S. \tag{2.8}$$

We argue analogously if (ii) is valid.

Now, assume that (iii) holds. Let $r_0 = p(e)$. Then, for every $s \in S$,

$$f(p(\Pi(s))) = f(p(e \star \Pi(s))) = f(p(\Pi(r_0) \star \Pi(s))) = f(r_0) * f(s). \tag{2.9}$$

This (with $s = r_0$) yields that $\iota := f(r_0)$ is an idempotent element of $(W, *)$ and therefore $\iota \in N_l \cup N_r \cup Z_l$.

Clearly, if $\iota \in Z_l$, then (2.9) implies that $f(p(\Pi(s))) = \iota$ for $s \in S$. Consequently,

$$\begin{aligned} f(s) * f(s) &= f(p(\Pi(s) \star \Pi(s))) = f(p(\Pi(p(\Pi(s))) \star \Pi(p(\Pi(s))))) \\ &= f(p(\Pi(s)) * f(p(\Pi(s))) = \iota, \qquad s \in S, \end{aligned}$$

whence $f(S) = \{\iota\}$ in view of (2.3). It is easily seen that in such situations we can write that $f = A \circ \Pi$ with $A(x) = \iota$ for every $x \in P_0$. Note that such A satisfies the equation

$$A(u \star v) = A(u) * A(v), \qquad u, v \in P_0, u \star v \in P_0.$$

So, consider the situation when ι is a left neutral element of $(W, *)$. Take $s, t \in S$ with $\Pi(s) = \Pi(t)$. Then

$$\begin{aligned} f(s) &= \iota * f(s) = f(p(\Pi(r_0) \star \Pi(s))) \\ &= f(p(\Pi(r_0) \star \Pi(t))) = \iota * f(t) = f(t). \end{aligned}$$

Thus we have shown that (2.7) holds, which yields (2.8).

If ι is a right neutral element of $(W, *)$, then

$$\begin{aligned} f(s) &= f(s) * \iota = f(s) * f(r_0) = f(p(\Pi(s) \star \Pi(r_0))) \\ &= f(p(\Pi(t) \star \Pi(r_0))) = f(t) * \iota = f(t) \end{aligned}$$

for every $s, t \in S$ with $\Pi(s) = \Pi(t)$, which again yields (2.8).

If (iv) holds, then we argue analogously as in the case of (iii), replacing (2.9) by the condition

$$f(p(\Pi(s))) = f(p(\Pi(s) \star e)) = f(p(\Pi(s) \star \Pi(r_0))) = f(s) * f(r_0), \qquad s \in S, \tag{2.10}$$

where $r_0 = p(e)$.

Finally assume that $A : P \to W$ is a solution to (2.6) and $f := A \circ \Pi$. Let $x, y \in S$ and $\Pi(x) \star \Pi(y) \in P$. Then

$$\begin{aligned} f(p(\Pi(x) \star \Pi(y))) &= A \circ \Pi(p(\Pi(x) \star \Pi(x))) = A(\Pi(x) \star \Pi(y)) \\ &= A(\Pi(x)) * A(\Pi(y)) = f(x) * f(y). \end{aligned}$$

Thus we have shown that f fulfils Eq. (2.5). Note yet that

$$A(x) = A(\Pi(p(x)) = f(p(x)), \qquad x \in P_0,$$

which yields the form and uniqueness of A. □

Remark 2.2. The following three examples show the independence of some of the conditions (i)–(iv) in the theorem.

1. Let $W \in \{[0, \infty), \mathbb{R}\}$ and

$$x * y := x(y + 1), \qquad x, y \in W.$$

   Then it is easily seen that $N_l = Z_r = \emptyset$, $Z_l = N_r = \{0\}$ and $I = \{0\}$. Consequently $I \subset N_r$ and (2.4) holds. In this way, we obtain a very simple example of a groupoid $(W, *)$ satisfying the assumptions of (iv) concerning it, which is not left cancellative (because $Z_l = \emptyset$).

   Further, if $W = \mathbb{R}$, then $(-1) * (-1) = 0 \in Z_l$ and $-1 \notin Z_l$, which means that (2.3) is not valid and therefore (iii) is not fulfilled. Note yet that $x * (-1) = 0$ for each $x \in \mathbb{R}$, whence $(\mathbb{R}, *)$ is neither right cancellative.

2. Let $(\mathscr{S}, \circ)$ be the semigroup of all surjective mappings $u\colon \mathbb{N} \to \mathbb{N}$. For $u, v, w \in \mathscr{S}$ assume $v \circ u = w \circ u$. For arbitrary n choose m such that $u(m) = n$. Then $v(n) = v(u(m)) = w(u(m)) = w(n)$. Thus we have shown that $(\mathscr{S}, \circ)$ is right cancellative (see (i)). But $\mathscr{S}$ is not left cancellative (see (ii)). In fact, let u be defined by $u(1) = 1$ and $u(n) = n - 1$ for $n > 1$. If $v = \mathrm{id}$, $w(1) = 2, w(2) = 1$, and $w(n) = n$ for $n > 2$, then it is immediately seen that $u \circ v = u \circ w$ (and $v \neq w$).

   Moreover, (iii) holds (for $(W, *) = (\mathscr{S}, \circ)$), which can be seen as follows. Let $u \in \mathscr{S}$ be idempotent: $u \circ u = u$. Then, given any $n \in \mathbb{N}$ there is some m with $u(m) = n$. Thus $n = u(m) = u(u(m)) = u(n)$, i.e., $u = \mathrm{id}$. Accordingly (iii) is satisfied (with suitable $(Y, \star)$ and P_0), since the set Z_l is empty. The same concerns (iv).

3. Let now $(M, \diamond)$ be a groupoid with a neutral element, which is not left cancellative. Assume that $(M, \diamond)$ has only one idempotent element and possesses neither left nor right zero elements. Define a binary operation $\triangleleft$

in $M' := M \times M$ by $(x, y) \triangleleft (x', y') := (x \diamond x', y' \diamond y)$. Then $(W, *) := (M', \triangleleft)$ satisfies the assumptions of (iii), concerning it, and it is neither left nor right cancellative.

Remark 2.3. Clearly, (2.1) holds with $P = P_0$ when P_0 is a subgroupoid of Y (i.e., $x \star y \in P_0$ for every $x, y \in P_0$); but we also have the following simple observation.

Let $P \subset Y$ be nonempty and $\mathcal{J} \subset 2^Y$ be an ideal, that is $A \cup B \in \mathcal{J}$ and $2^A \subset \mathcal{J}$ for every $A, B \in \mathcal{J}$. Write $P_1 := Y \setminus P$,

$$Y_{\mathcal{J}} := \{z \in Y : z \star Y \notin \mathcal{J}\}, \qquad Y_0 := \{x \in Y : x \star P_1 \in \mathcal{J}\}. \tag{2.11}$$

Take $y \in Y_{\mathcal{J}} \cap Y_0$ and assume that $P_1 \in \mathcal{J}$. Then

$$\begin{aligned}(y \star Y) \setminus (P \cap (y \star P)) &= [P_1 \cap (y \star Y) \cup P \cap (y \star Y)] \setminus [P \cap (y \star P)] \\ &\subset P_1 \cup (y \star P_1) \in \mathcal{J}.\end{aligned}$$

Consequently, $P \cap (y \star P) \neq \emptyset$, because $y \star Y \notin \mathcal{J}$. Thus we have proved the following simple observation.

Proposition 2.4. *If $P_1 \in \mathcal{J}$, then*

$$P \cap (y \star P) \neq \emptyset, \qquad y \in Y_{\mathcal{J}} \cap Y_0.$$

Proposition 2.4 implies that (2.1) holds in particular when there is an ideal $\mathcal{J} \subset 2^Y$ such that $P_1 \in \mathcal{J}$ and

$$P_0 \subset Y_{\mathcal{J}} \cap Y_0. \tag{2.12}$$

Remark 2.5. Note that the following two conditions:

$$x \star Y \notin \mathcal{J}, \qquad x \in Y, \tag{2.13}$$

$$x \star T \in \mathcal{J}, \qquad x \in Y, T \in \mathcal{J}, \tag{2.14}$$

imply that $Y_{\mathcal{J}} \cap Y_0 = Y$ (provided $P_1 \in \mathcal{J}$).

Below we describe several natural examples of ideals $\mathcal{J} \subset 2^Y$ satisfying (2.13) and (2.14).

(a) Y is left cancellative and not of finite cardinality and

$$\mathcal{J} = \{B \in 2^Y : \operatorname{card} B < \operatorname{card} Y\}.$$

(b) d is a metric in Y that is left invariant (i.e., $d(y \star x, y \star z) = d(x, z)$ for $x, y, z \in Y$), $\sup_{x,y \in Y} d(x, y) = \infty$ and $\mathcal{J}$ is the family of all sets $B \in 2^Y$ that are bounded (i.e., $\sup_{x,y \in B} d(x, y) < \infty$).

(c) Y is a subsemigroup of a topological group G, Y is of the second category of Baire in G, and $\mathcal{J}$ is the family of all subsets of Y that are of the first category in G.

(d) Y is a subsemigroup of a locally compact topological group G, with the left Haar measure μ (see, e.g., [14]), such that there does not exist any Borel set $D \in 2^G$ with $\mu(D) < \infty$ and $Y \subset D$, and

$$\mathcal{J} = \{B \in 2^Y : \text{ there is a Borel set } D \in 2^G \text{ with } \mu(D) < \infty \text{ and } B \subset D\}.$$

(e) Y is a subsemigroup of a locally compact topological group G, with the left Haar measure μ, such that there is no Borel set $D \in 2^G$ with $\mu(D) = 0$ and $Y \subset D$, and

$$\mathcal{J} = \{B \in 2^Y : \text{ there is a Borel set } D \in 2^G \text{ with } \mu(D) = 0 \text{ and } B \subset D\}.$$

(f) Y is a subsemigroup of a Polish abelian group G, Y is not a Christensen zero set in G, and $\mathcal{J}$ is the family of all subsets of Y that are Christensen zero sets in G (see [8,13]).

(g) Y is a subsemigroup of a Polish abelian group G, Y is not a Haar meager set in G, and $\mathcal{J}$ is the family of all subsets of Y that are Haar meager in G (see [9]).

It is easily seen that similar reasonings are also true for (2.2).

3. Extensions of conditional homomorphisms

Now, there arises a natural question when a function $A : P_0 \to W$ satisfying the conditional Eq. (2.6) can be extended to a solution $h : Y \to W$ of the functional equation

$$h(u \star v) = h(u) * h(v). \tag{3.1}$$

We have in particular the following result that can be derived from, e.g., [4, Lemma 2].

Proposition 3.1. *Assume that $(W, *)$ is an abelian group, $(Y, \star)$ is an abelian semigroup, $\mathcal{J} \subset 2^Y$ is an ideal satisfying (2.14) such that $Y \notin \mathcal{J}$ and*

$$x \star U \notin \mathcal{J}, \qquad x \in Y, U \notin \mathcal{J}, \tag{3.2}$$

$D \in 2^Y$, $Y \setminus D \in \mathcal{J}$, and $A : D \to W$ satisfies

$$A(u \star v) = A(u) * A(v), \qquad u, v \in D, u \star v \in D. \tag{3.3}$$

Then there exists a unique solution $h : Y \to W$ of the equation

$$h(x \star y) = h(x) * h(y) \tag{3.4}$$

such that $h(x) = A(x)$ for $x \in D$.

For analogous results on possible extensions of A, in the case where $(Y, \star)$ is a group and $D = P_0$ is a subsemigroup of it (see Remark 2.3), we refer to, e.g., [1]. We also have the subsequent simple observation.

Proposition 3.2. *Assume that* $(W, *)$ *is a group,* $(Y, \star)$ *is a semigroup,* $P_0 \subset Y$, P *is a subsemigroup of* Y *with* $P \subset P_0$,

$$u \star x = x \star u, \qquad x \in P, u \in Y, \tag{3.5}$$

$$P \cap (u \star P) \neq \emptyset, \qquad u \in Y, \tag{3.6}$$

and $A : P_0 \to W$ *satisfies* (2.6)*. Then there is a unique solution* $h : Y \to W$ *of Eq.* (3.4) *such that*

$$h(x) = A(x), \qquad x \in P_0. \tag{3.7}$$

Proof. Clearly, (2.6) implies that

$$A(u \star v) = A(u) * A(v), \qquad u, v \in P, \tag{3.8}$$

and next, by (3.5), $A(z) * A(y) = A(y) * A(z)$ for $y, z \in P$. Consequently

$$\begin{aligned} A(z) * A(y)^{-1} &= A(y)^{-1} * \big(A(y) * A(z)\big) * A(y)^{-1} \\ &= A(y)^{-1} * A(z), \qquad z, y \in P. \end{aligned} \tag{3.9}$$

Take $u \in Y$. In view of (3.6), there are $x, y \in P$ with $x = u \star y$. Next, if also $z, w \in P$ and $z = u \star w$, then we have

$$\begin{aligned} A(x) * A(w) &= A(x \star w) = A(u \star y \star w) = A(u \star w \star y) \\ &= A(z \star y) = A(z) * A(y), \end{aligned}$$

whence (3.9) yields $A(x) * A(y)^{-1} = A(z) * A(w)^{-1}$.

Therefore, we can define $h : Y \to W$ by

$$h(u) = A(x) * A(y)^{-1}$$

for every $u \in Y$ and $x, y \in P$ such that $x = u \star y$; since $A(u) = A(x) * A(y)^{-1}$ when $u \in P_0$, we have (3.7).

Now, we prove that h is a solution to Eq. (3.4). To this end, take $u, v \in Y$. There exist $x, y, z, w \in P$ with $x = u \star y$ and $z = v \star w$. Clearly, by (3.5), $x \star z = (u \star y) \star (v \star w) = (u \star v) \star (w \star y)$, whence (3.9) implies that

$$h(u \star v) = A(x) * A(z) * A(y)^{-1} * A(w)^{-1} = h(u) * h(v).$$

It remains to show the uniqueness of h. So, let $h_0 : Y \to W$ also be a solution to Eq. (3.4) with $h_0(z) = A(z)$ for $z \in P_0$. Take $u \in Y$ and $x, y \in P$ with $x = u \star y$. Then $h_0(u) * h_0(y) = h_0(u \star y) = h_0(x)$, whence

$$h(u) = A(x) * A(y)^{-1} = h_0(x) * h_0(y)^{-1} = h_0(u).$$

□

Note that in the case when $(Y, \star)$ is a group, condition (3.6) means that $Y = \{x \star y^{-1} : x, y \in P\}$.

Finally, let us yet consider a similar issue (of extending A) in the case depicted in assumption (iii) of Theorem 2.1, when $(Y, \star)$ has a neutral element $e \in P$. Then we can use for instance the outcomes in [10, Theorem 4.1], [17, Theorem 1.1, Ch. XVIII, p. 468] and [6, Lemma 2]. We present below a result that can be derived from the latter one.

In the next proposition we assume that $(Y,\star)$ is a commutative semigroup which is uniquely divisible by 2, i.e., for each $y\in Y$ there is a unique $z\in Y$ with $2z := z\star z = y$. Next, given $D\subset Y$ and $n\in\mathbb{N}$ we write $2^nD := \{2^nz : z\in D\}$, where $2^0z = z$ and $2^{k+1}z = 2(2^kz)$ for $k\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. From [6, Lemma 2] we can easily deduce the following.

Proposition 3.3. *Assume that $(W,*)$ is a group, $\{z\in Y : 2z\in P_0\}\subset P_0$ and*

$$Y = \bigcup_{n\in\mathbb{N}_0} 2^nP_0.$$

If $A : P_0\to W$ satisfies (2.6) with $P = P_0$, then there exists a unique solution $h : Y\to W$ of Eq. (3.4) such that $h(x) = A(x)$ for $x\in P_0$.

Clearly, if Y is a linear topological space and we take $(Y,\star) = (Y,+)$, then every balanced neighbourhood U of the origin fulfils the conditions

$$\{z\in Y : 2z\in U\}\subset U, \qquad Y = \bigcup_{n\in\mathbb{N}_0} 2^nU.$$

4. Applications

In this section we assume that $\mathbb{R}_+ := [0,\infty)$, $\mathcal{J}_1$ denotes the family of all first category subsets of $\mathbb{R}$ and $\mathcal{J}_0$ stands for the family of all subsets of $\mathbb{R}$ that are of the finite outer Lebesgue measure (see, e.g., [17]). The following theorem is a simple consequence of the results from the previous sections.

Theorem 4.1. *Let $n\in\mathbb{N}$, $T := \{x^n : x\in\mathbb{R}\}$, $S\subset\mathbb{R}$, $P_0 := \{x^n : x\in S\}$, and $P\subset P_0$. Assume that $(W,*)$ is an abelian group,*

$$T\backslash P\in\mathcal{J}_1\cup\mathcal{J}_0,$$

and, in the case of even n, $|x|\in S$ for $x\in S$. Then a function $f : S\to W$ satisfies the conditional functional equation

$$f\big(\sqrt[n]{x^n+y^n}\,\big) = f(x)*f(y), \qquad x,y\in S, x^n+y^n\in P, \tag{4.1}$$

if and only if there exists a function $h_0 : \mathbb{R}\to W$ such that

$$h_0(x+y) = h_0(x)*h_0(y), \qquad x,y\in\mathbb{R}, \tag{4.2}$$

$$f(x) = h_0\big(x^n\big), \qquad x\in S. \tag{4.3}$$

Moreover, such a function h_0 is unique.

Proof. According to Proposition 2.4 and Remark 2.5, with $(Y,\star) = (T,+)$ and $\mathcal{J}$ being either $\mathcal{J}_1\cap 2^T$ or $\mathcal{J}_0\cap 2^T$, respectively,

$$P\cap(u+P)\neq\emptyset, \qquad u\in T. \tag{4.4}$$

Consequently, by Theorem 2.1, there exists a unique solution $A : P_0 \to W$ of the conditional equation

$$A(u+v) = A(u) * A(v), \qquad u, v \in P_0, u+v \in P, \tag{4.5}$$

such that $f(x) = A(x^n)$ for $x \in S$.

Clearly, (4.5) yields

$$A(u+v) = A(u) * A(v), \qquad u, v \in P, u+v \in P, \tag{4.6}$$

whence Proposition 3.1, with $D = P$, implies that there exists a unique solution $h : T \to W$ of the equation

$$h(x+y) = h(x) * h(y), \qquad x, y \in T, \tag{4.7}$$

such that $h(x) = A(x)$ for $x \in P$.

Take $u \in P_0$. Since $P_0 \subset T$, there exist $x, y \in P$ with $u + x = y$ (see (4.4)). Clearly, $h(y) = h(u + x) = h(u) * h(x)$ and

$$h(y) = A(y) = A(u+x) = A(u) * A(x) = A(u) * h(x).$$

Consequently

$$A(u) = h(y) * h(x)^{-1} = h(u).$$

Thus we have proved that $A(u) = h(u)$ for each $u \in P_0$, which means that

$$f(x) = A(x^n) = h\big(x^n\big), \qquad x \in S. \tag{4.8}$$

This completes the proof in the case $T = \mathbb{R}$ (i.e., when n is odd).

If n is even (i.e., $T = \mathbb{R}_+$), then it is enough to notice that, in view of Proposition 3.2 (with $Y = \mathbb{R}$ and $P = P_0 = T$), there exists a unique solution $h_0 : \mathbb{R} \to W$ of the equation

$$h_0(x+y) = h_0(x) * h_0(y) \tag{4.9}$$

with $h(x) = h_0(x)$ for each $x \in T$. $\square$

In the case of odd n we can obtain even a bit stronger condition than (4.3). Namely, we have the following:

Corollary 4.2. *Let $(W, *)$ be an abelian group, $n \in \mathbb{N}$ be odd, $S \subset \mathbb{R}$ and*

$$\mathbb{R} \backslash S \in \mathcal{J}_1 \cup \mathcal{J}_0.$$

Then a function $f : \mathbb{R} \to W$ satisfies the functional equation

$$f\big(\sqrt[n]{x^n + y^n}\big) = f(x) * f(y), \qquad x, y \in S, \tag{4.10}$$

if and only if there exists a solution $h_0 : \mathbb{R} \to W$ to (4.2) such that

$$f(x) = h_0\big(x^n\big), \qquad x \in \mathbb{R}. \tag{4.11}$$

Moreover, such a function h_0 is unique.

Proof. On account of Theorem 4.1 (with $P = P_0$), there exists a unique solution $h_0 : \mathbb{R} \to W$ to (4.2) such that (4.3) holds. Take $z \in \mathbb{R}$. Let $S_0 := \{-x : x \in S\}$ and $S_n := \{x^n : x \in S \cap S_0\}$. Clearly, $\mathbb{R}\backslash S_0 \in \mathcal{J}_1 \cup \mathcal{J}_0$, whence $\mathbb{R}\backslash S_n \in \mathcal{J}_1 \cup \mathcal{J}_0$ and therefore, by Proposition 2.4 and Remark 2.5,

$$(z^n + S_n) \cap S_n \neq \emptyset.$$

Hence, there exist $x, y \in S$ with $z^n + x^n = y^n$ and $-x \in S$. Consequently,

$$\begin{aligned} f(z) = f\big(\sqrt[n]{(-x)^n + y^n}\,\big) &= f(-x) * f(y) = h_0((-x)^n) * h_0(y^n) \\ &= h_0((-x)^n + y^n) = h_0(z^n). \end{aligned}$$

□

In the sequel we assume additionally that X is a normed space, $\rho \geq 0$, $\widehat{\mathcal{J}} \in \{\mathcal{J}_1, \mathcal{J}_0\}$, R is a subsemigroup of the semigroup $(\mathbb{R}_+, +)$ with

$$[\rho, \infty)\backslash R \in \widehat{\mathcal{J}}, \tag{4.12}$$

$q : \mathbb{R}_+ \to X$ is a function with

$$\|q(a)\| = a, \qquad a \in \mathbb{R}_+, \tag{4.13}$$

$\alpha > 0$, and $S \subset X$ is nonempty and has the property

$$P_0 = \{\|x\|^\alpha : x \in S\} \subset R.$$

Moreover, $P \subset P_0$ is such that $R\backslash P \in \widehat{\mathcal{J}}$ (this means in particular that also $R\backslash P_0 \in \widehat{\mathcal{J}}$).

Theorem 4.3. *Let $(W, *)$ be an abelian group. A function $f : S \to W$ satisfies the conditional functional equation*

$$f\Big(q\big(\big(\|x\|^\alpha + \|y\|^\alpha\big)^{1/\alpha}\big)\Big) = f(x) * f(y), \qquad x, y \in S, \|x\|^\alpha + \|y\|^\alpha \in P, \tag{4.14}$$

if and only if there exists a function $h : \mathbb{R} \to W$ such that

$$h(x + y) = h(x) * h(y), \qquad x, y \in \mathbb{R}, \tag{4.15}$$

$$f(x) = h\big(\|x\|^\alpha\big), \qquad x \in S. \tag{4.16}$$

Moreover, such a function h is unique.

Proof. Let f be a solution to (4.14). Note that (4.14) is Eq. (2.5) with $(Y, \star) = (R, +)$, $\Pi(x) = \|x\|^\alpha$ for $x \in S$, and $p(a) = q\big(a^{1/\alpha}\big)$ for $a \in P_0$. Next, by Proposition 2.4 and Remark 2.5 (with $(Y, \star) = (R, +)$ and $\mathcal{J} = \widehat{\mathcal{J}} \cap 2^R$),

$$P \cap (x + P) \neq \emptyset, \qquad x \in R. \tag{4.17}$$

Hence, in view of Theorem 2.1, there exists a unique solution $A : P_0 \to W$ of the conditional equation

$$A(u + v) = A(u) * A(v), \qquad u, v \in P_0, u + v \in P, \tag{4.18}$$

such that $f = A \circ \Pi$. The remaining reasonings are analogous as in the proof of Theorem 4.1, but for the convenience of the reader we present them below.

Clearly, we have

$$A(u+v) = A(u) * A(v), \qquad u, v \in P, u+v \in P. \tag{4.19}$$

Consequently, according to Proposition 3.1 (with $D = P$, $(Y, \star) = (\mathbb{R}, +)$ and $\mathcal{J} = \{B \subset \mathbb{R} : B \backslash (-\infty, \nu) \in \widehat{\mathcal{J}}$ for some $\nu \in \mathbb{R}\}$), there exists a unique solution $h : \mathbb{R} \to W$ of the equation

$$h(u+v) = h(u) * h(v), \qquad u, v \in \mathbb{R}, \tag{4.20}$$

such that $A(x) = h(x)$ for $x \in P$.

Take $u \in P_0$. By (4.17), there is $x, y \in P$ with $x = u + y$, whence

$$\begin{aligned} A(u) * A(y) &= A(u+y) = A(x) = h(x) = h(u+y) \\ &= h(u) * h(y) = h(u) * A(y) \end{aligned}$$

and consequently $A(u) = h(u)$. Thus we have shown that $A(v) = h(v)$ for each $v \in P_0$, which yields (4.16).

It remains to show the uniqueness of h. So, let $h_1 : \mathbb{R} \to W$ be a solution of (4.15) with $f(x) = h_1(\|x\|^\alpha)$ for $x \in S$. Clearly, $h_1(u) = h(u) = A(u)$ for $u \in P$, whence we obtain $h_1 = h$.

Since it is easy to check that (4.15) and (4.16) imply (4.14), this completes the proof. □

Note that if a function $f : S \to W$ has form (4.16), then it fulfils Eq. (4.14) with every function $q : \mathbb{R}_+ \to X$ such that (4.13) holds.

Remark 4.4. Theorem 4.1 shows that the general solution $f \colon \mathbb{R} \to \mathbb{R}$ of

$$f\left(\sqrt{x^2 + y^2}\,\right) = f(x) + f(y) \tag{4.21}$$

is of the form $f(x) = A(x^2)$ with $A \colon \mathbb{R} \to \mathbb{R}$ additive.

Clearly this f is quadratic, i.e., f satisfies

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \qquad x, y \in \mathbb{R}. \tag{4.22}$$

It seems worthwhile to note that there are quadratic functions which are not solutions of (4.21): Let $B \colon \mathbb{R} \to \mathbb{R}$ be additive. Then $x \mapsto B(x)^2$ is of the form $x \mapsto A(x^2)$, with additive A, only if B is continuous, thus of the form $x \mapsto B(x) = cx$ for some real c.

Proof. If $A(x^2) = B(x)^2$ for all $x \in \mathbb{R}$, then A is nonnegative for positive arguments and thus (see, e.g., [17, ch. IX]) there is some real $d \geq 0$ such that $A(x) = dx$ for all $x \in \mathbb{R}$. Hence $|B(x)| = \sqrt{d}|x|$ for $x \in \mathbb{R}$, which again (see [17, ch. IX]) implies that $B(x) = cx$ for all $x \in \mathbb{R}$, where $c^2 = d$.

Therefore, if B is a discontinuous additive function and $f(x) = B(x)^2$ for $x \in \mathbb{R}$, then f is quadratic and does not satisfy (4.21). □

The following remark concerns generalizations of functional equations of the form

$$f\left(\sqrt[n]{x^n+y^n}\right)=f(x)*f(y), \tag{4.23}$$

with odd $n\in\mathbb{N}$.

Remark 4.5. Let X be a nonempty set and $N\colon X\to\mathbb{R}$, $q\colon\mathbb{R}\to X$ be such that $N\circ q=\mathrm{id}_{\mathbb{R}}$. For $\alpha>0$ let $p_\alpha\colon\mathbb{R}\to\mathbb{R}$ be defined by $p_\alpha(a):=\operatorname{sgn}(a)|a|^\alpha$ where sgn denotes the sign-function. Then a function $f\colon X\to W$ satisfies the equation

$$f(q\circ p_{1/\alpha}(p_\alpha(N(x))+p_\alpha(N(y))))=f(x)*f(y),\qquad x,y\in X, \tag{4.24}$$

if and only if there is some $h:\mathbb{R}\to W$ satisfying (4.15) such that

$$f(x)=h(p_\alpha(N(x))),\qquad x\in X. \tag{4.25}$$

Proof. Observe that (4.24) reads as

$$f(p(\Pi(x)+\Pi(y)))=f(x)*f(y),\qquad x,y\in X \tag{4.26}$$

with $p=q\circ p_{1/\alpha}$, $\Pi=p_\alpha\circ N$ and that $\Pi(p(a))=a$ for $a\in\mathbb{R}$, because $N\circ q=\mathrm{id}_{\mathbb{R}}$ and $p_\alpha\circ p_{1/\alpha}=\mathrm{id}_{\mathbb{R}}$. So, it is enough to use Theorem 2.1 (iii) (with $S=X$, $P=P_0=\mathbb{R}$, and $(Y,\star)=(\mathbb{R},+)$). □

Let us yet present an example of results connected with Eq. (1.2). We use a bit simpler approach than before. Namely, we have the following:

Corollary 4.6. *Let $n\in\mathbb{N}$, $n>1$, X be a linear space over a field $\mathbb{F}$, $\phi\in X$, $\alpha,\beta\in\mathbb{F}$, and $a,b,c\in\mathbb{R}_+$. Assume that $\alpha\neq0$ or $\beta\neq0$. Then a function $f:\mathbb{R}\to X$ satisfies the functional equation*

$$f\left(\sqrt[n]{ax^n+by^n+c}\right)=\alpha f(x)+\beta f(y)+\phi \tag{4.27}$$

if and only if there is a solution $A:P_0\to X$ of the functional equation

$$A(ax+by+c)=\alpha A(x)+\beta A(y)+\phi \tag{4.28}$$

such that $f(x)=A(x^n)$ for $x\in\mathbb{R}$, where $P_0:=\{x^n:x\in\mathbb{R}\}$.

Proof. Write

$$x\star y:=ax+by+c,\qquad x,y\in P_0,$$

$$u*v:=\alpha u+\beta v+\phi,\qquad u,v\in W,$$

where $W:=X$. Then one of conditions (i) and (ii) of Theorem 2.1 is valid (with $\Pi(x)\equiv x^n$, $p(u)\equiv\sqrt[n]{u}$, and $Y=P=P_0$). So, $f:\mathbb{R}\to X$ satisfies (4.27) if and only if there is a solution $A\colon P_0\to X$ of (4.28) such that $f(x)\equiv A(x^n)$. □

For some information and further references on solutions to (4.28) we refer to [17, ch. XIII, §10].

We end the paper with three simple corollaries concerning solutions to functional Eqs. (1.3)–(1.5).

Corollary 4.7. *A function $f : (0,\infty) \to (0,\infty)$ satisfies Eq.* (1.3) *if and only if there exists $c \in (0,\infty)$ such that*

$$f(x) = \frac{c}{x^2}, \qquad x \in (0,\infty). \tag{4.29}$$

Proof. Let $f : (0,\infty) \to (0,\infty)$ be a solution to Eq. (1.3) and $* : (0,\infty)^2 \to (0,\infty)$ be given by

$$x * y = \frac{xy}{x+y}, \qquad x, y \in (0,\infty).$$

It is easy to check that $*$ is cancellative. Hence, by Theorem 2.1 (with $P_0 = P = (0,\infty)$), there exists a solution $h : (0,\infty) \to (0,\infty)$ to the functional equation

$$h(x+y) = h(x) * h(y) = \frac{h(x)h(y)}{h(x)+h(y)} \tag{4.30}$$

with $f(x) = h(x^2)$ for $x \in (0,\infty)$. Next, note that $g(x) = 1/h(x)$ for $x \in (0,\infty)$ fulfils

$$\begin{aligned} g(x+y) &= \frac{h(x)+h(y)}{h(x)h(y)} \\ &= \left(\frac{1}{g(x)} + \frac{1}{g(y)}\right) g(x)g(y) \\ &= g(x) + g(y), \qquad x, y \in (0,\infty). \end{aligned}$$

Consequently, by Proposition 3.2 (with $(W,*) = (Y,\star) = (\mathbb{R},+)$, $P_0 = P = (0,\infty)$), there exists a unique function $A : \mathbb{R} \to \mathbb{R}$ such that

$$A(x+y) = A(x) + A(y), \qquad x, y \in \mathbb{R}, \tag{4.31}$$

and $g(x) = A(x)$ for $x \in (0,\infty)$. Clearly, $A((0,\infty)) \subset (0,\infty)$, whence there is $c_0 > 0$ with $A(x) = c_0 x$ for $x \in \mathbb{R}$ (see, e.g., [17]). This yields (4.29) with $c := 1/c_0$.

The converse is easy to check. □

Corollary 4.8. *Assume that W is left cancellative and $S = [1,\infty)$. Then $f : S \to W$ satisfies Eq.* (1.4) *if and only if there exists $\alpha \in W$ with $\alpha^{m+n} = \alpha^m * \alpha^n$ for $m, n \in \mathbb{N}$ and $f(x) = \alpha^{\lfloor x \rfloor}$ for $x \in [1,\infty)$, where $\alpha^1 := \alpha$ and $\alpha^{m+1} := \alpha^m * \alpha$ for $m \in \mathbb{N}$.*

Proof. Let f be a solution to Eq. (1.4). Theorem 2.1 (with $P_0 = \mathbb{N}$) implies that there exists a solution $h : \mathbb{N} \to W$ to the functional equation

$$h(x+y) = h(x) * h(y) \tag{4.32}$$

with $f(x) = h(\lfloor x \rfloor)$ for $x \in [1, \infty)$. Let $\alpha := h(1)$. Then it is easy to show by induction that $h(n) = \alpha^n$ for $n \in \mathbb{N}$, whence $f(x) = \alpha^{\lfloor x \rfloor}$ for $x \in [1, \infty)$. Moreover,

$$\alpha^{m+n} = f(m+n) = f(m) * f(m) = \alpha^m * \alpha^n, \qquad m, n \in \mathbb{N}.$$

The converse is easy to check. □

Corollary 4.9. *Let W be a commutative group and $S = (0, \infty) \backslash \mathbb{N}$. Then $f : S \to W$ satisfies Eq. (1.5) if and only if there exists a solution $h : \mathbb{R} \to W$ to Eq. (4.32) such that $f(x) = h(\{x\})$ for $x \in (0, \infty) \backslash \mathbb{N}$.*

Proof. Let f be a solution to Eq. (1.5). Then, by Theorem 2.1, there exists a function $g : (0, 1) \to W$ such that

$$g(x+y) = g(x) * g(y), \qquad x, y \in (0,1), x+y \in (0,1), \tag{4.33}$$

and $f(x) = g(\{x\})$ for $x \in (0, \infty) \backslash \mathbb{N}$. Next, by Proposition 3.3, there is $g_0 : (0, \infty) \to W$ with

$$g_0(x+y) = g_0(x) * g(y), \qquad x, y \in (0, \infty),$$

and $g(x) = g_0(x)$ for $x \in (0, 1)$. Finally, we deduce from Proposition 3.2 (with $(Y, \star) = (\mathbb{R}, +)$ and $P = P_0 = (0, \infty)$) that there exists a unique solution $h : \mathbb{R} \to W$ to (4.32) with $g_0(x) = h(x)$ for $x \in (0, \infty)$. This yields the form of f.

The converse is easy to check. □

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