Aequat. Math. 92 (2018), 801–872

0001-9054/18/050801-72
published online July 10, 2018
https://doi.org/10.1007/s00010-018-0564-5

Aequationes Mathematicae

Invariance of means

Justyna Jarczyk and Witold Jarczyk

Dedicated to Professor Zoltán Daróczy on the occasion of his 80th birthday

Abstract. We give a survey of results dealing with the problem of invariance of means which, for means of two variables, is expressed by the equality $K \circ (M, N) = K$. At the very beginning the Gauss composition of means and its strict connection with the invariance problem is presented. Most of the reported research was done during the last two decades, when means theory became one of the most engaging and influential topics of the theory of functional equations. The main attention has been focused on quasi-arithmetic and weighted quasi-arithmetic means, also on some of their surroundings. Among other means of great importance Bajraktarević means and Cauchy means are discussed.

Mathematics Subject Classification. Primary: 26E60, Secondary: 39B22.

Keywords. Mean, Invariance, Weighted quasi-arithmetic mean, Cauchy mean, Lagrangian mean, Bajraktarević mean, Gauss composition, Convergence of successive iterates.

Contents

Birkhäuser

1. Introduction

The idea of a mean is as old in human cognition as that one of a number. Given quantities $x_1, \dots, x_p$ one can intuitively look for a mean of them as any number $M(x_1, \dots, x_p)$ lying somewhere between the extreme values of $x_1, \dots, x_p$:

$$\min\{x_1, \dots, x_p\} \le M(x_1, \dots, x_p) \le \max\{x_1, \dots, x_p\}. \tag{1.1}$$

This is, in fact, probably the first formal definition of a mean, proposed by Cauchy [28] in 1821. Inequalities (1.1) are known today as the *internality Cauchy condition*. Three classical examples, well known already in the antiquity, are (in what follows $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^n$ is the cartesian product of p copies of $\mathbb{R}$) the *arithmetic mean* $A\colon \mathbb{R}^p \to \mathbb{R}$:

$$A(x_1, \dots, x_p) = \frac{x_1 + \dots + x_p}{p},$$

the *geometric mean* $G\colon (0, +\infty)^p \to (0, +\infty)$:

$$G(x_1, \dots, x_p) = \sqrt[p]{x_1 \dots x_p},$$

the *harmonic mean* $H\colon (0, +\infty)^p \to (0, +\infty)$:

$$H(x_1, \dots, x_p) = \frac{p}{\frac{1}{x_1} + \dots \frac{1}{x_p}}.$$

So, given an interval I, any function $M\colon I^p \to I$ satisfying inequalities (1.1) for all $x_1, \dots, x_p \in I$ is called a *mean* (more precisely: a *mean of* p *numbers*) *on* I. A mean $M\colon I^p \to I$ is said to be *strict* if inequalities (1.1) are sharp whenever $\min\{x_1, \dots, x_p\} < \max\{x_1, \dots, x_p\}$. In general, a *mean on* I is any function $M\colon \bigcup_{p=1}^{\infty} I^p \to I$ such that condition (1.1) holds for all numbers $x_1, \dots, x_p \in I$ and $p \in \mathbb{N}$. For some variants of the notion of a mean, definitions of various families of means and relationships between them see the monograph [21] and its previous versions [135,136] and [22]. A wealth of further information on means can be found also in Hardy, Littlewood and Pólya [74], in the book [20] by Borweins and the survey [52] by Daróczy and Páles; see also the quite recent book [153] by Gh. Toader and Costin.

For any $x, y \in (0, +\infty)$ we have

$$\frac{x+y}{2} \cdot \frac{2}{\frac{1}{x}+\frac{1}{y}} = (x+y)\frac{xy}{x+y} = xy,$$

whence

$$G\left(A\left(x,y\right),H\left(x,y\right)\right) = G(x,y). \tag{1.2}$$

This is the celebrated equality expressing the invariance of the geometric mean with respect to the pair (A, H) of the arithmetic and harmonic means. This is a good starting point to study the main problem of the paper which can be formulated as follows:

Given an interval I and a positive integer p we are interested in means $K\colon I^p \to I$ and $M_1, \ldots, M_p\colon I^p \to I$ satisfying the *invariance equation*

$$K\left(M_1\left(x_1,\ldots,x_p\right),\ldots,M_p\left(x_1,\ldots,x_p\right)\right) = K\left(x_1,\ldots,x_p\right). \tag{1.3}$$

Considering that question we can try to solve two problems in fact. Namely, given means $M_1, \ldots, M_p$ we are looking for a mean K satisfying Eq. (1.3) for all $x_1, \ldots, x_p \in I$. Another possibility is to fix K and then to ask about means $M_1, \ldots, M_p$ such that (1.3) holds for $x_1, \ldots, x_p$ running through I. In both tasks we say that the mean K is *invariant with respect to the mean-type mapping* $(M_1, \ldots, M_p)$ or, simply, $(M_1, \ldots, M_p)$-invariant. Most often the invariance problem is studied in classes of means described with the aid of function generators and some parameters. This causes that the invariance Eq. (1.3) takes different forms and becomes a functional equation in several variables, with a number of unknown functions (the generators of the means) and parameters to be determined. Since those equations contain superpositions of unknown functions, as a rule they are hard to solve. For that reason the authors of the results presented here studied mostly the case $p = 2$. Then the invariance Eq. (1.3) takes the form

$$K\left(M\left(x,y\right),N\left(x,y\right)\right) = K\left(x,y\right) \tag{1.4}$$

where K, M and N are means in two variables, defined on the same interval. Equality (1.4) can also be considered from another point of view. Namely, given means K and M on I one can ask about a mean $N\colon I^2 \to I$ such that (1.4) holds for all $x, y \in I$. A positive answer to that question was given by Matkowski [108, Remark 1]:

Theorem 1.1. *Let I be an interval and K be a symmetric mean on I, that is*

$$K(x,y) = K(y,x), \quad x, y \in I,$$

which is continuous and strictly increasing in each variable. Then for every mean M on I there exists a unique function $N\colon I^2 \to I$ such that equality (1.4) *holds for every $x, y \in I$; moreover, N is a mean on I.*

The (unique) mean N described by Theorem 1.1 is called the *K-complementary mean to M* and is denoted by $M^{(K)}$. Note that $\left(M^{(K)}\right)^{(K)} = M$ and the mean K is $\left(M, M^{(K)}\right)$-invariant. For these and some other basic properties of the operation $M \longmapsto M^{(K)}$ the reader is referred to [108, Sec. 1]. The notion of complementary means is not a main focus of our attention in this survey. However, it has pretty rich literature; the reader can consult the recent paper [134] by Matkowski, Nowicka and Witkowski and the references therein.

In what follows the term *interval* always refers to a nonempty connected set of reals which is not a singleton. This convention will not be repeated in the sequel.

We would like to pay attention mainly to three classes of means, namely weighted quasi-arithmetic means, Bajrakterević means and Cauchy means. Of course, the selection we made reflects our personal preference only and no doubt this survey does not pretend to be comprehensive in any way.

2. Gauss composition

Equality (1.2) is, in fact, only an excuse to deal with the invariance Eq. (1.3) or simply (1.4). Now we present a more grave reason for it, going 2 or even 3 centuries back.

It was Gauss (see [64] or [65]) who, following or rediscovering some ideas of Lagrange [99], came to the following observation. Taking any $x, y \in (0, +\infty)$ and putting $x_1 = x$, $y_1 = y$ and then

$$x_{n+1} = A\left(x_n, y_n\right), \quad y_{n+1} = G\left(x_n, y_n\right) \tag{2.1}$$

for all $n \in \mathbb{N}$, one can check that both sequences converge to a common limit, say $A \otimes G(x, y)$. The function $A \otimes G$ is a mean on $(0, +\infty)$. Gauss named it the *arithmetic-geometric mean* (*medium arithmeticum-geometricum*). It turns out that

$$A \otimes G\left(A\left(x, y\right), G\left(x, y\right)\right) = A \otimes G(x, y), \quad x, y \in (0, +\infty),$$

that is the limit mean $A \otimes G$ is invariant with respect to the pair (A, G) which defines the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ tending to $A \otimes G(x, y)$ (see (2.1)). It was also observed by Gauss, and probably even sooner by Lagrange [99], that

$$A \otimes G\,(x,y) = \left(\frac{2}{\pi} \int\limits_0^{\pi/2} \frac{1}{\sqrt{x^2 \cos^2 t + y^2 \sin^2 t}} dt \right)^{-1},$$

for all $x, y \in (0,+\infty)$ (cf. [20,64,65] and [52]). The above equality shows a rather surprising connection of invariant means and mean iterations to elliptic integrals.

Using iterates of the map $(A,G)\colon (0,+\infty)^2 \to (0,+\infty)^2$ we can rewrite the convergence of Gaussian recurrences (2.1) in the form

$$(A,G)^n \longrightarrow (A \otimes G, A \otimes G).$$

During the last 50 years this procedure was considerably generalized by a number of mathematicians. Some important steps to this aim were made by Lehmer [100] in 1971, Schoenberg [147] in 1982, and Foster and Phillips [63] in 1984. Their ideas were followed and extended by Borweins in the book [20] three years later. A percipient reader can find the following result there.

Theorem 2.1. (Generalized Gaussian Algorithm). *Let I be an interval and let $M_1, \ldots M_p\colon I^p \to I$ be continuous means such that*

$$\min \{M_1(x_1,\ldots,x_p), \ldots, M_p(x_1,\ldots,x_p)\} = \min\{x_1,\ldots,x_p\} \tag{2.2}$$

and

$$\max \{M_1(x_1,\ldots,x_p), \ldots, M_p(x_1,\ldots,x_p)\} = \max\{x_1,\ldots,x_p\} \tag{2.3}$$

imply $x_1 = \ldots = x_p$ for all $x_1,\ldots,x_p \in I$. Then there exists a continuous mean $K\colon I^p \to I$ such that

$$\lim_{n\to\infty} (M_1,\ldots,M_p)^n = (K,\ldots,K)$$

uniformly on every compact subset of I^p; moreover, K is the unique continuous $(M_1,\ldots,M_p)$-invariant mean:

$$K \circ (M_1,\ldots,M_p) = K.$$

The mean K uniquely determined in Theorem 2.1 is called the *Gauss composition of the means* $M_1,\ldots,M_p$ denoted by $M_1 \otimes \ldots \otimes M_p$. Observe that now equality (1.3) assumed for all $x_1,\ldots,x_p \in I$ can be rewritten in the form

$$(M_1 \otimes \ldots \otimes M_p) \circ (M_1,\ldots,M_p) = M_1 \otimes \ldots \otimes M_p,$$

or simply

$$(M_1 \otimes M_2) \circ (M_1, M_2) = M_1 \otimes M_2,$$

if $p = 2$. The reader looking for the proof in Borweins' book [20] should compile it combining some particular results, viz. [20, Theorems 8.2 and 8.3 jointly with two sentences just before Theorem 8.2, Example 1 on p. 247, Theorem 8.8, the paragraph just before Comments and Exercises on p. 269 and Example 7 on p. 272].

The Generalized Gaussian Algorithm, in its consolidated form, was rediscovered by Matkowski [111]: firstly in 1999 for $p = 2$, under slightly stronger assumptions (see Remark 1.2 below), and in the form presented in Theorem 2.1 in 2009 (see [118]). Notice that an important argument for the equality of some basic limits in the proof presented in [111] was missing. That gap was filled in [117, Proof of part 2, pp. 186-188]. However, already in 1988 Páles used the Gauss iterates for two arbitrary strict means in two variables to solve a problem dealing with some functional inequalities (see [140, proof of the Theorem]). In 1996 his reasoning was repeated by Matkowski and Wróbel in the paper [133] to prove a generalization of Páles' result. But none of them indicated that the common limit of the mixed iterates is a mean that is invariant with respect to the given pair of means (cf. also [35, Lemma 1]). In 2013 Matkowski generalized the "moreover" part of Theorem 2.1 proving the uniqueness of the $(M_1, \ldots, M_p)$-invariant mean in the class of all, not necessarily continuous, means on I (see [127]). A lot of information about the Gaussian procedure as well as some examples of Gauss compositions are provided in the survey article [52] by Daróczy and Páles, published in 2002. The reader is also referred to the paper [37] by Daróczy where a nice elementary proof of the existence of $\sqrt{2}$ is presented making use of the Gaussian iteration built with the harmonic and arithmetic means. This is, in fact, a classical Babylonian method of approximately extracting the square root of 2. For that procedure as well as some other iterative algorithms involving the arithmetic, geometric and harmonic means the reader is referred to [27] by Carlson, published in 1971.

A version of Theorem 2.1 for semigroups of pairs of means and the invariance with respect to such semigroups was proved by Matkowski [112, Theorem 2].

Remark 2.2. Matkowski (cf. also [52, Theorem 1.5]) proving in [111] his version of Theorem 2.1, assumed that $p = 2$ and at least one of the means M_1 and M_2 is strict. For an arbitrary p that assumption can be reformulated as follows: *at most one of the means $M_1, \ldots, M_p$ is not strict.* Observe that this implies the property of $M_1, \ldots, M_p$ postulated in Theorem 2.1. This is obvious if $p = 1$ as the unique mean on I in one variable is the identity function. So consider the case $p \geq 2$. Without loss of generality we may assume that the means $M_1, \ldots, M_{p-1}$ are strict. Fix any $x_1, \ldots, x_p \in I$ such that equalities (2.2) and (2.3) hold and suppose that $x_1 = \ldots = x_p$ is not true. Then $\min\{x_1, \ldots, x_p\} < \max\{x_1, \ldots, x_p\}$ whence

$$\min\{x_1, \ldots, x_p\} < M_i(x_1, \ldots, x_p) < \max\{x_1, \ldots, x_p\}$$

for all $i = 1, \ldots, p-1$. Thus (2.2) and (2.3) imply

$$M_p(x_1, \ldots, x_p) = \min\{x_1, \ldots, x_p\}$$

and

$$M_p(x_1, \ldots, x_p) = \max\{x_1, \ldots, x_p\},$$

respectively, which is impossible. Consequently, $x_1 = \ldots = x_p$ and we are done.

Recently, the Generalized Gauss Algorithm has been extended to means depending on parameter (see [84]). Given an interval I, a positive integer p and a non-void set Ω, a function $M\colon I^p \times \Omega \to I$ is called a *parametrized mean on* I if $M(\cdot, \omega)$ is a mean for all $\omega \in \Omega$. The main result of [84], viz. Theorem 3.5, makes use of iterates $(M_1, \ldots, M_p)^n$ of the mapping $(M_1, \ldots, M_p) : I^p \times \Omega \to I^p$. They are defined according to the following definition coming from the paper [14] by Baron and Kuczma (see also [60] by Diamond for a parallel notion introduced independently).

For a fixed set Ω put $\Omega^\infty \colon = \Omega^{\mathbb{N}}$. Given a set X and a function $f\colon X \times \Omega \to X$, define the iterates $f^n\colon X \times \Omega^\infty \to X$, $n \in \mathbb{N}$, by the equality

$$f^1(x, \omega_1, \omega_2, \ldots) \colon = f(x, \omega_1)$$

and the recurrence

$$f^{n+1}(x, \omega_1, \omega_2, \ldots) \colon = f\left(f^1(x, \omega_1, \omega_2, \ldots), \omega_{n+1}\right)$$

postulated for all $x \in X$, $(\omega_1, \omega_2, \ldots) \in \Omega^\infty$ and $n \in \mathbb{N}$. Observe that the n-th iterate $f^n(\cdot, \omega_1, \omega_2, \ldots)$ depends, in fact, on the first parameters $\omega_1, \ldots, \omega_n$ only.

Theorem 3.5 of [84] generalizes Theorem 2.1 to the case when the set Ω of parameters is a compact topological space. That assumption is essential for the validity of Theorem 3.5 as follows from Example 3.4 presented in [84].

3. Invariance in the class of weighted quasi-arithmetic means

3.1. Quasi-arithmetic means

They as well as more general weighted quasi-arithmetic means constitute classes naturally extending the arithmetic mean A. It seems that the idea of a quasi-arithmetic mean was formed in [94] by Knopp already in 1928. Then the notion was formally introduced independently and almost simultaneously by Kolmogoroff [97], Nagumo [137] in 1930 and by de Finetti [62] a year later. Given an interval I we denote by $\mathcal{CM}(I)$ the class of continuous strictly monotonic functions mapping I into $\mathbb{R}$. A mean $M\colon \bigcup_{n=1}^\infty I^n \to I$ is called *quasi-arithmetic* if there is a $\varphi \in \mathcal{CM}(I)$ such that

$$M(x_1, \ldots, x_n) = \varphi^{-1}\left(\frac{\varphi(x_1) + \ldots + \varphi(x_n)}{n}\right)$$

for all $x_1, \ldots, x_n \in I$ and $n \in \mathbb{N}$. Any such φ is called a *generator* of the mean M. In what follows the quasi-arithmetic mean generated by φ will be denoted by A^φ. Observe that the mean A^φ is conjugated to the arithmetic mean by

φ. Namely, if $n \in \mathbb{R}$ is fixed and we consider A^φ as a mean in n variables: $A^\varphi\colon I^n \to I$, then

$$A^\varphi = \varphi^{-1} \circ A \circ (\varphi, \dots, \varphi)$$

where we take n copies of φ in the parentheses. There is at least one name more, namely that of Chisini, when thinking about the origins of the notion of quasi-arithmetic mean; de Finetti based his paper [62] mainly on some ideas of Chisini [29] presented in 1929, that is still a year before papers [97] and [137] by Kolmogoroff and Nagumo, respectively. Nowadays quasi-arithmetic means are sometimes also called *Kolmogoroff means*.

The classical means of the ancient world: the arithmetic mean A, the geometric mean G and the harmonic mean H are quasi-arithmetic. Indeed, since

$$A(x,y) = \frac{x+y}{2}, \qquad x, y \in \mathbb{R},$$
$$G(x,y) = \sqrt{xy} = \exp\left(\frac{\log x + \log y}{2}\right), \qquad x, y \in (0+\infty),$$

and

$$H(x,y) = \frac{2xy}{x+y} = \frac{1}{\frac{\frac{1}{x}+\frac{1}{y}}{2}}, \qquad x, y \in (0+\infty),$$

their generators φ are given by $\varphi(x) = x$, $\varphi(x) = \log x$ and $\varphi(x) = 1/x$, respectively.

There is a vast literature dealing with quasi-arithmetic means. First of all, chronologically, we mention the paper [74] by Hardy, Littlewood and Pólya. Next, the late forties has a rich bibliography relating to quasi-arithmetic means: [1,5,76,86–88,93,146,152] (see also [30] and [31]). Some newer results have been described for instance in [96,107,142–145]. The last one deals with the speed of convergence of Gauss iterations for a class of mean-type mappings built with some quasi-arithmetic means. Also some books deal with this kind of means: [2] by Aczél, [22] by Bullen, Mitrinović and Vasić, [4] by Aczél and Dhombres, finally [21] due to P.S. Bullen.

The quasi-arithmetic means considered in the present paper are in two variables. The below famous characterization of quasi-arithmetic means was proved by Aczél (cf. [1] and [2, 6.4.1]):

Theorem 3.1. *Let I be an interval and let $M\colon I^2 \to I$. The function M is a quasi-arithmetic mean on I if and only if M is a continuous, strictly increasing in each variable, reflexive*

$$M(x,x) = x, \quad x \in I,$$

and symmetric

$$M(x,y) = M(y,x), \quad x, y \in I,$$

solution of the bisymmetry equation

$$M\left(M(x,y), M(u,v)\right) = M\left(M(x,u), M(y,v)\right).$$

The following useful notion us allows to simplify formulations and proofs of results dealing with means having function generators, so, in particular, concerning the problem of invariance in the class of quasi-arithmetic means. Given a set X we say that functions $\varphi\colon X \to \mathbb{R}$ and $\psi\colon X \to \mathbb{R}$ are *equivalent* or φ is *equivalent to* ψ if there are numbers $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that

$$\psi(x) = a\varphi(x) + b, \quad x \in X;$$

then we write

$$\varphi(x) \sim \psi(x), \qquad x \in X,$$

or, simply, $\varphi \sim \psi$. Clearly, $\sim$ is an equivalence relation in the set $\mathbb{R}^X$ of real-valued functions defined on X.

Using this notion one can give the below answer to the equality problem for quasi-arithmetic means (see [74,95] and [2, Sec. 6.4, Theorem 2] also [89,90], [35, Theorem 2] and [52, Theorem 2.3]).

Theorem 3.2. *Let I be an interval and $\varphi, \psi \in \mathcal{CM}(I)$. Then $A^\varphi = A^\psi$ if and only if $\varphi \sim \psi$.*

We will refer to this result while studying the invariance problem in the class of quasi-arithmetic means.

3.2. The Matkowski-Sutô problem

Fix a non-trivial interval $I \subset \mathbb{R}$. During the 5th International Conference on Functional Equations and Inequalities held in Muszyna-Złockie (Poland) in 1995 Matkowski asked about all functions $\varphi, \psi \in \mathcal{CM}(I)$ such that the pair (φ, ψ) satisfies the functional equation

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) + \psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) = x + y. \tag{3.1}$$

This problem was published in [108] but only in 1998. A year later he gave the following partial answer (see [109, Theorem 1]).

Theorem 3.3. *Let at least one of functions $\varphi, \psi \in \mathcal{CM}(I)$ be twice continuously differentiable. Then the pair (φ, ψ) satisfies Eq. (3.1) if and only if either*

$$\varphi(x) \sim x, \quad x \in I, \quad \textit{and} \quad \psi(x) \sim x, \quad x \in I, \tag{3.2}$$

or

$$\varphi(x) \sim \mathrm{e}^{ax}, \quad x \in I, \quad \textit{and} \quad \psi(x) \sim \mathrm{e}^{-ax}, \quad x \in I, \tag{3.3}$$

with some $a \in \mathbb{R} \setminus \{0\}$.

Observe that Eq. (3.1) can be rewritten in the equivalent form

$$A \circ \left(A^{\varphi}, A^{\psi}\right) = A,$$

which expresses the invariance of the arithmetic mean A with respect to the pair $\left(A^{\varphi}, A^{\psi}\right)$ of quasi-arithmetic means. So Theorem 3.3 can be reformulated as follows (cf. also Theorem 3.2).

Theorem 3.4. *Let at least one of functions $\varphi, \psi \in \mathcal{CM}(I)$ be twice continuously differentiable. Then the arithmetic mean A is invariant with respect to the pair $\left(A^{\varphi}, A^{\psi}\right)$ if and only if either $A^{\varphi} = A^{\psi} = A$, or*

$$A^{\varphi}(x,y) = \frac{1}{a}\log\left(\frac{\mathrm{e}^{ax}+\mathrm{e}^{ay}}{2}\right) \text{ and } A^{\psi}(x,y) = -\frac{1}{a}\log\left(\frac{\mathrm{e}^{-ax}+\mathrm{e}^{-ay}}{2}\right) \tag{3.4}$$

for every $x, y \in I$, with some $a \in \mathbb{R}\setminus\{0\}$.

The regularity assumption about the generators φ and ψ is not quite natural since the formulation of the original problem does not involve regularity conditions at all. But it was very useful in Matkowski's proof. The crucial tool in his reasoning is the following fact.

Lemma 3.5. *If $\varphi, \psi \in \mathcal{CM}(I)$ are twice differentiable functions and the pair (φ, ψ) satisfies Eq. (3.1), then*

$$\varphi'(x)\psi'(x) = c, \qquad x \in I,$$

with some $c \in \mathbb{R}\setminus\{0\}$.

Simultaneously, Hungarian colleagues were also working on Matkowski's problem. First of all they discovered the elderly and forgotten two-part paper [149,150] written by Sutô and published in the Tôhoku Mathematical Journal in 1914. It deals with a number of functional equations, in particular with Eq. (3.1). Sutô's result (see [150]) requires stronger regularity assumptions than those made by Matkowski in Theorem 3.3; it reads as follows.

Theorem 3.6. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be analytic. Then the pair (φ, ψ) satisfies Eq. (3.1) if and only if either condition (3.2), or (3.3) with some $a \in \mathbb{R}\setminus\{0\}$, holds.*

A little bit after the paper [108] was accepted for publication, viz. at the end of 1998, Daróczy and Páles substantially weakened the assumptions of Theorem 3.3 by omitting the word *twice* and achieved the below generalization.

Theorem 3.7. *Let at least one of functions $\varphi, \psi \in \mathcal{CM}(I)$ be continuously differentiable. Then the pair (φ, ψ) satisfies Eq. (3.1) if and only if either condition (3.2), or (3.3) with some $a \in \mathbb{R}\setminus\{0\}$, holds.*

It was published only in 2001 in their paper [49], where the so-called *conjugate arithmetic means*, that is means on I of the form

$$\varphi^{-1}\left(\varphi(x)+\varphi(y)-\varphi\left(\frac{x+y}{2}\right)\right)$$

with $\varphi \in \mathcal{CM}(I)$, have been studied (see also subsection 2.5.b). The main result of [49], that is Theorem 1, provides all continuously differentiable functions φ generating conjugate arithmetic means which are simultaneously quasi-arithmetic. Making use of this the authors deduced Theorem 3.7. Its immediate proof was presented by Daróczy and Páles in the article [52] a year later. The following extension theorem, proved by Daróczy, Maksa and Páles in [47] turned out to be a useful tool there (see also [52, Theorem 3.14]), and in the subsequent research which finally solved the Matkowski-Sutô problem (see [52, Sec. 4]).

Proposition 3.8. (Extension theorem) *Let $\varphi, \psi \in \mathcal{CM}(I)$. If the pair (φ, ψ) satisfies Eq. (3.1) and there is a non-trivial interval $K \subset I$ such that*

$$\varphi(x) \sim x, \quad x \in K, \qquad \text{and} \qquad \psi(x) \sim x, \quad x \in K, \tag{3.5}$$

or

$$\varphi(x) \sim \mathrm{e}^{ax}, \quad x \in K, \qquad \text{and} \qquad \psi(x) \sim \mathrm{e}^{-ax}, \quad x \in K, \tag{3.6}$$

with some $a \in \mathbb{R} \setminus \{0\}$, then either condition (3.2) or (3.3) holds.

Making some simple calculations one can show that when proving Theorem 3.7 we can confine ourselves to the case when both functions φ, ψ are continuously differentiable. The next step is to observe that $\varphi'(x) \neq 0$ and $\psi'(x) \neq 0$ for x's running through a non-trivial subinterval of I. In fact the following much more general fact holds true (see [52, Theorem 4.8]).

Proposition 3.9. *Let $\varphi, \psi \in \mathcal{CM}(I)$ and assume that the pair (φ, ψ) satisfies Eq. (3.1). Then there exists a non-trivial interval $K \subset I$ on which φ and ψ are differentiable and $\varphi'(x) \neq 0$ and $\psi'(x) \neq 0$ for all $x \in K$.*

The above theorem was an important tool while solving the Matkowski-Sutô problem in the general form. It allows us to make the step from the continuity of a solution to its differentiability on a subinterval. We postpone the description of next steps in proving Theorem 3.7 for a moment to point out main facts resulting in a proof of Proposition 3.9.

First of all, making use of Lebesgue's theorem on differentiating monotonic functions almost everywhere (with respect to the Lebesgue measure) and the fact that the Lebesgue integral of the derivative of an absolutely continuous function over an interval is the increment of the given function, one can come to the following important fact (see [52, Theorems 4.1 and 4.3]).

Proposition 3.10. *Let $\varphi, \psi \in \mathcal{CM}(I)$ and assume that the pair (φ, ψ) satisfies Eq. (3.1). Then the functions $\varphi, \psi, \varphi^{-1}, \psi^{-1}$ are locally Lipschitz.*

Using this result and again Lebesgue's theorem we can prove the next crucial fact providing points of differentiability of φ and ψ (see [52, Theorem 4.7].

Proposition 3.11. *Let $\varphi, \psi \in \mathcal{CM}(I)$ and $t_0 \in J\colon = \varphi(I)$. If the pair (φ, ψ) satisfies Eq. (3.1), then either*

$$\varphi^{-1}(t_0+t)+\varphi^{-1}(t_0-t)=2\varphi^{-1}(t_0), \qquad t \in (J-t_0)\cap(t_0-J), \tag{3.7}$$

or the function φ^{-1} is differentiable at t_0.

Since the set of all t's from J satisfying condition (3.7) is a closed subset of J it can be deduced from Proposition 3.11 that both φ and ψ are differentiable on a non-trivial interval $K \subset I$. Moreover, it follows from Proposition 3.10 that $\varphi'(x) \neq 0$ and $\psi'(x) \neq 0$ whenever $x \in K$. This gives the assertion of Proposition 3.9.

Now coming back to the sketch of the proof of Theorem 3.7, we may assume that the functions φ and ψ are continuously differentiable and, in addition, their derivatives do not vanish in the interval K. Since the derivative of any function defined on an interval has the Darboux property, we may assume that, in fact, $\varphi'(x) > 0$ and $\psi'(x) > 0$ for all $x \in K$. By virtue of Proposition 3.8 it is enough to prove that either condition (3.5), or (3.6) with some $a \in \mathbb{R}\setminus\{0\}$, holds. This, however, can be easily obtained having the following series of lemmas (see [52, Theorem 3.7, and Lemmas 3.8 and 3.9], also [49, Lemma 3]).

Lemma 3.12. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be differentiable functions with positive derivatives. If the pair (φ, ψ) satisfies Eq. (3.1), then (f, g), where $f = \varphi' \circ \varphi^{-1}$ and $g = \psi' \circ \varphi^{-1}$, is a solution of the equation*

$$2f\left(\frac{u+v}{2}\right)(g(v)-g(u)) = f(u)g(v)-f(v)g(u). \tag{3.8}$$

Actually this results was formulated under the additional assumption of the continuity of φ' and ψ' which, however, was not used in its proof (cf. [52, Theorem 3.7]).

Lemma 3.13. *If f and g are functions mapping continuously an interval J into $(0, +\infty)$ and the pair (f, g) satisfies Eq. (3.8), then there exists a $c \in (0, +\infty)$ such that*

$$f(u)g(u) = c, \qquad u \in J.$$

The last lemma deals with the equation

$$\left(f\left(\frac{u+v}{2}\right) - \frac{f(u)+f(v)}{2}\right)(f(u)-f(v)) = 0 \tag{3.9}$$

containing only one unknown function.

Lemma 3.14. *If f is a continuous real-valued solution of Eq. (3.9), defined on an interval J, then there exist $a, b \in \mathbb{R}$ such that*

$$f(u) = au + b, \qquad u \in J.$$

A little bit earlier the Hungarian group, working hard to remove the regularity assumption in Theorem 3.7, also tried to steer the research in a different direction. This other approach was connected with the notion of strict comparability of means (cf. [74] also [98, Sec. VIII. 3]. Two means M and N on I are said to be *strictly comparable* if

$$M(x,y) \lhd N(x,y), \qquad x,y \in I,\, x \neq y,$$

where $\lhd$ is one of the relations $=, <, >$. Making use of this notion, already at the beginning of 1999, Daróczy and Maksa proved what follows (see [46, Theorem 3]).

Theorem 3.15. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be functions generating strictly comparable means A^{φ} and A^{ψ}. Then the pair (φ, ψ) satisfies Eq. (3.1) if and only if either condition (3.2), or (3.3) with some $a \in \mathbb{R} \setminus \{0\}$, holds.*

Now we briefly report on the ultimate answer to the Matkowski-Sutô problem that was given by Daróczy and Páles in 2002 and reads as follows (see [52, Theorem 4.12]).

Theorem 3.16. *Let $\varphi, \psi \in \mathcal{CM}(I)$. Then the pair (φ, ψ) satisfies Eq. (3.1) if and only if either condition (3.2), or (3.3) with some $a \in \mathbb{R} \setminus \{0\}$, holds.*

First of all observe that, according to Propositions 3.9 and 3.8 , we may assume that the functions φ and ψ are differentiable and have nonvanishing derivatives. The derivative of any function defined on an interval has the Darboux property. Thus, replacing if necessary φ by $-\varphi$ and/or ψ by $-\psi$ (cf. Theorem 3.2), we may additionally assume that $\varphi'(x) > 0$ and $\psi'(x) > 0$ for all $x \in I$. Then, putting $f = \varphi' \circ \varphi^{-1}$, $g = \psi' \circ \varphi^{-1}$ and applying Lemma 3.12, we see that the pair (f, g) is a solution of Eq. (3.8). Let $J = \varphi(I)$ and notice that f and g are elements of the set $\mathcal{D}(J)$ of all the compositions $d \circ \chi$, where $\chi \in \mathcal{CM}(J)$ and d is the positive derivative of a function defined on the interval $\chi(J)$. The crucial role, in the next argument leading to the assertion of Theorem 3.16, is played by the following result (see [52, Theorem 4.10]).

Proposition 3.17. *If $f, g \in \mathcal{D}(J)$ and the pair (f, g) satisfies Eq. (3.8), then the function f is continuous in a non-trivial subinterval of the interval J.*

While proving this result one may assume that both f and g are constant on no non-trivial subinterval of J. Since derivatives are of Baire class 1, so are elements of the class $\mathcal{D}(J)$. Thus, by virtue of Baire's theorem (cf. for instance, [138, Theorem 7.3], the set of all points of continuity of the function g is a dense G_δ subset of the interval J. Using this fact one can show the existence of such $u_0, v_0 \in J$ that g is continuous at u_0, v_0 and $g(u_0) \neq g(v_0)$. Therefore, since the pair (f, g) satisfies Eq. (3.8), we see that

$$f\left(\frac{u+v}{2}\right) = \frac{1}{2}\frac{f(u)g(v) - f(v)g(u)}{g(v) - g(u)}$$

for u and v running through neighbourhoods U and V of u_0 and v_0, respectively. Now, using one of the results of Járai, viz. [77, Theorem 8.6], important in proving regularity, one can prove the continuity of f on a non-trivial interval. Now, making use of Proposition 3.17 we see that the function $\varphi' \circ \varphi^{-1}$ is continuous on a non-trivial subinterval of $\varphi(I)$. Consequently, φ is continuously differentiable on a non-trivial subinterval of I, and thus the assertion of Theorem 3.16 follows from Theorem 3.7.

We complete this subsection with the below reformulation of Theorem 3.16 in the language of the invariance of means. It also generalizes Theorem 3.4.

Theorem 3.18. *Let $\varphi, \psi \in \mathcal{CM}(I)$. Then the arithmetic mean A is invariant with respect to the pair $\left(A^\varphi, A^\psi\right)$ if and only if either $A^\varphi = A^\psi = A$, or the means A^φ and A^ψ are given by condition (3.4) with some $a \in \mathbb{R} \setminus \{0\}$.*

3.3. Some supplementary remarks

At the very beginning we solve the problem of invariance in the class of quasi-arithmetic means. The below result is an almost immediate consequence of Theorem 3.18.

Theorem 3.19. *Let $\varphi, \psi, \chi \in \mathcal{CM}(I)$. The mean A^χ is invariant with respect to the pair $\left(A^\varphi, A^\psi\right)$, that is*

$$A^\chi = A^\varphi \otimes A^\psi, \tag{3.10}$$

if and only if either $A^\varphi = A^\psi = A^\chi$, or

$$A^\varphi(x, y) = \chi^{-1}\left(\frac{1}{a}\log\left(\frac{\mathrm{e}^{a\chi(x)} + \mathrm{e}^{a\chi(y)}}{2}\right)\right)$$

and

$$A^\psi(x, y) = \chi^{-1}\left(-\frac{1}{a}\log\left(\frac{\mathrm{e}^{-a\chi(x)} + \mathrm{e}^{-a\chi(y)}}{2}\right)\right)$$

for every $x, y \in I$, with some $a \in \mathbb{R} \setminus \{0\}$.

Equation (3.10) can be seen as that expressing the generalized Matkowski-Sutô problem in the class of quasi-arithmetic means. Clearly, in different classes of means the answer to this problem can vary. Among quasi-arithmetic means especially important are *power means* or *Hölder means* which, in fact, constitute the class of all homogeneous quasi-arithmetic means (see [74]). Given a real number p we denote by H^p the power mean on the interval $(0, +\infty)$, defined by

$$H^p(x, y) = \begin{cases} \left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}}, & \text{if } p \neq 0, \\ \sqrt{xy}, & \text{if } p = 0. \end{cases}$$

So $H^p = A^{h_p}$, where $h_p\colon (0,+\infty) \to \mathbb{R}$ is given by

$$h_p(x) = \begin{cases} x^p, & \text{if } p \neq 0, \\ \log x, & \text{if } p = 0, \end{cases} \tag{3.11}$$

for each $p \in \mathbb{R}$. The invariance problem (the generalized Matkowski-Sutô problem) in the class of power means was solved in [100] by Lehmer. There one can find the following result (see also [52, Theorem 2.9]).

Theorem 3.20. *Let $p, q, s \in \mathbb{R}$. Then mean H^s is invariant with respect to the pair (H^p, H^q), that is*

$$H^s = H^p \otimes H^q, \tag{3.12}$$

if and only if either $s \neq 0$ and $p = q = s$, or $s = 0$ and $p + q = 0$.

This was rediscovered by Kahlig and Matkowski (see [91, Theorem 1] in 1997. In fact, in Theorem 1 from [91] one can find more. In fact, instead of the equation

$$H^s \circ (H^p, H^q) = H^s,$$

which is actually (3.12), the equation

$$H^s \circ (H^p, H^q) = H^r$$

with unknown real numbers p, q, r, s was solved there.

Among papers connected with the Matkowski-Sutô problem there is [71] by Głazowska, the second present author and Matkowski, which should be also mentioned. There the authors determined all quasi-arithmetic means A^φ, A^ψ generated by twice continuously differentiable φ, ψ, and the real numbers r, s such that

$$rA^\varphi + sA^\psi = A.$$

Actually we may assume that $r + s = 1$. Indeed, even more generally: if M and N are means on a common non-trivial interval I and $rM + sN$ is a mean with some $r, s \in \mathbb{R}$, then, by the reflexivity of the means, we have

$$rx + sx = rM(x, x) + sN(x, x) = (rM + sN)(x, x) = x$$

for all $x \in I$, whence $r + s = 1$ as I is non-trivial. So, in fact, we deal with the equation

$$rA^\varphi + (1 - r)A^\psi = A$$

or, equivalently,

$$r\varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right) + (1 - r)\psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right) = \frac{x + y}{2} \tag{3.13}$$

for all x, y running through the given interval. Observe that putting here $r = 1/2$ we come to Eq. (3.1) completely solved in the present section.

For some applications of the results presented in Section 3.3, both in the theory of means and in economy, as well as for some further facts and open problems concerning invariance in the class of quasi-arithmetic means, the reader is referred to [52, Chap. 5].

3.4. Solution in the class of weighted quasi-arithmetic means

Given an interval I, a function $\varphi \in \mathcal{CM}(I)$ and positive numbers $p_1, \ldots, p_n$ summing up to 1 we define the *weighted quasi-arithmetic mean* $A^{\varphi}_{(p_1,\ldots,p_n)}\colon I^n \to I$ putting

$$A^{\varphi}_{(p_1,\ldots,p_n)}(x_1,\ldots,x_n) = \varphi^{-1}\left(p_1\varphi(x_1) + \ldots + p_n\varphi(x_n)\right)$$

for all $x_1,\ldots,x_n \in I$. The function φ is called its *generator* and $p_1,\ldots,p_n$ are the *weights* of the mean. Clearly, the arithmetic mean A^{φ} in n variables is weighted with $p_1 = \ldots = p_n = 1/n$. In what follows we focus on weighted quasi-arithmetic means in two variables. Then instead of $A^{\varphi}_{p,1-p}$ we write A^{φ}_{p}, so given $\varphi \in \mathcal{CM}(I)$ and $p \in (0,1)$ we have

$$A^{\varphi}_{p}(x,y) = \varphi^{-1}\left(p\varphi(x) + (1-p)\varphi(y)\right)$$

for all $x, y \in I$.

The below extension of Theorem 3.2 to weighted quasi-arithmetic means will be useful in the next discussion (see [2, Sec. 6.4, Theorem 2], also [106]).

Theorem 3.21. *Let I be an interval, $\varphi, \psi \in \mathcal{CM}(I)$ and $p, q \in (0,1)$. Then $A^{\varphi}_{p} = A^{\psi}_{q}$ if and only if $\varphi \sim \psi$ and $p = q$.*

The problem of invariance in the class of weighted quasi-arithmetic means on the given interval I is to look for all functions $\varphi, \psi, \chi \in \mathcal{CM}(I)$ and numbers $p, q, r \in (0,1)$ such that the mean A^{χ}_{r} is the Gauss composition of A^{φ}_{p} and A^{ψ}_{q}:

$$A^{\chi}_{r} = A^{\varphi}_{p} \otimes A^{\psi}_{q}. \tag{3.14}$$

Considering the definition of Gauss composition and the form of weighted quasi-arithmetic means one can write Eq. (3.14) in the following equivalent form:

$$\begin{aligned} &r\chi\left(\varphi^{-1}\left(p\varphi(x) + (1-p)\varphi(y)\right)\right) + (1-r)\chi\left(\psi^{-1}\left(q\psi(x) + (1-q)\psi(y)\right)\right) \\ &\quad = r\chi(x) + (1-r)\chi(y). \end{aligned} \tag{3.15}$$

Observe that putting here $\chi = \mathrm{id}|_I$ and $p = q = r = 1/2$ we come to Eq. (3.1).

It seems that it was paper [53] by Daróczy and Páles where a first step in solving the general form of (3.15) was made. There the authors determined all continuously differentiable, with nonvanishing derivatives functions $\varphi, \psi \in \mathcal{CM}(I)$, satisfying the equation

$$\begin{aligned} &r\varphi^{-1}\left(r\varphi(x)+(1-r)\varphi(y)\right)+(1-r)\psi^{-1}\left(r\psi(x)+(1-r)\psi(y)\right) \\ &\quad = rx+(1-r)y \end{aligned} \tag{3.16}$$

in the case when $r \in (0,1)\setminus\{1/2\}$ (see [53, Theorem]). Since (3.16) for $r = 1/2$ is simply (3.1), it follows from Theorem 3.16 that in the case $r \in (0,1)$ we are done in the considered class of functions. Clearly, Eq. (3.16) is a particular case of (3.15) with $\chi = \mathrm{id}_I$ and $p = q = r$.

In the same year, 2003, Daróczy and Páles gave the complete solution of Eq. (3.16) in the class $\mathcal{CM}(I)$, proving the following result in [54]. The set of solutions in this class is exactly the same as that described in [53].

Theorem 3.22. *Let $\varphi, \psi \in \mathcal{CM}(I)$ and $r \in (0,1)\setminus\{1/2\}$. Then the pair (φ,ψ) satisfies Eq. (3.16) if and only if condition (3.2) holds.*

Consequently, it follows from Theorems 3.16 and 3.22 that Eq. (3.16), although formally more general than (3.1), admits the same solutions (φ,ψ). One of the tools allowing us to pass from the result of [53] to Theorem 3.22 was an extension theorem for solutions of Eq. (3.16) obtained by Daróczy, Hajdu and Ng [43] in 2003.

Another particular case of Eq. (3.15) different from (3.16), viz. the equation

$$\varphi^{-1}\left(p\varphi(x)+(1-p)\varphi(y)\right)+\varphi^{-1}\left((1-p)\varphi(x)+p\varphi(y)\right)=x+y, \tag{3.17}$$

was investigated by Burai in [23]. Clearly, we come to (3.17) putting $\chi = \mathrm{id}|_I$, $q = 1-p$ and $r = 1/2$ in Eq. (3.15). On the other hand (3.17) with $p = 1/2$ becomes Eq. (3.1). It turns out that also this problem has the same set of solutions as the cases considered previously. Its detailed description is given in the following result (see [23, Theorem 6]).

Theorem 3.23. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be continuously differentiable on a nontrivial subinterval of the interval I and let $p \in (0,1)$. Then the pair (φ,ψ) satisfies Eq. (3.17) if and only if either condition (3.2), or (3.3) with some $a \in \mathbb{R}\setminus\{0\}$, holds.*

Among the tools, useful in proving this result, there is an extension theorem (see [23, Theorem 2]) generalizing the one formulated here as Proposition 3.8.

After a short break, the investigations came back to Poland. In 2006 the first present author and Matkowski published the following result dealing with the equation

$$\begin{aligned} &r\varphi^{-1}\left(p\varphi(x)+(1-p)\varphi(y)\right)+(1-r)\psi^{-1}\left(q\psi(x)+(1-q)\psi(y)\right) \\ &\quad = rx+(1-r)y, \end{aligned} \tag{3.18}$$

which is (3.15) with $\chi = \mathrm{id}_I$ (see [85, Theorem 1]).

Theorem 3.24. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be twice continuously differentiable and $p, q, r \in (0,1)$. Then the pair (φ, ψ) and the triple (p, q, r) satisfy Eq. (3.18) if and only if*
(i) $r = \frac{q}{1-p+q}$
and
(ii) *condition (3.2) holds or $r = 1/2$, $p + q = 1$ and condition (3.3) with some $a \in \mathbb{R} \setminus \{0\}$ holds.*

A key role in the proof of Theorem 3.24 is played by the below description of conditionally homogeneous weighted quasi-arithmetic means (see [44], also [85, Proposition 1]).

Proposition 3.25. *Assume that $I \subset (0, +\infty)$. Let $\sigma \in \mathcal{CM}(I)$ and $s \in (0,1)$. The mean A_s^σ is conditionally homogeneous:*

$$A_s^\sigma(tx, ty) = tA_s^\sigma(x, y)$$

for all $x, y \in I$ and $t \in (0, +\infty)$ with $tx, ty \in I$ if and only if either

$$\sigma(x) \sim x^p, \qquad x \in I,$$

for some $p \in \mathbb{R} \setminus \{0\}$, or

$$\sigma(x) \sim \log x, \qquad x \in I,$$

that is A_s^σ is the weighted power mean: either

$$A_s^\sigma(x, y) = (sx^p + (1-s)y^p)^{\frac{1}{p}}, \qquad x, y \in I,$$

for some $p \in \mathbb{R} \setminus \{0\}$, or

$$A_s^\sigma(x, y) = x^s y^{1-s}, \quad x, y \in I.$$

This result generalizes the classical theorem stating that power means H^p, where $p \in \mathbb{R}$, are the only homogeneous quasi-arithmetic means on $(0, +\infty)$ (see [74]). For further generalizations the reader is referred to paper [25] by Burai and the first author, where the conditional homogeneity of Makó–Páles means of the form

$$I^2 \ni (x, y) \longmapsto \varphi^{-1}\left(\int_0^1 \varphi\left(tx + (1-t)y\right) d\mu(t)\right),$$

where $\varphi \in \mathcal{CM}(I)$ and μ is a probability Borel measure on $[0, 1]$, is examined (see [25, Theorem 3.1]). This class is a common extension of the classes of weighted quasi-arithmetic means and Lagrangian means. For more information on Makó–Páles means the reader is referred to Subsection 6.1, which is devoted mainly to the invariance of the arithmetic mean A as well as the geometric mean G with respect to a pair of Makó–Páles means.

Besides Proposition 3.25 the below three analogues of Lemmas 3.12–3.14 play a crucial role while proving Theorem 3.24 (see [85, Lemmas 2–4]).

Lemma 3.26. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be differentiable functions with positive derivatives and let $p, q, r \in (0,1)$. If the pair (φ, ψ) satisfies Eq. (3.18), then (f, g) where $f = \varphi' \circ \varphi^{-1}$ and $g = \psi' \circ \varphi^{-1}$, is a solution of the equation*

$$\begin{aligned} &f\left(pu + (1-p)v\right)\left[(1-q)g(v) - (1-p)g(u)\right] \\ &\quad = p(1-q)f(u)g(v) - q(1-p)f(v)g(u). \end{aligned} \tag{3.19}$$

In the case when $p = q = 1/2$ Eq. (3.19) becomes (3.8), so Lemma 3.26 generalizes Lemma 3.12. The authors of [85] assumed in Lemma 2 there that φ' and ψ' are continuous but this is superfluous (cf. also the comment just after Lemma 3.12). The next result, although deals with solutions of Eq. (3.19), which is more general than Eq. (3.8), does not extend Lemma 3.13. This is because of the stronger assumption imposed on solutions here.

Lemma 3.27. *Let $p, q \in (0,1)$. If f and g are continuously differentiable functions mapping an interval J into $(0, +\infty)$ and the pair (f, g) satisfies equation (3.19), then there exists a $c \in (0, +\infty)$ such that*

$$f(u)^p g(u)^{1-q} = c, \qquad u \in J. \tag{3.20}$$

No doubt the last lemma, originating from paper [85], has the most involved argument. It deals with the equation

$$\begin{aligned} &f\left(pu + (1-p)v\right)\left[(1-q)f(v)^{-p/(1-q)} - (1-p)f(u)^{-p/(1-q)}\right] \\ &\quad = p(1-q)f(u)f(v)^{-p/(1-q)} - q(1-p)f(v)f(u)^{-p/(1-q)}, \end{aligned} \tag{3.21}$$

which, after setting $p = q = 1/2$, reduces to Eq. (3.9). Therefore the below lemma generalizes Lemma 3.14.

Lemma 3.28. *Let $p, q \in (0,1)$. If f is a continuous real-valued solution of Eq. (3.21), defined on an interval J, then there exists $a, b \in \mathbb{R}$ such that*

$$f(u) = au + b, \qquad u \in J;$$

if, in addition, $p + q \neq 1$, then $a = 0$.

Finally the problem of invariance in the class of weighted quasi-arithmetic means, i.e. Eq. (3.14), was completely solved in [78] by the first present author, in 2007. It turned out that if we impose no regularity assumptions on functions $\varphi, \psi \in \mathcal{CM}(I)$, then the set of solutions of the crucial Eq. (3.18) does not extend at all. The suitable counterpart of Theorem 3.24 reads as follows (see [78, Theorem 1]).

Theorem 3.29. *Let $\varphi, \psi \in \mathcal{CM}(I)$ and $p, q, r \in (0,1)$. Then the pair (φ, ψ) and the triple (p, q, r) satisfy Eq. (3.18) if and only if*
(i) $r = \frac{q}{1-p+q}$
and
(ii) *condition (3.2) holds or $r = 1/2$, $p + q = 1$ and condition (3.3) with $a \in \mathbb{R} \setminus \{0\}$ holds.*

To prove this result the argument used by Daróczy and Páles in validating Theorem 3.16 has been extended substantially. Below we formulate Proposition 3.30, Lemma 3.31 and Proposition 3.32, the main tools of [78]. They are generalizations of Proposition 3.17, Lemma 3.13 and Proposition 3.8, respectively (cf. [78, Lemmas 3–5]). The form of Eq. (3.18) is significantly more complicated than that of Eq. (3.1), and thus the proofs are now essentially longer and more involved. Also methods elaborated by Járai in monograph [77] are used to a larger extent than in the proof of Theorem 3.16.

Proposition 3.30. *Let f, g be positive functions defined on a non-trivial interval J. If f is Lebesgue measurable, g is of Baire class* 1 *and the pair* (f, g) *satisfies Eq.* (3.19)*, then the functions f and g are continuous in a non-trivial subinterval of the interval J.*

This result (see [78, Lemma 3]) essentially extends Proposition 3.17 in two directions. First of all it deals with Eq. (3.19) instead of its particular case (3.8). Secondly, it admits a larger class of solutions (f, g). Its rather long proof is an enhancement of that of Proposition 3.17 given by Daróczy and Páles in [52] (cf. also the sketch of its proof given in the present section).

Lemma 3.31. *Let $p, q \in (0, 1)$. If f and g are functions mapping continuously an interval J into $(0, +\infty)$ and the pair (f, g) satisfies Eq.* (3.19)*, then there exists a $c \in (0, +\infty)$ such that condition* (3.20) *holds.*

Also here one can observe two aspects of generalizing previous results. On the one hand the above lemma (see [78, Lemma 4]) extends Lemma 3.13 dealing with Eq. (3.8) to the more general Eq. (3.19). On the other hand it generalizes Lemma 3.27 weakening regularity assumptions imposed on the solution (f, g) there. The proof of Lemma 3.31 given in [78] runs in another way than that of Lemma 3.13 presented in [52]. If $p = q$ then Eq. (3.19) takes the form

$$f\left(pu + (1-p)v\right)\left(g(v) - g(u)\right) = p\left(f(u)g(v) - f(v)g(u)\right)$$

(which is (3.8) when we put $p = 1/2$) and the assertion follows from Theorem 2 by Daróczy and Páles as published in their paper [55] in 2003. In the case when $p \neq q$ we may argue as follows. At first, making use of Theorem 11.6 from the monograph [77], one can show that the function f is locally Lipschitzian, and thus, on account of Rademacher's theorem, it is differentiable almost everywhere (with respect to the Lebesgue measure) in J. Now, according to another result by Járai (see [77, Theorem 14.2]), f is continuously differentiable and, by Lemma 3.27, we are done.

Finally we formulate an extension theorem (see [78, Lemma 5]) generalizing some earlier results of this type, among others Proposition 3.8 as well as the theorem published in [43] dealing with solutions of Eq. (3.16).

Proposition 3.32. (Extension theorem) *Let $\varphi, \psi \in \mathcal{CM}(I)$ and $p, q, r \in (0, 1)$. If the pair (φ, ψ) satisfies Eq.* (3.18) *and there is a non-trivial interval $K \subset I$*

such that either condition (3.5)*, or* (3.6) *with some* $a \in \mathbb{R} \setminus \{0\}$ *is true, then either condition* (3.2)*, or* (3.3)*, holds.*

In the special case when $p = q = r$ in Eq. (3.18) the assertion of Proposition 3.32 follows immediately from [42, Theorem 4] where the extension theorem was proved for the equation

$$\begin{aligned} &r\varphi^{-1}\left(p\varphi(x) + (1-p)\varphi(y)\right) + (1-r)\psi^{-1}\left(p\psi(x) + (1-p)\psi(y)\right) \\ &\quad = px + (1-p)y. \end{aligned} \tag{3.22}$$

The proof of Theorem 3.29 presented in [78] starts with showing that the functions $\varphi, \varphi^{-1}, \psi, \psi^{-1}$ are locally Lipschitzian and their derivatives do not vanish wherever they exist. Since they have the Darboux property we may assume without loss of generality that they take positive values only. The next step is to prove that φ, ψ are differentiable on a non-trivial interval $I_0 \subset I$. This initial part of the proof takes the pattern of the argument used by Daróczy and Páles while proving Theorem 3.16. It requires only a bit of care as the complexity of Eq. (3.18) is higher than that of Eq. (3.1). Now, defining $f\colon J_0 \to (0, +\infty)$ and $g\colon J_0 \to (0, +\infty)$ by $f = \varphi' \circ \varphi^{-1}$ and $g = \psi' \circ \varphi^{-1}$, respectively, where $J_0 = \varphi(I_0)$, we come to a solution (f, g) of Eq. (3.19). By virtue of Proposition 3.30 and Lemma 3.31 there exists a positive number c such that condition (3.20) holds. Taking into account condition (3.20) we can rewrite Eq. (3.19) in the form of (3.21). Then, making use of Lemma 3.28 and the definition of f, we obtain the desired form of the function φ, and then also ψ, on a non-trivial subinterval of I. To complete the proof it is enough to apply Proposition 3.32.

We complete this Subsection with the following corollary from Theorem 3.29 providing the form of all weighted quasi-arithmetic means satisfying Eq. (3.14). This gives the full answer to the problem of invariance in the class of these means and generalizes Theorem 3.19.

Theorem 3.33. *Let* $\varphi, \psi, \chi \in \mathcal{CM}(I)$ *and* $p, q, r \in (0,1)$*. The mean* A_r^χ *is invariant with respect to the pair* $\left(A_p^\varphi, A_q^\psi\right)$*, that is the triplet* (φ, ψ, χ) *satisfies Eq.* (3.14) *if and only if*
(i) $r = \frac{q}{1-p+q}$
and
(ii) *either* $A_p^\varphi = A_p^\chi$ *and* $A_q^\psi = A_q^\chi$*, or* $r = 1/2$*,* $p + q = 1$*,*

$$A_p^\varphi(x,y) = \chi^{-1}\left(\frac{1}{a}\log\left(p\mathrm{e}^{a\chi(x)} + (1-p)\mathrm{e}^{a\chi(y)}\right)\right)$$

and

$$A_q^\psi(x,y) = \chi^{-1}\left(-\frac{1}{a}\log\left(q\mathrm{e}^{-a\chi(x)} + (1-q)\mathrm{e}^{-a\chi(y)}\right)\right)$$

for every $x, y \in I$*, with some* $a \in \mathbb{R} \setminus \{0\}$*.*

3.5. Generalized weighted quasi-arithmetic means in the sense of Matkowski

This generalization seems to be especially important and promising for further research. Given an interval I and strictly increasing (or strictly decreasing) functions $\varphi, \psi \in \mathcal{CM}(I)$ we define the mean $A^{[\varphi,\psi]}\colon I^2 \to I$ by the formula

$$A^{[\varphi,\psi]}(x,y) = (\varphi+\psi)^{-1}(\varphi(x)+\psi(y)).$$

It was introduced by Matkowski [114] in 2003. The first invariance results were obtained six years later by Matkowski and Volkmann in the paper [132] and by Baják and Páles in [6] (cf. also [124]). Taking any $\varphi \in \mathcal{CM}(I)$ and $p \in (0,1)$ we see that

$$A^{[p\varphi,(1-p)\varphi]}(x,y) = \varphi^{-1}(p\varphi(x)+(1-p)\varphi(y)) = A_p^{\varphi}(x,y),$$

that is the considered notion extends that of weighted quasi-arithmetic mean. For this reason $A^{[\varphi,\psi]}$ is called *generalized weighted quasi-arithmetic mean with the generators φ and ψ*. In fact, while defining the mean $A^{[\varphi,\psi]}$, it is enough to assume less, namely that the functions $\varphi, \psi\colon I \to \mathbb{R}$ are such that $\varphi(I)+\psi(I) \subset (\varphi+\psi)(I)$ and $\varphi+\psi \in \mathcal{CM}(I)$.

Many different properties of generalized weighted quasi-arithmetic means have been extensively studied by Matkowski in his papers [119,128,130] and [131]. Most of them deal with a natural generalization of $A^{[\varphi,\psi]}$ to the mean $A^{[\varphi_1,\ldots,\varphi_n]}$ in n variables, where $n \geq 2$ is a fixed integer. Given functions $\varphi_1, \ldots, \varphi_n \in \mathcal{CM}(I)$ of the same type of monotonicity we put

$$A^{[\varphi_1,\ldots,\varphi_n]}(x_1,\ldots,x_n) = (\varphi_1+\ldots+\varphi_n)^{-1}(\varphi_1(x_1)+\ldots+\varphi_n(x_n))$$

whenever $x_1, \ldots, x_n \in I$. To formulate a result solving the equality problem for generalized weighted quasi-arithmetic means we extend the notion of equivalence of functions described in Subsection 3.1. Given a set X we say that n-tuples $(\varphi_1, \ldots, \varphi_n)$ and $(\psi_1, \ldots, \psi_n)$ of real-valued functions defined on X are *equivalent* or $(\varphi_1, \ldots, \varphi_n)$ is *equivalent to* $(\psi_1, \ldots, \psi_n)$ if there are numbers $a \in \mathbb{R}\setminus\{0\}$ and $b_1, \ldots, b_n \in \mathbb{R}$ such that

$$\psi_i(x) = a\varphi_i(x) + b_i, \qquad x \in X;$$

then we write

$$(\varphi_1(x_1), \ldots, \varphi_n(x_n)) \sim (\psi_1(x_1), \ldots, \psi_n(x_n)), \qquad x \in X,$$

or, simply, $(\varphi_1, \ldots, \varphi_n) \sim (\psi_1, \ldots, \psi_n)$. Observe that $\sim$ is an equivalence relation in the set of functions mapping X into $\mathbb{R}^n$.

Theorem 3.34. *Let I be an interval and $\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_n \in \mathcal{CM}(I)$, where $n \geq 2$. Assume that $\varphi_1, \ldots, \varphi_n$ are of the same type of monotonicity and so are $\psi_1, \ldots, \psi_n$. Then $A^{[\varphi_1,\ldots,\varphi_n]} = A^{[\psi_1,\ldots,\psi_n]}$ if and only if $(\varphi_1, \ldots, \varphi_n) \sim (\psi_1, \ldots, \psi_n)$.*

This result (see [119, Theorem 2] for $n = 2$ and [128, Theorem 3] for the general case) extends Theorems 3.2 and 3.21 solving the equality problem for quasi-arithmetic and weighted quasi-arithmetic means, respectively, as well as similar results dealing with the equality of conjugate means proved in [13, 35, 41, 50].

What concerns invariance problems the list of results is so far rather short. The first attempt was made by Matkowski and Volkmann [132] in 2008. It answers the question on the invariance of the arithmetic mean A with respect to the pair $\left(A^{[\varphi,\psi]}, A^{[\psi,\varphi]}\right)$ considered on a fixed interval I.

Theorem 3.35. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be functions of the same type of monotonicity. Then the pair (φ, ψ) satisfies the equation*

$$(\varphi+\psi)^{-1}(\varphi(x)+\psi(y))+(\varphi+\psi)^{-1}(\varphi(y)+\psi(x))=x+y \tag{3.23}$$

if and only if

$$\varphi(x)+\psi(x) \sim x, \qquad x \in I.$$

The proof is quite elementary and consists in verifying that the Gauss iteration of the mean-type mapping $\left(A^{[\varphi,\psi]}, A^{[\psi,\varphi]}\right)$ starting from an arbitrary point $(x, y) \in I^2$ converges to $A(x, y)$. Of course Eq. (3.23) is a particular case of the equation

$$(\varphi_1+\psi_1)^{-1}(\varphi_1(x)+\psi_1(y))+(\varphi_2+\psi_2)^{-1}(\varphi_2(x)+\psi_2(y))=x+y \tag{3.24}$$

expressing the invariance of the mean A with respect to the mean-type mapping $\left(A^{[\varphi_1,\psi_1]}, A^{[\varphi_2,\psi_2]}\right)$; to see this it is enough to put $\varphi_1 = \psi_2 = \varphi$ and $\varphi_2 = \psi_1 = \psi$. The study of Eq. (3.24), unlike its special case (3.23), has turned out to be more complicated. The following result was proved by Baják and Páles in the paper [6].

Theorem 3.36. *Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{CM}(I)$ be four times continuously differentiable functions with nonvanishing first derivatives. Assume that φ_1, φ_2 are of the same type of monotonicity and so are ψ_1, ψ_2. Then the quadruple $(\varphi_1, \varphi_2, \psi_1, \psi_2)$ satisfies Eq. (3.24) if and only if either*

$$\varphi_1(x)+\varphi_2(x) \sim x, \qquad x \in I,$$

and

$$(\varphi_1(x), \varphi_2(x)) \sim (\psi_2(x), \psi_1(x)), \qquad x \in I,$$

or

$$\begin{aligned} &\varphi_1(x) \sim e^{ax}, \quad x \in I, \qquad \varphi_2(x) \sim e^{ax}, \quad x \in I, \\ &\psi_1(x) \sim e^{-ax}, \quad x \in I, \qquad \psi_2(x) \sim e^{-ax}, \quad x \in I, \end{aligned}$$

with some $a \in \mathbb{R} \setminus \{0\}$ and

$$(\varphi_1(x)-\varphi_1(y))(\psi_1(x)-\psi_1(y))=(\varphi_2(x)-\varphi_2(y))(\psi_2(x)-\psi_2(y))$$

for all $x, y \in I$.

The problem of relaxing the regularity assumption in Theorem 3.36 is still open.

The next result, proved by Matkowski in [130], provides an interesting invariance formula in the class of generalized weighted quasi-arithmetic means (see [130, Theorem 2]).

Theorem 3.37. *Let $n \geq 2$ and $\varphi_1, \ldots, \varphi_{2n-1} \in \mathcal{CM}(I)$ be of the same type of monotonicity. Then the mean $A^{[\phi_1,\ldots,\phi_n]}$, where*

$$\phi_i \colon = \varphi_i + \ldots + \phi_{n+i-1}, \qquad i = 1, \ldots, n,$$

is invariant with respect to the mean-type mapping $\left(A^{[\varphi_1,\ldots,\varphi_n]}, A^{[\varphi_2,\ldots,\varphi_{n+1}]}, \ldots, A^{[\varphi_n,\ldots,\varphi_{2n-1}]}\right)$, that is

$$A^{[\phi_1,\ldots,\phi_n]} \circ \left(A^{[\varphi_1,\ldots,\varphi_n]}, A^{[\varphi_2,\ldots,\varphi_{n+1}]}, \ldots, A^{[\varphi_n,\ldots,\varphi_{2n-1}]}\right) = A^{[\phi_1,\ldots,\phi_n]}.$$

As an immediate consequence, taking $\varphi_{n+i} = \varphi_i$ for $i = 1, \ldots, n-1$, one can obtain another invariance equality (see [130, Corollary 3] also [128, Remark 4] and [131, Theorem 4]).

Corollary 3.38. *Let $n \geq 2$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{CM}(I)$ be of the same type of monotonicity. Then the mean $A^{\varphi_1+\ldots+\varphi_n}$ is invariant with respect to the cyclic mean-type mapping $\left(A^{[\varphi_1,\ldots,\varphi_n]}, A^{[\varphi_2,\ldots,\varphi_{n+1},\varphi_1]}, \ldots, A^{[\varphi_n,\varphi_1\ldots,\varphi_{n-1}]}\right)$, that is*

$$A^{\varphi_1+\ldots+\varphi_n} \circ \left(A^{[\varphi_1,\ldots,\varphi_n]}, A^{[\varphi_2,\ldots,\varphi_{n+1},\varphi_1]}, \ldots, A^{[\varphi_n,\varphi_1\ldots,\varphi_{n-1}]}\right) = A^{\varphi_1+\ldots+\varphi_n}.$$

In other words, the quasi-arithmetic mean $A^{\varphi_1+\ldots+\varphi_n}$ is the Gauss composition of the means $A^{[\varphi_1,\ldots,\varphi_n]}$, $A^{[\varphi_2,\ldots,\varphi_n,\varphi_1]}, \ldots,$ $A^{[\varphi_n,\varphi_1\ldots,\varphi_{n-1}]}$:

$$A^{\varphi_1+\ldots+\varphi_n} = A^{[\varphi_1,\ldots,\varphi_n]} \otimes A^{[\varphi_2,\ldots,\varphi_n,\varphi_1]} \otimes \ldots \otimes A^{[\varphi_n,\varphi_1\ldots,\varphi_{n-1}]}.$$

In 2015, Matkowski and Páles gave a characterization of generalized quasi-arithmetic means which involves this fact and a generalized bisymmetry equation (see [131, Theorem 5]).

3.6. Around weighted quasi-arithmetic means

a. We begin with a recent paper [92] by Kahlig and Matkowski where the weighted arithmetic, geometric and harmonic means (but not only this one) are the main heros. Given a number $p \in (0,1)$ we define the means $A_p \colon \mathbb{R}^2 \to \mathbb{R}$, $G_p \colon (0,+\infty)^2 \to (0,+\infty)$ and $H_p \colon (0,+\infty)^2 \to (0,+\infty)$ by

$$A_p(x,y) = px + (1-p)y,$$
$$G_p(x,y) = x^p y^{1-p}$$

and

$$H_p(x,y) = \frac{1}{p\frac{1}{x} + (1-p)\frac{1}{y}},$$

respectively. Clearly $A_{1/2} = A$, $G_{1/2} = G$ and $H_{1/2} = H$. In paper [92] Kahlig and Matkowski solved the following problem: taking any pair $(p, q) \in (0, 1)^2$ and means $K_p \in \{A_p, G_p, H_p\}$ and $M_q \in \{A_q, G_q, H_q\}$ determine all functions $N\colon (0, +\infty)^2 \to (0, +\infty)$ such that equality (1.4) holds for every $x, y \in (0, +\infty)$. It turned out that there is exactly one such function, i.e. the unique $N_{p,q} : (0, +\infty)^2 \to (0, +\infty)$ satisfying

$$K_p \circ (M_q, N_{p,q}) = K_p.$$

Moreover, they characterized those (p, q)'s for which $N_{p,q}$ is a mean. In other words, we know all pairs (p, q) for which the mean M_p has a K_p-complementary mean, that is $N_{p,q}$, and the form $N_{p,q}$.

In the rest of Subsection 3.6 we deal with some generalizations of weighted quasi-arithmetic means. Some others will be discussed later. In particular, in Subsection 6.1 the so-called *Makó–Páles means*, which are a common generalization of both weighted quasi-arithmetic means as well as *Lagrangian* means, are considered.

b. It seems that the notion of a *conjugate mean* was originally introduced in paper [35] by Daróczy in 1999 who was inspired by Matkowski [108] (see also [50] by Daróczy and Páles). Given a mean L on an interval I a mean $M\colon I^2 \to I$ was called by him *L-conjugate* if there exists a function $\varphi \in \mathcal{CM}(I)$ such that

$$M(x, y) = \varphi^{-1} \left(\varphi(x) + \varphi(y) - \varphi \left(L(x, y)\right)\right), \qquad x, y \in I.$$

Later, in 2001, Daróczy and Páles generalized it setting parameters into the above equality (see [49], also [56] and [39]). Namely, M is said by them to be *L-conjugate*, if

$$M(x, y) = \varphi^{-1} \left(p\varphi(x) + q\varphi(y) + (1 - p - q)\varphi \left(L(x, y)\right)\right), \qquad x, y \in I,$$

with some function $\varphi \in \mathcal{CM}(I)$ and numbers $p, q \in [0, 1]$. Putting here $p = q = 1$ we come to the previous version of the definition. The mean M of the above form is denoted by $L_\varphi^{(p,q)}$. The function φ is called its *generator* and the numbers p, q are its *weights*. Observe that $L_\varphi^{(0,0)} = L$ and $L_\varphi^{(p,1-p)} = A_p^\varphi$ for an arbitrary mean L and all $\varphi \in \mathcal{CM}(I)$, $p \in (0, 1)$. Notice also that the equation $L_\varphi^{(p,q)} = A_\nu^\psi$ with $L = A_\mu^\varphi$ was completely solved in [83] with no regularity assumption imposed on the generators φ and ψ.

The invariance of the arithmetic mean A with respect to the pair $\left(A_\varphi^{(1,1)}, A_\psi^{(1,1)}\right)$ of two A-conjugated means, that is the equation

$$A_\varphi^{(1,1)}(x, y) + A_\psi^{(1,1)}(x, y) = x + y,$$

or

$$\varphi^{-1}\left(\varphi(x)+\varphi(y)-\varphi\left(\frac{x+y}{2}\right)\right)+\psi^{-1}\left(\psi(x)+\psi(y)-\psi\left(\frac{x+y}{2}\right)\right) = x+y, \tag{3.25}$$

was considered by Daróczy [36] in 2000. He found all pairs (φ,ψ) of twice continuously differentiable functions satisfying Eq. (3.25). An extension theorem for solutions of Eq. (3.25) was proved by Hajdu [73] in 2002. She showed that each solution (φ,ψ) of (3.25) can be uniquely extended from any non-trivial interval $K\subset I$ to the whole I. Fifteen years later, in 2015, Sonubon and Orankitjaroen [148] proved the following result trying to make a progress in this research.

Theorem 3.39. *Let $\varphi,\psi\in\mathcal{CM}(I)$ be three times continuously differentiable and $p,q,r,s,t\in(0,1)$. Assume that $p\neq q$, $p+q\neq 1$, $r\neq s$, $r+s\neq 1$, $ts=(1-t)r$, and either $p+q=r+s$, or $p+q=2(r+s)$. If the pair (φ,ψ) satisfies the equation*

$$\begin{aligned}\varphi^{-1}\left(p\varphi(x)+q\varphi(y)+(1-p-q)\varphi\left(tx+(1-t)y\right)\right)\\ +\psi^{-1}\left(r\psi(x)+s\psi(y)+(1-r-s)\psi\left(tx+(1-t)y\right)\right)=x+y,\end{aligned} \tag{3.26}$$

then condition (3.2) holds.

Unfortunately, the assumption $p\neq q$ and $r\neq s$ made in Theorem 3.39 implies that this result does not generalize Daróczy's one dealing with Eq. (3.25). Moreover, the implication stated in Theorem 3.39 cannot be reversed, as the pair of identity functions defined on the interval I does not satisfy Eq. (3.26) with $p=s=1/3$, $q=r=1/2$ and $t=3/5$. Therefore the problem of invariance of the arithmetic mean A with respect to pairs of parametrized A_t-conjugated means is still open, even in the class of pairs of three times continuously differentiable functions. By the way, it is a curio that Theorem 2.1 from [148], which is the main tool of the proof of Theorem 3.39, is apparently recalled as Theorem 8 from [105] but it is cited improperly, in an incomplete form.

It is worth noting that some invariance problems for means are particular cases of the equality problem for conjugate means which is widely studied in the literature (cf., for example, [13,35,39,41,50,126]). If, for instance, $\varphi,\psi\in\mathcal{CM}(I)$ and $(p,q)\in(0,1)^2$ satisfy the equation

$$\varphi^{-1}\left(p\varphi(x)+(1-p)\varphi(y)\right)+\psi^{-1}\left(q\psi(x)+(1-q)\psi(y)\right)=x+y,$$

expressing the invariance of the arithmetic mean A with respect to the pair $\left(A_p^{\varphi},A_q^{\psi}\right)$ of weighted quasi-arithmetic means, then, putting

$$L(u,v)=qu+(1-q)v,\qquad u,v\in\psi(I),$$

and $\alpha = \psi^{-1}$, $\beta = \varphi \circ \psi^{-1}$, we see that

$$\alpha^{-1}\left(\alpha(u)+\alpha(v)-\alpha\left(L(u,v)\right)\right)=\beta^{-1}\left(p\beta(u)+(1-p)\beta(v)\right),$$

that is $L_{\alpha}^{(1,1)} = A_p^{\beta} = L_{\beta}^{(p,1-p)}$ or $L_{\psi^{-1}}^{(1,1)} = L_{\varphi\circ\psi^{-1}}^{(p,1-p)}$.

c. In 2002 Daróczy and Páles [51] introduced the following other generalization of the notion of quasi-arithmetic mean. Given an interval I and a real number $\alpha \geq -1$, a mean $M\colon I^2 \to I$ is called *quasi-arithmetic of order* α if there exists a function $\varphi \in \mathcal{CM}(I)$ such that

$$M(x,y)=\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)+\alpha\varphi\left(\frac{x+y}{2}\right)}{2+\alpha}\right),\qquad x,y\in I.$$

In such a case the mean M is denoted by $A_{\varphi}^{(\alpha)}$, the function φ is called its *generator* and α its *order*. Observe that

$$A_{\varphi}^{(0)}(x,y)=\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right)=A^{\varphi}(x,y),\qquad x,y\in I,$$

and

$$A_{\varphi}^{(-1)}(x,y)=\varphi^{-1}\left(\varphi(x)+\varphi(y)-\varphi\left(\frac{x+y}{2}\right)\right)=A_{\varphi}^{(1,1)}(x,y),\qquad x,y\in I,$$

so means of order 0 coincide with quasi-arithmetic means and means of order -1 are conjugate quasi-arithmetic means. In [51] the authors studied the invariance of the arithmetic mean with respect to the pair of quasi-arithmetic means of fixed order α, that is the equation

$$A_{\varphi}^{(\alpha)}(x,y)+A_{\psi}^{(\alpha)}(x,y)=x+y \tag{3.27}$$

or

$$\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)+\alpha\varphi\left(\frac{x+y}{2}\right)}{2+\alpha}\right)+\psi^{-1}\left(\frac{\psi(x)+\psi(y)+\alpha\psi\left(\frac{x+y}{2}\right)}{2+\alpha}\right)=x+y. \tag{3.28}$$

Theorems 1 and 2 from [51] provide the form of solutions of Eq. (3.27) which are regular enough. The result reads as follows.

Theorem 3.40. *Let at least one of functions $\varphi, \psi \in \mathcal{CM}(I)$ be continuously differentiable and $\alpha \in [-1, +\infty)$. Then the pair (φ, ψ) satisfies Eq. (3.28) if and only if either condition (3.2), or (3.3) with some $a \in \mathbb{R} \setminus \{0\}$, holds.*

As an almost immediate consequence we obtain the following description of all pairs $\left(A_{\varphi}^{(\alpha)}, A_{\psi}^{(\alpha)}\right)$ of quasi-arithmetic means of order α such that the invariance Eq. (3.27) is satisfied, that is the arithmetic mean A is the Gauss composition of $A_{\varphi}^{(\alpha)}$ and $A_{\psi}^{(\alpha)}$.

Theorem 3.41. *Let at least one of functions $\varphi, \psi \in \mathcal{CM}(I)$ be continuously differentiable and $\alpha \in [-1,+\infty)$. Then the arithmetic mean A is invariant with respect to the pair $\left(A_{\varphi}^{(\alpha)}, A_{\psi}^{(\alpha)}\right)$ if and only if either $A_{\varphi}^{(\alpha)} = A_{\psi}^{(\alpha)} = A$, or*

$$A_{\varphi}^{(\alpha)}(x,y) = \frac{1}{a}\log\left(\frac{e^{ax}+e^{ay}+\alpha e^{a\frac{x+y}{2}}}{2+\alpha}\right)$$

and

$$A_{\psi}^{(\alpha)}(x,y) = -\frac{1}{a}\log\left(\frac{e^{-ax}+e^{-ay}+\alpha e^{-a\frac{x+y}{2}}}{2+\alpha}\right)$$

for every $x, y \in I$, with some $a \in \mathbb{R}\setminus\{0\}$.

Originally Theorems 1 and 2 in [51] were formulated as implications. However, one can easily check that, in fact, their converses are also true. The set of invariance means described in Theorem 3.41 is larger than that in Subsection 3.2. Nevertheless, as it follows from Theorem 3.40, Eq. (3.28) is equivalent to its special case (3.1). Notice also that Theorem 3.40 extends the result obtained by Daróczy in [36] (the case $\alpha = -1$) as well as Theorem 3.7 dealing with the classical Matkowski-Sutô problem (the case $\alpha = 0$).

d. We complete this Subsection with an invariance result concerning the so-called *symmetrized weighted quasi-arithmetic mean*. This notion was introduced by Daróczy and Páles in 2006 (see [57] and [58]). Let I be a non-trivial interval. Given a function $\varphi \in \mathcal{CM}(I)$ and a parameter $p \in (0,1)$ we define the mean $A_p^{*\varphi}\colon I^2 \to I$ by

$$A_p^{*\varphi} = \frac{A_p^{\varphi} + A_{1-p}^{\varphi}}{2};$$

in other words

$$A_p^{*\varphi}(x,y) = \frac{1}{2}\left(\varphi^{-1}\left(p\varphi(x)+(1-p)\varphi(y)\right)+\varphi^{-1}\left((1-p)\varphi(x)+p\varphi(y)\right)\right)$$

for all $x, y \in I$. Some fundamental properties of symmetrized quasi-arithmetic means were proved in another paper [58] by Daróczy and Páles. There the authors gave necessary and sufficient conditions for the comparison, equality and homogeneity of means of the form

$$M\left(A_{\omega_1}^{\varphi}(x,y),\ldots,A_{\omega_n}^{\varphi}(x,y)\right), \tag{3.29}$$

where $M\colon I^n \to I$ is an arbitrary mean, $\varphi \in \mathcal{CM}(I)$ and $\omega_1,\ldots,\omega_n\colon I^2 \to (0,1)$; here $A_{\omega_i}^{\varphi}$ is the weighted quasi-arithmetic mean generated by φ with function weight ω_i:

$$A_{\omega_i}^{\varphi}(x,y) = \varphi^{-1}\left(\omega_i(x,y)\varphi(x)+(1-\omega_i(x,y))\varphi(y)\right), \qquad x,y \in I.$$

Putting here $n = 2$, $M = A$, and taking the constant weights $\omega_1 = p$ and $\omega_2 = 1-p$ we come to the symmetrized weighted quasi-arithmetic mean $A_p^{*\varphi}$.

The equality problem for means of form (3.29) has the following solution (see [58, Theorem 3.3]).

Theorem 3.42. *Let $M\colon I^n \to I$ be a mean and assume that it is strictly increasing in each variable. Let $\varphi, \psi \in \mathcal{CM}(I)$ and $\omega_1, \ldots, \omega_n\colon I^2 \to (0,1)$. Then*

$$M\left(A^{\varphi}_{\omega_1}(x,y), \ldots, A^{\varphi}_{\omega_n}(x,y)\right) = M\left(A^{\psi}_{\omega_1}(x,y), \ldots, A^{\psi}_{\omega_n}(x,y)\right)$$

for all $x, y \in I$ if and only if $\varphi \sim \psi$.

The detailed discussion of invariance for weighted quasi-arithmetic means is postponed to Section 3.

It seems that invariance problems for means of form (3.29) are, in general, complex. However, in the particular case of symmetrized weighted quasi-arithmetic means the situation is much easier. In 2013 Burai proved what follows (see [24, Theorem 2]).

Theorem 3.43. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be two times continuously differentiable. Assume that at least one of φ, ψ is four times continuously differentiable and*

$$\left|p - \frac{1}{2}\right| \neq \frac{\sqrt{21}}{14}. \tag{3.30}$$

Then the pair (φ, ψ) satisfies the equation

$$A^{*\varphi}_p(x,y) + A^{*\psi}_p(x,y) = x + y \tag{3.31}$$

if and only if either condition (3.2), or (3.3) with some $a \in \mathbb{R} \setminus \{0\}$, holds.

The assumption (3.30) looks rather strange and it is not known if it is essential for the validity of Theorem 3.43. The proof is technical, tedious and comes down to solving some complicated differential equations derived on a nontrivial subinterval of I. Condition (3.30) allows us to exclude a hopeless case while studying one of these equations. To omit the exclusion (3.30) Burai needed to presume higher order regularity of the generators φ, ψ (see [24, Theorem 3]).

Theorem 3.44. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be two times continuously differentiable. Assume that at least one of φ, ψ is six times differentiable. Then the pair (φ, ψ) satisfies Eq. (3.31) if and only if either condition (3.2), or (3.3) with some $a \in \mathbb{R} \setminus \{0\}$, holds.*

Another tool while proving both results is an extension theorem for solutions of Eq. (3.31) (see [24, Theorem 4]). On the one hand it allows us to spread the information obtained on the generators when solving the derived differential equations to the whole interval I. On the other hand, using this extension theorem, one can formulate Theorems 3.43 and 3.44 in a more general form where the regularity assumptions are imposed on the generators only on a subinterval of I.

3.7. Close to invariance

Some natural generalizations of the invariance of the Matkowski-Sutô problem lead to equations not expressing the invariance of the mean. Such is, for instance, the problem described by Eq. (3.13) studied in [71] by Głazowska, the second present author and Matkowski. The below result (see [71, Theorem]), published in 2002, provides all solutions (φ, ψ) of (3.13), consisting of functions which are regular enough. In what follows I denotes a fixed non-trivial real interval.

Theorem 3.45. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be twice continuously differentiable and $r \in \mathbb{R}$. Then the pair (φ, ψ) satisfies Eq.* (3.13) *if and only if one of the following cases occurs:*

(i) *either $r = 1$ and $\varphi(x) \sim x$, $x \in I$, or $r = 0$ and $\psi(x) \sim x$, $x \in I$;*
(ii) *condition* (3.2) *holds;*
(iii) *$r = 1/2$ and condition* (3.3) *with some $a \in \mathbb{R} \setminus \{0\}$ holds;*
(iv) *either $r = -1$ and*

$$\varphi(x) \sim \log|x - x_0|, \quad x \in I, \qquad \text{and} \qquad \varphi(x) \sim \sqrt{|x - x_0|}, \quad x \in I,$$

or $r = 2$ and

$$\varphi(x) \sim \sqrt{|x - x_0|}, \quad x \in I, \qquad \text{and} \qquad \varphi(x) \sim \log|x - x_0|, \quad x \in I,$$

with some $x_0 \in \mathbb{R} \setminus I$.

A year later Daróczy and Páles [55] extended the research to the equation

$$r\varphi^{-1}\left(p\varphi(x) + (1-p)\varphi(y)\right) + (1-r)\psi^{-1}\left(p\psi(x) + (1-p)\psi(y)\right) = px + (1-p)y. \tag{3.32}$$

Clearly, Eq. (3.13) is a particular case of (3.32) where $p = 1/2$. The authors of [55] also relaxed the regularity assumptions imposed on the solutions and proved what follows (see [55, Theorem 6]).

Theorem 3.46. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be continuously differentiable with nonvanishing derivatives and $p \in (0, 1)$, $r \in \mathbb{R}$. Then the pair (φ, ψ) satisfies Eq.* (3.32) *if and only if one of the conditions* (i)-(iv) *holds.*

It is interesting that in spite of adding a parameter to Eq. (3.13) and weakening the regularity assumptions the set of solutions does not enlarge. When proving Theorem 3.46 Daróczy and Páles patterned after their argument used in [52] to solve the Matkowski-Sutô problem under the assumption of continuous differentiability (see Theorem 3.7 and the sketch of its proof presented in Subsection 3.2). Roughly speaking the procedure runs according to the following schedule:

1. *The pair (f, g), where $f = \varphi' \circ \varphi^{-1}$ and $g = \psi' \circ \varphi^{-1}$, satisfies the equation*

$$f(pu + (1-p)v)(g(v) - g(u)) = r(f(u)g(v) - f(v)g(u)) \tag{3.33}$$

(see [55, Theorem 4]). Fix an interval J.

2. *If* $f, g\colon J \to (0,+\infty)$ *are continuous,* $r \in \mathbb{R}\setminus\{0,1\}$ *and the pair* (f,g) *satisfies Eq.* (3.33)*, then there exists* $c \in (0,+\infty)$ *such that*

$$f(u)^r g(u)^{1-r} = c, \qquad u \in J,$$

and f *is a solution of the equation*

$$f(pu+(1-p)v)\left(f(u)^{\frac{r}{1-r}} - f(v)^{\frac{r}{1-r}}\right) = r(f(u)^{\frac{1}{1-r}} - f(v)^{\frac{1}{1-r}}) \quad (3.34)$$

(see [55, Theorem 2 and Corollary 1]).

3. *If* $f\colon J \to (0,+\infty)$ *is a continuous solution of Eq.* (3.34) *with* $r \in (0,+\infty)\setminus\{1\}$*, then either*

$$p = 1/2, \quad r = 1/2 \quad \textit{and} \quad f(u) \sim u, \qquad u \in J,$$

or

$$p = 1/2, \quad r = 2 \quad \textit{and} \quad 1/f(u) \sim u, \qquad u \in J,$$

or f *is constant*

(see [55, Theorems 3 and 5]). Notice that the weighted arithmetic mean A_p, occuring on the left-hand side of equalities (3.33) and (3.34), is replaced by an arbitrary strict mean in the above quoted Theorems 2, 3 and Corollary 1 from [55]. Observe also that taking $p = r = 1/2$ in equalities (3.33) and (3.34) we come to equations (3.8) and (3.9), respectively, so the argument described above generalizes that provided by Lemmas 3.12–3.14 used in the proof of Theorem 3.7.

After a few years break, in 2009, Daróczy and Dascăl reactivated the research and studied another extension of Eq. (3.31), viz.

$$r\varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right) + (1-r)\psi^{-1}\left(q\psi(x)+(1-q)\psi(y)\right) = sx+(1-s)y. \quad (3.35)$$

Clearly, putting $q = s = 1/2$ in (3.35) we come to Eq. (3.31). The papers [40] and [59] provide the form of all solutions (φ,ψ) of Eq. (3.35) under the assumptions that the functions $\varphi,\psi \in \mathcal{CM}(I)$ are differentiable with nonvanishing derivatives and the parameters q, r, s are assumed to satisfy the following conditions:

$$q \in (0,1)\setminus\left\{\frac{1}{2}\right\}, \quad r \in \mathbb{R}\setminus\{0,1\}, \quad s \in (0,1), \quad s \neq q, \quad \text{and} \quad r = \frac{2(s-q)}{1-2q}.$$

The invariance results of both papers have been generalized by Daróczy in [38] by relaxing both the regularity assumption and the conditions imposed on the parameters. Namely, he determined all functions $\varphi,\psi \in \mathcal{CM}(I)$ such that (φ,ψ) satisfies Eq. (3.35) assuming only that

$$q \in (0,1)\setminus\left\{\frac{1}{2}\right\} \quad \text{and} \quad r, s \in \mathbb{R}\setminus\{0,1\}.$$

It is interesting that the method, applied by him to solve the problem there, essentially differs from those used before. Roughly speaking it relies on Theorems 1 and 2 from [38] providing complete answers to the following two questions.

Given a nonsymmetric weighted quasi-arithmetic mean M on I and nonzero real numbers α, β, γ, $\alpha \neq \beta$, find necessary and sufficient conditions for the function $N\colon I^2 \to \mathbb{R}$ defined by

$$N(x,y) = \alpha x + \beta y + \gamma M(x,y)$$

to be

(a) *symmetric:*

$$N(x,y) = N(y,x), \qquad x,y \in I;$$

(b) *reflexive:*

$$N(x,x) = x, \qquad x \in I,$$

symmetric, and bisymmetric:

$$N(N(x,y), N(u,v)) = N(N(x,u), N(y,v)), \qquad x,y,u,v \in I$$

(cf. also Aczél's characterization of quasi-arithmetic means contained in Theorem 3.1). When answering these questions Daróczy made use of results of paper [45] by him and Maksa.

In 2010 the first author of the present paper completed her research concerning the equation

$$\begin{aligned} &r\varphi^{-1}\left(p\varphi(x) + (1-p)\varphi(y)\right) + (1-r)\psi^{-1}\left(q\psi(x) + (1-q)\psi(y)\right) \\ &\quad = sx + (1-s)y \end{aligned} \tag{3.36}$$

which is a common extension of equations (3.32) and (3.35): putting $p = q = s$ in (3.36) we obtain (3.32), wheras taking $p = 1/2$ we come to (3.35). Observe that Eq. (3.36) can be rewritten in the form

$$rA_p^{\varphi} + (1-r)A_q^{\psi} = A_s. \tag{3.37}$$

Solving this equation (treating φ, ψ as unknown functions and p, q, r, s as unknown parameters) we answer the question when a linear combination of two weighted quasi-arithmetic means is a weighted arithmetic mean (cf. also the last but one paragraph of Subsection 3.3.).

The main result of paper [81] allows the full spectrum of possible functions φ, ψ and parameters p, q, r, s, practically with no restrictions, and reads as follows (cf. [81, Theorem 1]).

Theorem 3.47. *Let $\varphi, \psi \in \mathcal{CM}(I)$ and $r, s \in \mathbb{R}$, $p, q \in (0,1)$. Then the pair (φ, ψ) satisfies Eq. (3.36) if and only if $s = r(p-q)+q$ and one of the following cases occurs:*

(a) *either $r = 1$ and $\varphi(x) \sim x$, $x \in I$, or $r = 0$ and $\psi(x) \sim x$, $x \in I$;*

(b) *condition* (3.2) *holds;*
(c) $p+q=1$, $r=s=1/2$ *and condition* (3.3) *with some* $a\in\mathbb{R}\setminus\{0\}$ *holds;*
(d) *either* $p=\frac{1}{2}$, $r=\frac{-2q(1-q)}{q^2+(1-q)^2}$, $s=\frac{q^2}{q^2+(1-q)^2}$ *and*

$$\varphi(x)\sim\log|x-x_0|,\quad x\in I,\qquad \text{and}\qquad \psi(x)\sim\sqrt{|x-x_0|},\quad x\in I.$$

or $q=\frac{1}{2}$, $r=\frac{1}{p^2+(1-p)^2}$, $s=\frac{p^2}{p^2+(1-p)^2}$ *and*

$$\varphi(x)\sim\sqrt{|x-x_0|},\quad x\in I,\qquad \text{and}\qquad \psi(x)\sim\log|x-x_0|,\quad x\in I,$$

with some $x_0\in\mathbb{R}\setminus I$*;*
(e) $p\neq q$, $p+q\neq 1$, $r=\frac{s-q}{p-q}$, $s=\frac{pq}{p+q-1}$ *and*

$$\varphi(x)\sim\sqrt{|x-x_0|},\quad x\in I,\qquad \text{and}\qquad \psi(x)\sim\sqrt{|x-x_0|},\quad x\in I,$$

with some $x_0\in\mathbb{R}\setminus I$.

Observe that if $I=\mathbb{R}$, i.e. there is no $x_0\in\mathbb{R}\setminus I$, then only cases (a), (b), (c) are possible.

Briefly speaking Theorem 3.47 can be easily obtained using two crucial results. The first of them (see [81, Theorem 2]) provides all the possible forms of local solutions of (3.36). One of the main tools used in its proof is the following regularity theorem published in paper [80] by the first present author.

Theorem 3.48. *Let* $\varphi,\psi\in\mathcal{CM}(I)$ *and* $r,s\in\mathbb{R}$, $p,q\in(0,+\infty)$. *If the pair* (φ,ψ) *satisfies Eq.* (3.36)*, then there exists a non-trivial interval* $K\subset I$ *such that the functions* $\varphi|_K,\psi|_K$ *are infinitely many times differentiable with non-vanishing first derivatives.*

Its proof follows ideas used by Daróczy and Páles while giving the ultimate answer to the Matkowski-Sutô problem in [52] (see Theorem 2.16 herein and the sketch of its proof) and their amplification applied in [78] to solving the problem of invariance in the class of weighted quasi-arithmetic means (see theorem 3.29 and part of Subsection 3.4 following it). As previously a key role is played by regularity improving results due to Járai (see [77, Theorems 8.6 and 11.6]).

Having determined all the local solutions of Eq. (3.36) the next problem is how to propagate the information about their forms to the whole interval I. The below extension result solves that problem. It generalizes Proposition 3.32, so consequently also 3.8.

Theorem 3.49. (Extension Theorem) *Let* $\varphi,\psi\in\mathcal{CM}(I)$ *and* $r,s\in\mathbb{R}$, $p,q\in(0,1)$. *If the pair* (φ,ψ) *satisfies Eq.* (3.36) *and one of conditions* (a)-(e) *holds in a non-trivial interval* $K\subset I$ *(with* $x_0\in\mathbb{R}\setminus K$ *in* (d) *and* (e)*), then it holds in* I *(with* x_0 *in* (d) *and* (e)*).*

The following schedule of the procedure of determining local solutions of (3.36) is a further expansion of that applied by Daróczy and Páles in the proof of Theorem 3.46 (see [55]) and, originally, in solving the Matkowski-Sutô problem in [52], or by the first present author while proving Theorem 3.29 (see [78]).

1. *The pair* (f, g)*, where* $f = \varphi' \circ \varphi^{-1}$ *and* $g = \psi' \circ \varphi^{-1}$*, satisfies the equation*

$$\begin{aligned} f(pu + (1-p)v)\left[(1-q)sg(v) - q(1-s)g(u)\right] \\ = r\left[p(1-q)f(u)g(v) - (1-p)qf(v)g(u)\right] \end{aligned} \tag{3.38}$$

(see [81, Lemma 2]). Fix an interval J.

2. *If* $f, g \colon J \to (0, +\infty)$ *are continuous,* $r \in \mathbb{R} \setminus \{0, 1\}$*,* $p, q \in (0, 1)$ *and the pair* (f, g) *satisfies Eq.* (3.38)*, then there exists* $c \in (0, +\infty)$ *such that*

$$f(u)^{p(s-q)-p(1-q)r} g(u)^{(1-p)qr-q(1-s)} = c, \qquad u \in J;$$

moreover, either $p = q = s$ *and* f *is a solution of Eq.* (3.34)*, or* $p \neq q$*,* $p \neq s$ *and* f *satisfies the equation*

$$\begin{aligned} f(pu + (1-p)v)\left[(1-q)sf(u)^{\frac{p(1-p)(s-q)}{q(1-q)(s-p)}} - q(1-s)f(v)^{\frac{p(1-p)(s-q)}{q(1-q)(s-p)}}\right] \\ = r\left[p(1-q)f(u)f(v)^{\frac{p(1-p)(s-q)}{q(1-q)(s-p)}} - (1-p)qf(v)f(u)^{\frac{p(1-p)(s-q)}{q(1-q)(s-p)}}\right] \end{aligned} \tag{3.39}$$

(see [81, Lemma 3]).

3. *If* $f \colon J \to (0, +\infty)$ *is a continuous solution of Eq.* (3.39) *with* $r, s \in (0, +\infty) \setminus \{0, 1\}$*,* $p, q \in (0, 1)$*,* $p \neq q$*,* $p \neq s$*, then either*

$$p = \frac{1}{2}, \quad s = \frac{q^2}{q^2 + (1-q)^2} \quad \text{and} \quad f(u) \sim e^{au}, \qquad u \in J,$$

with some $a \in \mathbb{R} \setminus \{0\}$*, or*

$$q = \frac{1}{2}, \quad s = \frac{p^2}{p^2 + (1-p)^2} \quad \text{and} \quad 1/f(u) \sim u, \qquad u \in J,$$

or $p + q = 1$*,* $s = \frac{1}{2}$ *and* $f(u) \sim u, u \in J$*,*
or $p + q \neq 1$*,* $s = \frac{pq}{p+q-1}$ *and* $1/f(u) \sim u, u \in J$*,*
or f *is constant*
(cf. [81, Theorem 4]. Observe that putting $p = q = s$ in (3.39) we come to Eq. (3.33).

Now we can give all solutions of Eq. (3.37). To list them we introduce some notations. Given a real number ξ we write $\xi < I$ and $\xi > I$ for

$$\xi < x, \quad x \in I, \qquad \text{and} \qquad \xi > x, \quad x \in I,$$

respectively. We put

$$S_p^a(x, y) = \frac{1}{a} \log\left(p\mathrm{e}^{ax} + (1-p)\mathrm{e}^{ay}\right)$$

with $a \in \mathbb{R} \setminus \{0\}$,

$$H^+_{p,\xi}(x,y) = \xi + \left(p\sqrt{|x-\xi|} + (1-p)\sqrt{|y-\xi|}\right)^2$$

and

$$H^-_{p,\xi}(x,y) = \xi - \left(p\sqrt{|\xi-x|} + (1-p)\sqrt{|\xi-y|}\right)^2$$

with $\xi < I$ and $\xi > I$, respectively (*weighted power means centered at* ξ),

$$G^+_{p,\xi}(x,y) = \xi + (x-\xi)^p\,(y-\xi)^{1-p}$$

and

$$G^-_{p,\xi}(x,y) = \xi - (\xi-x)^p\,(\xi-y)^{1-p}$$

with $\xi < I$ and $\xi > I$, respectively (*weighted geometric means centered at* ξ). The result below is an immediate consequence of Theorem 3.47.

Theorem 3.50. *Let $\varphi,\psi \in \mathcal{CM}(I)$ and $r,s \in \mathbb{R}$, $p,q \in (0,1)$. Then the pair (φ,ψ) and the quadruple (p,q,r,s) satisfy Eq. (3.37) if and only if $s \neq 0$, $s \neq 1$ and one of the following cases occurs:*

(a) *either $r = 1$, $p = s$ and $A^\varphi_p = A_s$, or $r = 0$, $q = s$ and $A^\psi_q = A_s$;*

(b) *$rp + (1-r)q = s$ and $A^\varphi_p = A_p$, $A^\psi_q = A_q$;*

(c) *$p + q = 1$, $r = s = 1/2$ and $A^\varphi_p = S^a_p$, $A^\psi_q = S^{-a}_{1-p}$ with some $a \in \mathbb{R} \setminus \{0\}$;*

(d) *either $p = \frac{1}{2}$, $r = \frac{-2q(1-q)}{q^2+(1-q)^2}$, $s = \frac{q^2}{q^2+(1-q)^2}$ and*

$$A^\varphi_p = H^\varepsilon_{p,\xi}, \quad A^\psi_q = G^\varepsilon_{q,\xi},$$

or $q = \frac{1}{2}$, $r = \frac{1}{p^2+(1-p)^2}$, $s = \frac{p^2}{p^2+(1-p)^2}$ and

$$A^\varphi_p = G^\varepsilon_{p,\xi}, \quad A^\psi_q = H^\varepsilon_{q,\xi}$$

with $\varepsilon \in \{+,-\}$ and some real $\xi < I$ if $\varepsilon = +$ and $\xi > I$ if $\varepsilon = -$;

(e) *$p \neq q$, $p + q \neq 1$, $r = \frac{s-q}{p-q}$, $s = \frac{pq}{p+q-1}$ and*

$$A^\varphi_p = H^\varepsilon_{p,\xi}, \quad A^\psi_q = H^\varepsilon_{q,\xi}$$

with $\varepsilon \in \{+,-\}$ and some real $\xi < I$ if $\varepsilon = +$ and $\xi > I$ if $\varepsilon = -$.

Remark 3.51. Recently, rather unexpectedly, Theorem 3.50 (or 3.47) has enabled us to solve the following problem which came from iteration theory.

Given a non-trivial interval I and a mean-type mapping $\left(A^F_p, A^G_q\right) : I^2 \to I^2$ find all pairs $\left(A^\varphi_\lambda, A^\psi_\mu\right)$ that are square iterative roots of $\left(A^F_p, A^G_q\right)$, that is satisfy the equation

$$\left(A^\varphi_\lambda, A^\psi_\mu\right) \circ \left(A^\varphi_\lambda, A^\psi_\mu\right) = \left(A^F_p, A^G_q\right).$$

A complete answer to this problem was given in [70] by Głazowska and the authors of this survey.

4. Invariance and Bajraktarević means

4.1. Generalities

Among the numerous extensions of the notion of weighted quasi-arithmetic mean one of the most important is that of the *Bajraktarević mean* (see [10]-[12]). Given an integer $n \geq 2$, an interval I, functions $\varphi \in \mathcal{CM}(I)$ and $\omega\colon I \to (0,+\infty)$, and positive numbers $p_1,\dots,p_n$ we define the weighted mean $B^{\varphi,\omega}_{(p_1,\dots,p_n)}\colon I^n \to I$ by

$$B^{\varphi,\omega}_{(p_1,\dots,p_n)}(x_1,\dots,x_n) = \varphi^{-1}\left(\frac{p_1\omega(x_1)\varphi(x_1)+\ldots+p_n\omega(x_n)\varphi(x_n)}{p_1\omega(x_1)+\ldots+p_n\omega(x_n)}\right).$$

Observe that, in the case when ω is a constant function: $\omega(x) = c \in (0,+\infty)$ for all $x \in I$, and $p_1+\ldots+p_n = 1$ we have actually $B^{\varphi,c}_{(p_1,\dots,p_n)} = A^{\varphi}_{(p_1,\dots,p_n)}$. Most of the research has been made here for the case when $p_1 = \ldots = p_n =: p$; then we write $B^{\varphi,\omega}$ instead of $B^{\varphi,\omega}_{(p,\dots,p)}$. In the general case, putting $f := \omega\varphi$ and $g := \omega$, we have

$$\begin{aligned} B^{\varphi,\omega}_{(p_1,\dots,p_n)}(x_1,\dots x_n) &= \varphi^{-1}\left(\frac{p_1\omega(x_1)\varphi(x_1)+\ldots+p_n\omega(x_n)\varphi(x_n)}{p_1\omega(x_1)+\ldots+p_n\omega(x_n)}\right) \\ &= \left(\frac{f}{g}\right)^{-1}\left(\frac{p_1f_1(x_1)+\ldots+p_nf_n(x_n)}{p_1g_1(x_1)+\ldots+p_ng_n(x_n)}\right) \\ &=: B^{[f,g]}_{(p_1,\dots,p_n)}(x_1,\dots x_n) \end{aligned}$$

for all $(x_1,\dots x_n) \in I^n$, which provides another form of the Bajraktarević mean. As previously we omit here the lower index $(p,\dots,p)$. In what follows we focus mainly on the case $n = 2$.

The equality problem for Bajraktarević means with $p_1 = \ldots = p_n$, postulated for all $n \geq 2$, was solved by Aczél and Daróczy in [3] already in 1963 (there Bajraktarević means are called *generalized quasi-linear means*). The comparison, and thus also the equality problem, in the general case was solved in [106] by Maksa and Páles. For some other rather partial results the reader is referred to [10,44,101,103] and [48]. The following result can be deduced from [106, Theorem 3].

Theorem 4.1. *Let I be an interval, $\varphi,\psi \in \mathcal{CM}(I)$, $\omega\colon I \to (0,+\infty)$ and $p,q \in (0,1)$. Assume that ω is continuous at a point $x_0 \in I$ and the function $\psi\circ\varphi^{-1}$ is differentiable at $\varphi(x_0)$ with non-zero derivative. Then $B^{\varphi,\omega}_{(p,1-p)} = B^{\psi,\omega}_{(q,1-q)}$ if and only if $\varphi \sim \psi$ and $p = q$.*

It seems that it is still an open problem if the regularity assumption made here can be removed.

A characterization of Bajraktarević means of the form $B^{\varphi,\omega}$ was given in [139]. The reader interested in one of the possible generalizations of such means should consult the paper [120], where $B^{\varphi,\omega}$'s are embedded in some one- and two-parameter families of means.

To discuss the invariance problem involving Bajraktarević means we start with a few results on the invariance of the arithmetic mean and some similar ones with respect to a pair or even a tuple of Bajraktarević means and some of their generalizations. Fix a real interval I. They were Domsta and Matkowski who studied it probably for the first time. In [61], assuming that $I \subset (0,+\infty)$, they solved the equation

$$\varphi^{-1}\left(\frac{x\varphi(x)+y\varphi(y)}{x+y}\right)+\psi^{-1}\left(\frac{x\psi(x)+y\psi(y)}{x+y}\right)=x+y, \tag{4.1}$$

expressing the invariance of the arithmetic mean with respect to the pair $\left(B^{\varphi,\mathrm{Id}}, B^{\psi,\mathrm{Id}}\right)$ of Bajraktarević means weighted by the identity function Id defined on I.

Theorem 4.2. *Assume that $I \subset (0,+\infty)$ and let $\varphi,\psi \in \mathcal{CM}(I)$ be four times continuously differentiable. Then the pair (φ,ψ) satisfies Eq. (4.1) if and only if*

$$\varphi(x) \sim 1/x, \quad x \in I, \qquad \textit{and} \qquad \psi(x) \sim 1/x, \quad x \in I. \tag{4.2}$$

In fact, the assertion of the above result remains true if we assume that at least one of functions $\varphi,\psi \in \mathcal{CM}(I)$ is four times countinuously differentiable. Notice also that if $\varphi(x) \sim 1/x$ for all $x \in I$, then

$$B^{\varphi,\mathrm{Id}}(x,y)=\frac{x+y}{x\frac{1}{x}+y\frac{1}{y}}=\frac{x+y}{2}, \qquad x,y \in I,$$

that is $B^{\varphi,\mathrm{Id}}$ is simply the arithmetic mean A.

A more general invariance problem, namely the invariance of the arithmetic mean with respect to the pair $\left(B^{\varphi,\omega}, B^{\psi,\omega}\right)$ of Bajraktarević means in two variables, was studied in [82]. A suitable equation has the form

$$\varphi^{-1}\left(\frac{\omega(x)\varphi(x)+\omega(y)\varphi(y)}{\omega(x)+\omega(y)}\right)+\psi^{-1}\left(\frac{\omega(x)\psi(x)+\omega(y)\psi(y)}{\omega(x)+\omega(y)}\right)=x+y. \tag{4.3}$$

The following result (see [82, Theorem 1]) gives a fundamental necessary condition for a pair (φ,ψ) to be a solution of Eq. (4.3), under additional assumptions imposed on the form of ω.

Theorem 4.3. *Let $\varphi,\psi \in \mathcal{CM}(I)$ and $\omega\colon I \to (0,+\infty)$ be four times differentiable functions. Assume that ω satisfies the equation of the harmonic oscillator*

$$\omega''(x)=c\omega(x) \tag{4.4}$$

with some $c \in \mathbb{R}$. *If the pair* (φ, ψ) *satisfies Eq.* (4.3), *then for any* $x_0 \in I$ *there exist numbers* $a, b \in \mathbb{R} \setminus \{0\}$ *and* $d \in \mathbb{R}$ *such that*

$$\varphi'(x) = \frac{a}{\omega(x)^2} \exp\left(d \int_{x_0}^{x} \omega(t)^{-4/3} dt\right)$$

and

$$\psi'(x) = \frac{b}{\omega(x)^2} \exp\left(-d \int_{x_0}^{x} \omega(t)^{-4/3} dt\right)$$

for all $x \in I$.

The next result (see [82, Theorems 3 and 4]) generalizes Theorem 4.2. Here the function ω is assumed to be a fundamental solution of Eq. (4.4), that is one of the functions given by

(a) $\omega(x) = x$ for $x \in I \subset (0, +\infty)$,

(b) $\omega(x) = \exp x$ for $x \in I$,

(c) $\omega(x) = \cos x$ for $x \in I \subset (0, \pi/2)$.

Theorem 4.4. *Let* $\varphi, \psi \in \mathcal{CM}(I)$ *and* $\omega\colon I \to (0, +\infty)$.
(i) *Assume that the functions* φ, ψ *are four times differentiable and* ω *is either of form* (a), *or* (b). *Then the pair* (φ, ψ) *satisfies Eq.* (4.3) *if and only if condition* (4.2) *holds in case* (a), *and*

$$\varphi(x) \sim \exp(-2x), \quad x \in I, \qquad \text{and} \qquad \psi(x) \sim \exp(-2x), \quad x \in I,$$

in case (b).
(ii) *Assume that functions* φ, ψ *are six times continuously differentiable and* ω *is of form* (c). *Then the pair* (φ, ψ) *satisfies Eq.* (4.3) *if and only if*

$$\varphi(x) \sim \tan x, \quad x \in I, \qquad \text{and} \qquad \psi(x) \sim \tan x, \quad x \in I.$$

A standard computation shows that the functions φ and ψ described in Theorem 4.4 generate again the arithmetic mean. It would be interesting to find a non-trivial pair $\left(B^{\varphi,\omega}, B^{\psi,\omega}\right)$, i.e. different from (A, A), such that the arithmetic mean is invariant with respect to $\left(B^{\varphi,\omega}, B^{\psi,\omega}\right)$.

Quite recently Páles and Zakaria have studied the equation

$$B^{[f,g]}_{(t,s)}(x, y) + B^{[h,k]}_{(s,t)}(x, y) = x + y, \tag{4.5}$$

where $f, g, h, k\colon I \to \mathbb{R}$ are continuous functions such that g, k do not vanish and the functions f/g and h/k are strictly monotonic, and s, t are positive numbers (see [141]). This is the equation expressing the invariance of the arithmetic mean with respect to the pair $\left(B^{[f,g]}_{(t,s)}, B^{[h,k]}_{(s,t)}\right)$. Substituting $\varphi := f/g$, $\chi := g$, $\psi := h/k$ and $\omega := k$ we can rewrite (4.5) as

$$\varphi^{-1}\left(\frac{t\chi(x)\varphi(x) + s\chi(y)\varphi(y)}{t\chi(x) + s\chi(y)}\right) + \psi^{-1}\left(\frac{s\omega(x)\psi(x) + t\omega(y)\psi(y)}{s\omega(x) + t\omega(y)}\right). \tag{4.6}$$

Taking here $\chi = \omega$ and $s = t$ we obtain Eq. (4.3). Notice, however, that the assumption $s \neq t$ is imposed on the parameters s, t in [141], everywhere the solutions of (4.6) are determined, so Theorems 4.3 and 4.4 above cannot be deduced from Theorem 1 proved in the paper [141]. There, under the assumption that φ, ψ are four times continuously differentiable and $s \neq t$, the authors found all solutions (φ, ψ) of (4.6): roughly speaking φ and ψ are of the form

$$\frac{P \circ (\omega_1, \omega_2, \omega_3)}{Q \circ (\omega_1, \omega_2, \omega_3)},$$

where P, Q are affine functions of three variables and $\omega_1, \omega_2, \omega_3$ are fundamental solutions of the equation of the harmonic oscillator, i.e. Eq. (4.4) with some $c \in \mathbb{R}$. The converse implication holds with no regularity assumptions imposed on φ, ψ and for all positive s, t (cf. [141, Theorem 1]).

Eq. (4.6) is a particular case of the equation

$$\begin{aligned} &\lambda(x,y)\varphi^{-1}\left(\mu(x,y)\varphi(x) + (1-\mu(x,y))\varphi(y)\right) \\ &\qquad + (1-\lambda(x,y))\psi^{-1}\left(\nu(x,y)\psi(x) + (1-\nu(x,y))\psi(y)\right) \\ &\quad = \lambda(x,y)x + (1-\lambda(x,y))y, \end{aligned} \tag{4.7}$$

where $\lambda, \mu, \nu \colon I^2 \to (0,1)$ are given weighted functions and the unknown functions φ and ψ are assumed to belong to the class $\mathcal{CM}(I)$; it is enough to take $\lambda(x,y) = 1/2$ and

$$\mu(x,y) = \frac{tg(x)}{tg(x) + sg(y)} \quad \text{and} \quad \nu(x,y) = \frac{sk(x)}{sk(x) + tk(y)}$$

for all $x, y \in I$. Extending the notion introduced in Section 3.4 by putting

$$A_\lambda^\varphi(x,y) = \varphi^{-1}\left(\lambda(x,y)\varphi(x) + (1-\lambda(x,y))\varphi(y)\right)$$

we can rewrite (4.7) in the form

$$\lambda(x,y)A_\mu^\varphi(x,y) + (1-\lambda(x,y))A_\nu^\psi(x,y) = \lambda(x,y)x + (1-\lambda(x,y))y. \tag{4.8}$$

Observe that for $\lambda \colon I^2 \to (0,1)$ defined by

$$\lambda(x,y) = \frac{\omega(x)}{\omega(x) + \omega(y)}$$

we have $A_\lambda^\varphi = B^{\varphi,\omega}$.

In the special case when λ is constant or, more generally, $\left(A_\mu^\varphi, A_\nu^\psi\right)$-invariant:

$$\lambda \circ \left(A_\mu^\varphi, A_\nu^\psi\right) = \lambda,$$

Eq. (4.8) is equivalent to

$$A_\lambda^{\mathrm{Id}} \circ \left(A_\mu^\varphi, A_\nu^\psi\right) = A_\lambda^{\mathrm{Id}},$$

which expresses the invariance of the mean A_λ^{Id} with respect to the pair $\left(A_\mu^\varphi, A_\nu^\psi\right)$. Then we come to

$$A_\lambda^\chi = A_\mu^\varphi \otimes A_\nu^\psi$$

with $\chi = \mathrm{Id}$. The above equation extends (3.14) to the case of quasi-arithmetic means with function weights.

In what follows we say that a function $\lambda\colon I^2 \to \mathbb{R}$ is *k-times differentiable in the first variable on the diagonal* if for any $x \in I$ the function $\lambda(\cdot, x)$ is k-times differentiable at x. The following result (see [79, Theorem 2]) allows us to reduce the problem of determining solutions of Eq. (4.7) to that of solving the differential Ricatti equation

$$\begin{aligned}(\mu(x,x) - \nu(x,x))\,\phi'(x)\\ = \frac{1-\mu(x,x)-\nu(x,x)}{1-\nu(x,x)}\phi(x)^2 + a(x)\phi(x) + b(x).\end{aligned} \tag{4.9}$$

Theorem 4.5. *Let $\lambda, \mu, \nu\colon I^2 \to (0,1)$ be three times differentiable in the first variable on the diagonal. Then there exist twice differentiable functions $a, b\colon I \to \mathbb{R}$ such that $\phi := \varphi''/\varphi'$ satisfies Eq. (4.9) whenever $\varphi, \psi\colon I \to \mathbb{R}$ are three times differentiable functions with non-vanishing first derivatives and the pair (φ, ψ)satisfies Eq. (4.7).*

It is well known that, in general, it is hard to solve the Ricatti equation effectively. However, as it follows from (4.9), in some particular cases we are able to manage the situation. This is, for instance, the case when

$$\mu(x,x) = \nu(x,x), \qquad x \in I,$$

or

$$\mu(x,x) + \nu(x,x) = 1, \qquad x \in I.$$

Then the Ricatti Eq. (4.9) becomes an algebraic quadratic equation or a linear differential one. In general, however, we are far from determining all the solutions of (4.7). If we take any pair (φ, ψ) satisfying (4.7), then Theorem 4.5 may give only the form of φ''/φ'. Integrating it twice we come to φ (and then also to ψ). It is usually difficult to verify if the obtained pair (φ, ψ) really satisfies Eq. (4.7). Some particular situations, when we are able to decide it, were described in [79].

Other problems deal with the invariance of Bajraktarević means with respect to pairs of some other means. One of them was solved in [125] by Matkowski. With no regularity assumptions the following result brings the form of all Bajraktarević means $B^{[f,g]}$ in two variables which are invariant with respect to the pair $\left(A^f, A^g\right)$ of quasi-arithmetic means:

$$B^{[f,g]} \circ \left(A^f, A^g\right) = B^{[f,g]} \tag{4.10}$$

(see [125, Theorem 1]).

Theorem 4.6. *Let $f, g \in \mathcal{CM}(I)$ be non-vanishing functions such that $f/g \in \mathcal{CM}(I)$. Then the pair (f, g) satisfies Eq. (4.10) if and only if either $f \sim g$ and $B^{[f,g]} = A^f = A^g$, or the function fg is constant and*

$$B^{[f,g]}(x,y) = f^{-1}\left(\sqrt{f(x)f(y)}\right) = g^{-1}\left(\sqrt{g(x)g(y)}\right),$$
$$A^f(x,y) = g^{-1}\left(\frac{2g(x)g(y)}{g(x)+g(y)}\right) \text{ and } A^g(x,y) = f^{-1}\left(\frac{2f(x)f(y)}{f(x)+f(y)}\right)$$

for all $x, y \in I$, that is

$$B^{[f,g]} = f^{-1} \circ G \circ (f \times f) = g^{-1} \circ G \circ (g \times g),$$
$$A^f = g^{-1} \circ H \circ (g \times g) \text{ and } A^g = f^{-1} \circ H \circ (f \times f)$$

where

$$(f \times f)(x,y) := (f(x), f(y))$$

for all $x, y \in I$.

If, in addition, f, g are regular enough, then one can prove a similar theorem dealing with the more general equation

$$B^{[f,g]} \circ \left(A_p^f, A_r^g\right) = B^{[f,g]}$$

which expresses the invariance of $B^{[f,g]}$ with respect to the pair $\left(A_p^f, A_r^g\right)$ of weighted quasi-arithmetic means (see [125, Theorem 2]).

We complete this section with a short report on the paper [123] by Matkowski. It seems that much more important than the invariance result (see [123, Theorem 3]) is the following notion of a generalized Bajraktarević mean proposed there. Given a positive integer k denote by σ_k the shift of the set $\{1,\dots,k\}$ mod k that is the permutation defined by

$$\sigma_k(j) = \begin{cases} j+1, & \text{if } j \in \{1,\dots,k-1\}, \\ 1, & \text{if } j = k, \end{cases}$$

and by σ_k^i, for $i = 0,\dots,k-1$, the ith iterate of σ_k:

$$\sigma_k^i(j) = \begin{cases} j+i, & \text{if } j \in \{1,\dots,k-i\}, \\ j+1-k, & \text{if } j \in \{k-i+1,\dots,k\}. \end{cases}$$

Assuming that $k \geq 2$ and given functions $\varphi \in \mathcal{CM}(I)$ and $\omega_1,\dots,\omega_k \colon I \to (0,+\infty)$ we put

$$B^{\varphi,\omega_1,\dots\omega_k;i}(x_1,\dots,x_k) := \varphi^{-1}\left(\frac{\omega_1(x_1)\varphi\left(x_{\sigma_k^i(1)}\right) + \ldots + \omega_k(x_k)\varphi\left(x_{\sigma_k^i(k)}\right)}{\omega_1(x_1) + \ldots + \omega_k(x_k)}\right)$$

for every $i \in \{0, \ldots, k-1\}$ and all $x_1, \ldots, x_k \in I$. One can check that $B^{\varphi,\omega_1,\ldots\omega_k;i}$ is a strict mean on I for each $i = 0, \ldots, k-1$. It generalizes the Bajraktarević mean: if $k = 2$ and $\omega_1 = \omega_2 =: \omega$, then

$$\begin{aligned} B^{\varphi,\omega_1,\omega_2;0}(x_1, x_2) &= \varphi^{-1}\left(\frac{\omega(x_1)\varphi(x_1) + \omega(x_2)\varphi(x_2)}{\omega(x_1) + \omega(x_2)}\right) \\ &= B^{\varphi,\omega}(x_1, x_2) \end{aligned}$$

and

$$\begin{aligned} B^{\varphi,\omega_1,\omega_2;1}(x_1, x_2) &= \varphi^{-1}\left(\frac{\omega(x_1)\varphi(x_2) + \omega(x_2)\varphi(x_1)}{\omega(x_1) + \omega(x_2)}\right) \\ &= \varphi^{-1}\left(\frac{\frac{1}{\omega(x_1)}\varphi(x_1) + \frac{1}{\omega(x_2)}\varphi(x_2)}{\frac{1}{\omega(x_1)} + \frac{1}{\omega(x_2)}}\right) \\ &= B^{\varphi,1/\omega}(x_1, x_2) \end{aligned}$$

for all $x_1, x_2 \in I$ (cf. [123, Remark 3]). Nevertheless, if $k \geq 3$ then the means $B^{\varphi,\omega_1,\ldots\omega_k;i}$ need not be of Bajraktarević type. For a simple example the reader is referred to [123, Remark 2].

The invariance result from [123], mentioned above, reads as follows.

Theorem 4.7. *Let $\varphi \in \mathcal{CM}(I)$ and $\omega_1, \ldots \omega_k \colon I \to (0, +\infty)$ be continuous. Then*

$$A^{\varphi} \circ \left(B^{\varphi,\omega_1,\ldots\omega_k;0}, \ldots, B^{\varphi,\omega_1,\ldots\omega_k;k-1}\right) = A^{\varphi},$$

that is the quasi-arithmetic mean A^{φ} is invariant with respect to the k-tuple $\left(B^{\varphi,\omega_1,\ldots\omega_k;0}, \ldots, B^{\varphi,\omega_1,\ldots\omega_k;k-1}\right)$.

4.2. Beckenbach-Gini means

Among (weighted) Bajraktarević means there are those generated by the identity function, named (weighted) *Beckenbach-Gini means*. So, given a function $\omega \colon I \to (0, +\infty)$ and a *probability vector* $(p_1, \ldots, p_n)$, that is a vector with nonnegative coordinates $(p_1, \ldots, p_n)$ summing up to 1, we have

$$\begin{aligned} B^{\omega}_{(p_1,\ldots,p_n)}(x_1, \ldots, x_n) &:= B^{\mathrm{Id},\omega}_{(p_1,\ldots,p_n)}(x_1, \ldots, x_n) \\ &= \frac{p_1 x_1 \omega(x_1) + \ldots + p_n x_n \omega(x_n)}{p_1 \omega(x_1) + \ldots + p_n \omega(x_n)} \end{aligned}$$

for all $x_1, \ldots, x_n \in I$; in the case when $p_1 = \ldots = p_n = 1/n$ we write simply

$$B^{\omega}(x_1, \ldots, x_n) := B^{\mathrm{Id},\omega}(x_1, \ldots, x_n) = \frac{x_1 \omega(x_1) + \ldots + x_n \omega(x_n)}{\omega(x_1) + \ldots + \omega(x_n)}.$$

Moreover, instead of $B^{\omega}_{(p,1-p)}$ we write B^{ω}_{p}. In the case when ω is a power function the mean B^{ω} was considered by Gini already in 1938 (cf. [66]) and then, in a more general setting, by Beckenbach (see [15]).

Classical means, viz. the arithmetic, geometric and harmonic ones, serve as typical examples of Beckenbach-Gini means. Indeed, if ω is constant: $\omega(x) = c \in (0, +\infty)$, then

$$B^{\omega}(x, y) = \frac{cx + cy}{c + c} = \frac{x + y}{2} = A(x, y), \qquad x, y \in \mathbb{R}.$$

In turn, taking $\omega(x) = 1/\sqrt{x}$ for each $x \in (0, +\infty)$ we get

$$B^{\omega}(x, y) = \frac{x/\sqrt{x} + y/\sqrt{y}}{1/\sqrt{x} + 1/\sqrt{y}} = \sqrt{xy}\frac{1/\sqrt{y} + 1/\sqrt{x}}{1/\sqrt{x} + 1/\sqrt{y}} = \sqrt{xy} = G(x, y)$$

for all $x, y \in (0, +\infty)$. Finally, putting $\omega(x) = 1/x$ for each $x \in (0, +\infty)$, we obtain

$$B^{\omega}(x, y) = \frac{x/x + y/y}{1/x + 1/y} = \frac{2}{1/x + 1/y} = H(x, y)$$

for all $x, y \in (0, +\infty)$.

The problem of invariance in the class of Beckenbach-Gini means leads to the question on triples (ω, α, β) of positive functions defined on the interval I and satisfying the functional equation

$$B^{\omega} \circ \left(B^{\alpha}, B^{\beta}\right) = B^{\omega}. \tag{4.11}$$

It seems that in general it is hard to answer this question. We start with the particular cases when B^{ω} is one of the means A, G, H. Then we are able to determine all the pairs satisfying (4.11) with no regularity assumptions. The result below has been proved by Matkowski (see [113, Theorems 1-3]).

Theorem 4.8. *Let* $\alpha, \beta \colon I \to (0, +\infty)$.
(i) *The pair* (α, β) *satisfies the equation*

$$A \circ \left(B^{\alpha}, B^{\beta}\right) = A$$

if and only if

$$\alpha(x)\beta(x) = c, \qquad x \in I,$$

with some $c \in (0, +\infty)$.
(ii) *Assume that* $I \subset (0, +\infty)$. *The pair* (α, β) *satisfies the equation*

$$G \circ \left(B^{\alpha}, B^{\beta}\right) = G$$

if and only if

$$\alpha(x)\beta(x) = \frac{c}{x}, \qquad x \in I,$$

with some $c \in (0, +\infty)$.

(iii) *Assume that* $I \subset (0, +\infty)$. *The pair* (α, β) *satisfies the equation*

$$H \circ \left(B^{\alpha}, B^{\beta}\right) = H$$

if and only if

$$\alpha(x)\beta(x) = \frac{c}{x^2}, \qquad x \in I,$$

with some $c \in (0, +\infty)$.

What concerns the general case of Eq. (4.11) we have the following necessary condition obtained under some regularity conditions (see [113, Theorem 4]).

Theorem 4.9. *Let* $\alpha, \beta \colon I \to (0, +\infty)$ *be differentiable and* $\omega \colon I \to (0, +\infty)$ *be twice differentiable. If the triple* (α, β, ω) *satisfies Eq.* (4.11)*, then*

$$\alpha(x)\beta(x) = c\omega(x)^2, \qquad x \in I,$$

with some $c \in (0, +\infty)$.

The converse is not true (cf. [113, Remark 4]), so the question about a sufficient condition arises. Another open problem concerns weakening regularity assumptions in Theorem 4.9.

The next problem involving Beckenbach-Gini means was solved by Matkowski in [129]. The main result proved there (cf. [129, Theorem 1]) deals with the invariance of the Bajraktarević the mean $B^{[f,g]}$ with the pair $\left(B^f, B^g\right)$ of Beckenbach-Gini means.

Theorem 4.10. *Let* $f, g \colon I \to (0, +\infty)$ *be three times differentiable and assume that* f/g *is one-to-one. Then the following conditions are pairwise equivalent:*

(i) *the pair* (f, g) *satisfies the equation*

$$B^{[f,g]} \circ \left(B^f, B^g\right) = B^{[f,g]};$$

(ii) *there are* $a \in \mathbb{R} \setminus \{0\}$ *and* $c, d \in (0, +\infty)$ *such that*

$$f(x) = c\mathrm{e}^{ax} \qquad \textit{and} \qquad g(x) = d\mathrm{e}^{-ax}, \qquad x \in I;$$

(iii) $B^{[f,g]} = A$ *and there is* $a \in \mathbb{R} \setminus \{0\}$ *such that*

$$B^f(x, y) = \frac{x\mathrm{e}^{ax} + y\mathrm{e}^{ay}}{\mathrm{e}^{ax} + \mathrm{e}^{ay}} \qquad \textit{and} \qquad B^g(x, y) = \frac{x\mathrm{e}^{-ax} + y\mathrm{e}^{-ay}}{\mathrm{e}^{-ax} + \mathrm{e}^{-ay}}$$

for all $x, y \in \mathbb{R}$.

Finally we mention the paper [32] by Costin and Gh. Toader, where invariance in the class of *weighted Lehmer means* C_p^a is examined. Given a number $a \in \mathbb{R}$ and a weight $p \in (0, 1)$ we define C_p^a on $(0, +\infty)$ by

$$C_p^a(x, y) = \frac{px^a + (1-p)y^a}{px^{a-1} + (1-p)y^{a-1}}.$$

Observe that C_p^0 is the weighted harmonic mean H_p and C_p^1 is the weighted arithmetic mean A_p. Setting $a=2$ we come to the weighted contraharmonic mean C_p defined by

$$C_p(x,y)=C_p^2(x,y)=\frac{px^2+(1-p)y^2}{px+(1-p)y}.$$

If we put $a=1/2$ we see that

$$C_p^{1/2}(x,y)=\frac{p\sqrt{x}+(1-p)\sqrt{y}}{p/\sqrt{x}+(1-p)/\sqrt{y}}=\sqrt{xy}\frac{p\sqrt{x}+(1-p)\sqrt{y}}{p\sqrt{y}+(1-p)\sqrt{x}}$$

for all $x,y\in(0,+\infty)$, so $C_{1/2}^{1/2}$ is the geometric mean G. Notice, however, that for $p\neq 1/2$ the mean $C_p^{1/2}$ is not the weighted geometric mean G_p. Clearly, all C_p^a are Beckenbach-Gini means: putting $\omega(x)=x^{a-1}$ for all $x\in(0,+\infty)$ we have $C_p^a=B_p^\omega$.

The main aim of the paper [32] is to look for solutions of the invariance equation

$$C_p^a\circ\left(C_q^s,C_r^t\right)=C_p^a, \tag{4.12}$$

where $a,s,t\in\mathbb{R}$ and $p,q,r\in[0,1]$. The authors presented a method of Taylor series expansion which is not used very often while studying the invariance of means.

4.3. Gini means

An important class of Bajraktarević means are those introduced by Gini still in 1938 in the paper [66]. Given parameters $r,s\in\mathbb{R}$ the two-variable *Gini mean* $G^{r,s}$ is defined on $(0,+\infty)$ by the formula

$$G^{r,s}(x,y)=\begin{cases}\left(\frac{x^r+y^r}{x^s+y^s}\right)^{\frac{1}{r-s}}, & \text{if } r\neq s,\\ \exp\left(\frac{x^s\log x+y^s\log y}{x^s+y^s}\right), & \text{if } r=s.\end{cases}$$

Defining $\omega\colon(0,+\infty)\to(0,+\infty)$ by $\omega(x)=x^s$ we see that $G^{r,s}=B^{h_{r-s},\omega}$. Observe that $G^{r,s}=G^{s,r}$ for all $r,s\in\mathbb{R}$. Moreover, for any $r\in\mathbb{R}$ we have

$$G^{r,0}(x,y)=\begin{cases}\left(\frac{x^r+y^r}{2}\right)^{\frac{1}{r}}, & \text{if } r\neq 0,\\ \sqrt{xy}, & \text{if } r=0,\end{cases}$$

so $G^{r,0}$ is the power (or Hölder) mean H^r and $G^{r,-r}(x,y)=\sqrt{xy}$ for all $x,y\in(0,+\infty)$, that is $G^{r,-r}$ is the geometric mean G. It can be also shown that $G^{a,b}=G^{c,d}$ if and only if $\{a,b\}=\{c,d\}$ for all $a,b,c,d\in\mathbb{R}$.

In the paper [7] Bajak and Páles solved the invariance equation

$$G^{r,s}\circ\left(G^{a,b},G^{c,d}\right)=G^{r,s} \tag{4.13}$$

proving the following result (see [7, Theorem]).

Theorem 4.11. *Let $r, s, a, b, c, d \in \mathbb{R}$. The 6-tuple (r, s, a, b, c, d) satisfies Eq. (4.13) if and only if one of the following possibilities holds:*

(i) $r+s=a+b=c+d=0$, *i.e.* $G^{r,s}, G^{a,b}$ *and* $G^{c,d}$ *are the geometric mean* G;
(ii) $\{r,s\}=\{a,b\}=\{c,d\}$, *i.e.* $G^{r,s}=G^{a,b}=G^{c,d}$;
(iii) $r+s=0$ *and* $\{a,b\}=\{-c,-d\}$, *i.e.* $G^{r,s}$ *is the geometric mean* G *and* $G^{a,b}=G^{-c,-d}$;
(iv) *there exist* $u, v \in \mathbb{R}$ *such that* $\{a,b\}=\{u+v,v\}$, $\{c,d\}=\{u-v,-v\}$ *and* $\{r,s\}=\{u,0\}$, *i.e.* $G^{r,s}$ *is the power mean* H^u *and we have* $G^{a,b}=G^{u+v,v}$ *and* $G^{c,d}=G^{u-v,-v}$;
(v) *there exists* $w \in \mathbb{R}$ *such that* $\{a,b\}=\{3w,w\}$, $c+d=0$ *and* $\{r,s\}=\{2w,0\}$, *i.e.* $G^{r,s}$ *is the power mean* H^{2w}, $G^{a,b}=G^{3w,w}$ *and* $G^{c,d}$ *is the geometric mean* G;
(vi) *there exists* $w \in \mathbb{R}$ *such that* $a+b=0$, $\{c,d\}=\{3w,w\}$ *and* $\{r,s\}=\{2w,0\}$, *i.e.* $G^{r,s}$ *is the power mean* H^{2w}, $G^{a,b}$ *is the geometric mean* G *and* $G^{c,d}=G^{3w,w}$.

As an immediate consequence one can deduce what follows (see [7, Corollary]). The result concerns the case of Eq. (4.13) when $\{r,s\}=\{1,0\}$, that is $G^{r,s}=A$. In other words, we deal with the equation

$$A \circ \left(G^{a,b}, G^{c,d}\right) = A \tag{4.14}$$

expressing the invariance of the arithmetic mean with respect to a pair of two Gini means.

Corollary 4.12. *Let $a, b, c, d \in \mathbb{R}$. The quadruple (a, b, c, d) satisfies Eq. (4.14) if and only if one of the following possibilities holds:*

(i) $\{a,b\}=\{c,d\}=\{1,0\}$, *i.e.* $G^{a,b}=G^{c,d}=A$;
(ii) *there exists* $v \in \mathbb{R}$ *such that* $\{a,b\}=\{1+v,v\}$ *and* $\{c,d\}=\{1-v,-v\}$, *i.e.* $G^{a,b}=G^{1+v,v}$ *and* $G^{c,d}=G^{1-v,-v}$;
(iii) $\{a,b\}=\left\{\frac{3}{2},\frac{1}{2}\right\}$ *and* $c+d=0$, *i.e.* $G^{c,d}$ *is the geometric mean* G *and* $G^{a,b}=2A-G$;
(iv) $a+b=0$ *and* $\{c,d\}=\left\{\frac{3}{2},\frac{1}{2}\right\}$, *i.e.* $G^{a,b}$ *is the geometric mean* G *and* $G^{c,d}=2A-G$.

Similarly, as in the case of solving Eq. (4.12), here, in the proof of the Theorem, the method of Taylor series expansion turned out to be useful. To compute the expansion up to 12th order the authors made use of the computer algebra system *Maple V Release 9*. It seems that the method of Taylor expansion is especially useful while studying the invariance of means described by numerical parameters only, not by function generators, just like Lehmer means or Gini means. The same concerns *Stolarsky means* (cf Subsection 5.3).

Gini means can be embedded into some one-parameter families of means defined on $(0,+\infty)$ which are their weighted versions. For instance, in the paper [120] Matkowski considered two different such families. For arbitrary numbers $r,s\in\mathbb{R}$ we define $G_t^{r,s}, t\in(0,+\infty)$, and $M_p^{r,s}$, $p\in(0,1)$, by the formulas

$$G_t^{r,s}(x,y)=\begin{cases}\left(\frac{t^s+1}{t^r+1}\frac{(tx)^r+y^r}{(tx)^s+y^s}\right)^{\frac{1}{r-s}}, & \text{if } r\neq s,\\ \exp\left(\frac{(tx)^s\log(tx)+y^s\log y}{(tx)^s+y^s}-\frac{t^s\log t}{t^s+1}\right), & \text{if } r=s,\end{cases}$$

and

$$M_p^{r,s}(x,y)=\begin{cases}\left(\frac{px^r+(1-p)y^r}{px^s+(1-p)y^s}\right)^{\frac{1}{r-s}}, & \text{if } r\neq s,\\ \exp\left(\frac{px^s\log x+(1-p)y^s\log y}{px^s+(1-p)y^s}\right), & \text{if } r=s,\end{cases}$$

respectively. Notice that $G_1^{r,s}=G^{r,s}$ and $M_{1/2}^{r,s}=G^{r,s}$ for all $r,s\in\mathbb{R}$. Matkowski observed (cf. [120, Proposition 2]) that the geometric mean G is invariant with respect to some pairs of the above means. Namely,

$$G\circ\left(B_t^{r,s},B_t^{-r,-s}\right)=G,\qquad r,s\in\mathbb{R},\ t\in(0,+\infty),$$

and

$$G\circ\left(M_p^{r,s},M_{1-p}^{-r,-s}\right)=G,\qquad r,s\in\mathbb{R},\ p\in(0,1).$$

For some further results concerning invariance in families of *weighted Gini means* the reader is referred to the papers [33] and [34] by Costin and Gh. Toader where the method of series expansion is applied again.

5. Invariance and Cauchy means

5.1. Generalities

Fix an interval $I\subset\mathbb{R}$ and differentiable functions $f,g\colon I\to\mathbb{R}$ such that g' does not vanish and the function f'/g' is one-to-one. Then the Cauchy mean value theorem implies that the formula

$$D^{f,g}(x,y)=\begin{cases}\left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x)-f(y)}{g(x)-g(y)}\right), & \text{if } x\neq y,\\ x, & \text{if } x\neq y,\end{cases}$$

defines a function $D^{f,g}\colon I^2\to I$ which is a strict mean on I. We call it the *Cauchy mean generated by* f *and* g. Observe that if also f' does not vanish, then $D^{f,g}=D^{g,f}$. The equality problem

$$D^{f_1,g_1}=D^{f_2,g_2},$$

not easy to solve, was treated by Losonczi in [102] under rather restrictive assumptions concerning, among others, the regularity of the generators. These restrictions have been removed by Matkowski in [115]. There one can also find a simple argument showing that under the above assumptions imposed on f and g we have, in fact, $f'/g' \in \mathcal{CM}(I)$ (cf. [115, Remark 1]).

There are some important subclasses of the class of Cauchy means. The first one consists of those of the form $D^{f,\mathrm{id}}$; such a mean is called the *Lagrangian mean generated by* f and denoted by L^f:

$$L^f(x,y) = \begin{cases} (f')^{-1}\left(\frac{f(x)-f(y)}{x-y}\right), & \text{if } x \neq y, \\ x, & \text{if } x = y, \end{cases}$$

for all $x, y \in I$. A classical example is the *logarithmic mean* $L^{\log}$:

$$L^{\log}(x,y) = \begin{cases} \frac{x-y}{\log x - \log y}, & \text{if } x \neq y, \\ x, & \text{if } x = y, \end{cases}$$

for all $x, y \in (0, +\infty)$.

Another class of Cauchy means has been proposed by Stolarsky [151]. Remembering the definition (3.11) of the function $h_p \colon (0, +\infty) \to \mathbb{R}$, $p \in \mathbb{R}$, and given real parameters p, q such that $p^2 + q^2 > 0$ define the *Stolarsky mean* $E^{p,q}$ on $(0, +\infty)$ by the equalities

$$E^{p,q} = \begin{cases} D^{h_p, h_q}, & \text{if } p \neq q, \\ D^{h_p h_0, h_q}, & \text{if } p = q. \end{cases}$$

Additionally we put $E^{0,0} = G$. In other words, we have

$$E^{p,q}(x,y) = \begin{cases} \left(\frac{q}{p}\frac{x^p - y^p}{x^q - y^q}\right)^{\frac{1}{p-q}}, & \text{if } p \neq q \text{ and } pq \neq 0, \\ \left(\frac{x^p - y^p}{p(\log x - \log y)}\right)^{\frac{1}{p}}, & \text{if } p \neq q \text{ and } q = 0, \\ \left(\frac{x^q - y^q}{q(\log x - \log y)}\right)^{\frac{1}{q}}, & \text{if } p \neq q \text{ and } p = 0, \\ \exp\left(-\frac{1}{p} + \frac{x^p \log x - y^p \log y}{x^p - y^p}\right), & \text{if } p = q \neq 0, \\ \sqrt{xy}, & \text{if } p = q = 0, \end{cases}$$

for all $x, y \in (0, +\infty), x \neq y$. Observe that $E^{p,q} = E^{q,p}$ for all $p, q \in \mathbb{R}$. The means $E^{p,0}$, $p \in R$, are also called *extended logarithmic means*, whereas the mean $E^{1,1}$ is called *identric*. Notice that $E^{1,0} = L^{\log}$ is simply the logarithmic mean. Moreover, $E^{p,-p}$ is the geometric mean G and $E^{2p,p}$ is the power mean H^p, $p \in \mathbb{R}$; in particular, $E^{2,1} = A$, $E^{0,0} = G$, and $E^{-2,-1} = H$.

Studying invariance in the class of Cauchy means leads to difficult problems and, as it seems, there are no results on solutions $(\varphi, \psi, f, g, h, k)$ of the equation

$$D^{\varphi,\psi} \circ \left(D^{f,g}, D^{h,k}\right) = D^{\varphi,\psi}$$

in the general case; only some of its particular cases have been considered up to now. The equation

$$G \circ \left(D^{f,g}, D^{h,k}\right) = G,$$

where one of the generators of each Cauchy mean is a power function, serves as an example of such a situation. It has been studied by Głazowska in [69]. The main result proved there reads as follows (see [69, Theorem 2]).

Theorem 5.1. *Let $p \in \mathbb{R} \setminus \{0\}$ and $f, g\colon I \to \mathbb{R}$ be differentiable functions such that f'/h_p and g'/h_{-p} are one-to-one. Then the triple (f, g, p) satisfies the equation*

$$G \circ \left(D^{f,h_p}, D^{g,h_{-p}}\right) = G \tag{5.1}$$

if and only if either there exists $r \in \mathbb{R} \setminus \{0, p\}$ such that

$$f(x) = a_f x^r + b_f x^p + c_f \quad \text{and} \quad g(x) = a_g x^{-r} + b_g x^{-p} + c_g$$

for all $x \in I$, or there exists $r \in \{0, p\}$ such that

$$f(x) = a_f x^r \log x + b_f x^p + c_f \quad \text{and} \quad g(x) = a_g x^{-r} \log x + b_g x^{-p} + c_g$$

for some $a_f, a_g \in \mathbb{R} \setminus \{0\}$ and $b_f, b_g, c_f, c_g \in \mathbb{R}$.

As it follows from Theorem 5.1 the only pairs $\left(D^{f,h_p}, D^{g,h_{-p}}\right)$ satisfying Eq. (5.1) are pairs of some Stolarsky means (see [69, Corollary 1]):

Corollary 5.2. *Let $p \in \mathbb{R} \setminus \{0\}$ and $f, g\colon I \to \mathbb{R}$ be differentiable functions such that f'/h_p and g'/h_{-p} are one-to-one. Then the pair $\left(D^{f,h_p}, D^{g,h_{-p}}\right)$ satisfies Eq. (5.1) if and only if $D^{f,h_p} = E^{p,r}$ and $D^{g,h_{-p}} = E^{-p,-r}$ with some $r \in \mathbb{R}$.*

The proof of Theorem 5.1 is very technical and long. Among the tools the reader can find the following regularity result (cf. [69, Theorem 1]). It concerns a little bit more general equation than Eq. (5.1).

Theorem 5.3. *Let $p, q \in \mathbb{R} \setminus \{0\}$ and $f, g\colon I \to \mathbb{R}$ be differentiable functions such that f'/h_p and g'/h_q are one-to-one. If the quadriple (f, g, p, q) satisfies the equation*

$$G \circ \left(D^{f,h_p}, D^{g,h_q}\right) = G, \tag{5.2}$$

then for every $x_0 \in I$ and $n \in \mathbb{N}$ there exists a neighbourhood U of x_0 such that $f|_U$ and $g|_U$ are n-times continuously differentiable except for a closed set with an empty interior.

Theorem 5.3 is a counterpart of a result by Matkowski [116, Theorem 1] used while considering the invariance of the arithmetic mean A with respect to a pair of Lagrangian means. It was proved with the use of the implicit function theorem. Some necessary conditions for the invariance of the geometric mean G with respect to the pair $\left(D^{f,h_p}, D^{g,h_q}\right)$ are given by Lemmas 1 and 2 proved in [69]. The first of these reads as follows.

Lemma 5.4. *Let $p, q \in \mathbb{R} \setminus \{0\}$ and $f, g\colon I \to \mathbb{R}$ be differentiable functions such that f'/h_p and g'/h_q are one-to-one. If the quadriple (f, g, p, q) satisfies Eq. (5.2), then for every interval $J \subset I$ there exist an interval $I_0 \subset J$ and a number $c \in \mathbb{R} \setminus \{0\}$ such that $f|_{I_0}$ and $g|_{I_0}$ are three times continuously differentiable and*

$$\left(\frac{f'(x)}{h_p'(x)}\right)' \left(\frac{g'(x)}{h_q'(x)}\right)' = \frac{c}{x^{2(p+q+1)}}$$

for all $x \in I_0$.

The second one (see [69, Lemmas 2 and 3]) has a long and complicated formulation. For that reason we present it in the particular case when $q = -p$.

Lemma 5.5. *Let $p \in \mathbb{R} \setminus \{0\}$ and $f, g\colon I \to \mathbb{R}$ be differentiable functions such that f'/h_p and g'/h_{-p} are one-to-one. If the triple (f, g, p) satisfies Eq. (5.1), then for every interval $J \subset I$ there exist an interval $I_0 \subset J$ and a number $c \in \mathbb{R} \setminus \{0\}$ such that $f|_{I_0}$ and $g|_{I_0}$ are five times continuously differentiable and*

$$x^2 f'''(x) - 2(p-1)xf''(x) + p(1-p)f'(x) = c\left(xf''(x) - (1-p)f'(x)\right)$$

for all $x \in I_0$.

Lemma 5.5, especially in its full form (cf. [69, Lemma 2]), has a tedious highly computational proof making use of both Theorem 5.3 and Lemma 5.4, and is the main tool while proving Theorem 5.1.

Difficulties, while solving the invariance problem in the class of Cauchy means, can be essentially reduced when we confine ourselves to some of its special subclasses, viz. to Lagrangian means or Stolarsky means.

5.2. Lagrangian means

Before a discussion of the invariance questions for Lagrangian means we deal with the equality problem concerning them. It was answered by Berrone and Moro [16, Corollary 7] and independently by Głazowska in [67, Lemma 2]. For another proof by Matkowski the reader is referred to [122, Lemma 1].

Theorem 5.6. *Let $f, g\colon I \to \mathbb{R}$ be differentiable functions with one-to-one derivatives f' and g'. Then $L^f = L^g$ if and only if $f' \sim g'$.*

It seems that the first paper devoted to invariance with respect to an arbitrary pair of Lagrangian means was published in 2005. It is [116] by Matkowski where all the pairs $\left(l^f, L^g\right)$ satisfying the equation

$$A \circ \left(L^f, L^g\right) = A \tag{5.3}$$

were determined. The main result of the paper (cf. [116, Theorem 2]) reads as follows.

Theorem 5.7. *Let $f, g\colon I \to \mathbb{R}$ be differentiable functions with one-to-one derivatives f' and g'. Then the pair (f, g) satisfies Eq. (5.3) if and only if either*

$$f'(x) \sim x, \quad x \in I, \quad \textit{and} \quad g'(x) \sim x, \quad x \in I,$$

or

$$f'(x) \sim \mathrm{e}^{ax}, \quad x \in I, \quad \textit{and} \quad g'(x) \sim \mathrm{e}^{-ax}, \quad x \in I,$$

with some $a \in \mathbb{R} \setminus \{0\}$.

As an immediate consequence of Theorem 5.7 we obtain its reformulation in terms of means. For each $a \in \mathbb{R} \setminus \{0\}$ we denote by L_a the Lagrangian mean generated by the function $I \ni x \longmapsto \mathrm{e}^{ax}$:

$$L_a(x, y) = \frac{1}{a} \log\left(\frac{\mathrm{e}^{ax} - \mathrm{e}^{ay}}{a(x - y)}\right) \tag{5.4}$$

for all $x, y \in I$. Additionally we put $L_0 = A$.

Corollary 5.8. *Let $f, g\colon I \to \mathbb{R}$ be differentiable functions with one-to-one derivatives f' and g'. Then the pair (f, g) satisfies Eq. (5.3) if and only if $L^f = L_a$ and $L^g = L_{-a}$ with some $a \in \mathbb{R}$.*

In the proof of Theorem 5.7 the following result improving the regularity of the generators f, g is useful (see [116, Theorem 1]).

Theorem 5.9. *Let $f, g\colon I \to \mathbb{R}$ be differentiable functions with one-to-one derivatives f' and g'. If the pair (f, g) satisfies Eq. (5.3), then the functions f, g are of class C^∞ except a nowhere dense subset of I.*

Making use of this regularity theorem one can reduce the problem of determining the solutions of (5.3) to solving the differential equation

$$f^{(3)}(x)^2 = f^{(2)}(x) f^{(4)}(x)$$

in a subinterval of I.

The equation

$$G \circ \left(L^f, L^g\right) = G, \tag{5.5}$$

which expresses the invariance of the geometric mean G with respect to a pair $\left(L^f, L^g\right)$ of Lagrangian means, is completely solved. We start with the research made by Głazowska in [67] under the assumption of the conditional homogeneity of the means L^f and L^g. Assuming that $I \subset (0, +\infty)$ we say that a mean $M\colon I^2 \to I$ is *conditionally homogeneous* if

$$M(tx, ty) = tM(x, y)$$

for all $x, y \in I$ and $t \in (0, +\infty)$ such that $tx, ty \in (0, +\infty)$. All conditionally homogeneous Lagrangian means are listed below (see [67, Theorem 2]). We use the following denotation:

$$L^{[p]}(x,y) = \begin{cases} \left(\frac{1}{p+1}\frac{x^{p+1}-y^{p+1}}{x-y}\right)^{\frac{1}{p}}, & \text{if } p \neq -1 \text{ and } p \neq 0, \\ \frac{x-y}{\log x - \log y}, & \text{if } p = -1, \\ \exp\left(-1 + \frac{x\log x - y\log y}{x-y}\right), & \text{if } p = 0, \end{cases}$$

for all $x, y \in I, x \neq y$, and $L^{[p]}(x,x) = x$ for all $x \in I$, $p \in \mathbb{R}$. Notice that $L^{[p]}$ is the Lagrangian mean generated by the function h_{p+1} whenever $p \neq -1$ and $p \neq 0$. Moreover, $L^{[-1]} = E^{1,0} = E^{0,1}$ is the logarithmic mean and $L^{[0]} = E^{1,1}$ is the identric one. In general, $L^{[p]}$ is called the *generalized logarithmic mean.*

Theorem 5.10. *Assume that $I \subset (0, +\infty)$. Let $f\colon I \to \mathbb{R}$ be a differentiable function with one-to-one derivative f'. Then the following statements are pairwise equivalent:*

(i) *the mean L^f is conditionally homogeneous;*
(ii) *there exists $p \in \mathbb{R}$ such that $f' = h_p$, that is either $p \neq 0$ and*

$$f'(x) \sim x^p, \qquad x \in I,$$

or $p = 0$ and

$$f'(x) \sim \log x, \qquad x \in I;$$

(iii) *there exists $p \in \mathbb{R}$ such that $L^f = L^{[p]}$.*

In the proof [110, Theorem 1] as well as some ideas of the proof of [85, Proposition 2] have been used.

Making use of Theorems 5.10 and 5.6 , and some calculus of derivatives, Głazowska proved the theorem below which is the main result of [67].

Theorem 5.11. *Assume that $I \subset (0, +\infty)$. Let $f, g\colon I \to \mathbb{R}$ be differentiable functions with one-to-one derivatives f' and g'. Assume that at least one of the means L^f, L^g is conditionally homogeneous. Then the following statements are pairwise equivalent:*

(i) *the pair (f, g) satisfies Eq. (5.5);*
(ii) *$f'(x) \sim \frac{1}{x^2}$, $x \in I$, and $g'(x) \sim \frac{1}{x^2}$, $x \in I$;*
(iii) *$L^f = L^g = G$.*

The next step in solving Eq. (5.5) was made in the paper [72] by Głazowska and Matkowski. There they resigned the assumption of conditional homogeneity of Lagrangian means and proved the following necessary condition for the $\left(L^f, L^g\right)$-invariance of the geometric mean (cf. [72, Theorem 4]).

Theorem 5.12. *Assume that $I \subset (0, +\infty)$. Let $f, g\colon I \to \mathbb{R}$ be differentiable functions with one-to-one derivatives f' and g'. If the pair (f, g) satisfies Eq. (5.5), then either*

$$f'(x) \sim \frac{1}{x^2}, \quad x \in I, \quad \textit{and} \quad g'(x) \sim \frac{1}{x^2}, \quad x \in I,$$

or

$$\log\left(x^3 f''(x)\right) \sim x^{-\frac{4}{9}}, \quad x \in I.$$

While proving this theorem some auxiliary results were obtained:

- a theorem improving the regularity of solutions (see [72, Theorem 1]) being a counterpart of Theorem 5.9;
- a result stating that, under the assumptions of Theorem 5.12, for every interval $J \subset I$ there exist an interval $I_0 \subset J$ and a number $c \in \mathbb{R} \setminus \{0\}$ such that
$$f''(x) g''(x) = \frac{c}{x^6}, \quad x \in I \tag{5.6}$$
(see [72, Theorem 2]);
- a theorem reducing the original problem to the differential equation
$$(3 + xw(x))\left(9x^2 w'(x) + 13xw(x) + 12\right) = 0$$
locally satisfied by the function f'''/f'' (see [72, Theorem 3]);
- a result providing the general solution of the above differential equation.

In [72] the functions f and g are actually assumed continuously differentiable. This is, in fact, superfluous since any function differentiable on an interval, with one-to-one derivative, is continuously differentiable there (cf. [116, Remark 1]).

In the case when $\log\left(x^3 f''(x)\right) \sim x^{-\frac{4}{9}}$, $x \in I$, (and of course also (5.6) holds) the mean L^f cannot be expressed by elementary functions. For this reason it was hard to decide if the geometric mean G is, in fact, invariant with respect to the pair $\left(L^f, L^g\right)$. This problem remained unsolved until 2011 when Głazowska [68] showed that this is not the case. This was done by calculating partial derivatives of order 7 of L^f and L^g satisfying (5.5). Some of the calculations were made using *Mathematica 4.1*.

Summarizing, the final answer to the question on solutions of Eq. (5.5) can be formulated as follows (cf. Theorem 5.12 and [68, Theorem 3.1]).

Theorem 5.13. *Assume that $I \subset (0, +\infty)$. Let $f, g\colon I \to \mathbb{R}$ be differentiable functions with one-to-one derivatives f' and g'. Then the pair (f, g) satisfies Eq. (5.5) if and only if*

$$f'(x) \sim \frac{1}{x^2}, \quad x \in I, \quad \textit{and} \quad g'(x) \sim \frac{1}{x^2}, \quad x \in I,$$

or, equivalently, $L^f = L^g = G$.

We conclude this section with a result of Matkowski [122] solving the equation

$$D^{f,g} \circ \left(L^f, L^g\right) = D^{f,g}, \tag{5.7}$$

expressing the invariance of the Cauchy mean $D^{f,g}$ with respect to a pair of Lagrangian means generated by the same functions f and g. Here the means L_a, $a \in \mathbb{R}\setminus\{0\}$, defined by (5.4) play an important role (see [122, Theorem 1]). In its original formulation there is a lack of the assumption that the functions f' and g' are one-to-one, needful to define the means L^f and L^g. It does not follow from the assumption that f'/g' is one-to-one: $f = \log$ and $g = \mathrm{id}$ serve as an example. On the other hand there is no need to assume that f, g are continuously differentiable. This is forced by the invertibility of f' and g'.

Theorem 5.14. *Let $f, g\colon I \to \mathbb{R}$ be differentiable functions such that g' does not vanish and the functions f', g' and f'/g' are one-to-one. Then the following statements are pairwise equivalent:*

(i) *the pair (f, g) satisfies Eq. (5.7);*

(ii) *either*

$$f'(x) \sim g'(x), \quad x \in I,$$

or

$$f(x) \sim \mathrm{e}^{ax}, \quad x \in I, \quad \textit{and} \quad g(x) \sim \mathrm{e}^{-ax}, \quad x \in I; \tag{5.8}$$

(iii) *either $D^{f,g} = L^f = L^g$, or there is $a \in \mathbb{R}\setminus\{0\}$ such that $D^{f,g} = L_0 = A$, $L^f = L_a$ and $L^g = l_{-a}$.*

In the proof the lemma below is very useful (see [122, Lemma 2]).

Lemma 5.15. *Let $f, g\colon I \to \mathbb{R}$ be differentiable functions with nonvanishing derivatives f' and g'. If*

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(x)}{g'(y)}, \quad x, y \in I, x \neq y,$$

then condition (5.8) holds with some $a \in \mathbb{R}\setminus\{0\}$.

5.3. Stolarsky means

The invariance problem in the class of Stolarsky means relies on solving the equation

$$E^{p,q} \circ \left(E^{a,b}, E^{c,d}\right) = E^{p,q}. \tag{5.9}$$

Its particular case when $p + q = 0$, that is the equation

$$G \circ \left(E^{a,b}, E^{c,d}\right) = G \tag{5.10}$$

was studied in [19] by Błasińska-Lesk (the former name of the first author), Głazowska and Matkowski. They solved Eq. (5.10) by finding quadruples

(a, b, c, d) satisfying it. Some of them were formally losed. Nevertheless, as $E^{p,q} = E^{q,p}$ for all $p, q \in \mathbb{R}$, all the means $E^{a,b}$ and $E^{c,d}$ were finally determined. Below we present a reformulation of Theorem 2 from [19] made also in terms of means, which seems to be more adequate then the “parameter only” attempt.

Theorem 5.16. *Let $a, b, c, d \in \mathbb{R}$. The quadruple (a, b, c, d) satisfies Eq. (5.10) if and only if one of the following possibilities holds:*

(i) *$a + b = c + d = 0$, i.e. $E^{a,b}$ and $E^{c,d}$ are the geometric mean G;*
(ii) *$\{a, b\} = \{-c, -d\}$, i.e. $E^{a,b} = E^{-c,-d}$.*

To prove this result the authors of [19] used classical methods of mathematical analysis like differentiation (up to eighth order derivatives) and taking limits.

Seven years later Theorem 5.16 was generalized by Baják and Páles in the paper [8] where they solved Eq. (5.9) in full generality. To prove the next theorem they used the computer algebra system *Maple Release 9* to compute the Taylor expansion of the approximation of some involved functions up to 12th order.

Theorem 5.17. *Let $p, q, a, b, c, d \in \mathbb{R}$. The 6-tuple (p, q, a, b, c, d) satisfies Eq. (5.9) if and only if one of the following possibilities holds:*

(i) *$p+q = a+b = c+d = 0$, i.e. $E^{p,q}$, $E^{a,b}$ and $E^{c,d}$ are the geometric mean G;*
(ii) *$\{p, q\} = \{a, b\} = \{c, d\}$, i.e. $E^{p,q} = E^{a,b} = E^{c,d}$;*
(iii) *$p + q = 0$ and $\{a, b\} = \{-c, -d\}$, i.e. $E^{p,q}$ is the geometric mean G and $E^{a,b} = E^{-c,-d}$.*

The method of Taylor series expansion was also used by Gh. Toader, Costin and S. Toader in [154] to study the invariance problem in the class of extended logarithmic means, that is the equation

$$E^{p,0} \circ \left(E^{a,0}, E^{c,0}\right) = E^{p,0},$$

which is a particular case of (5.9). Of course the final result of [154] is covered by Theorem 5.16.

At the very end of the section we say some words about the mixed case where each of the means K, M, N satisfying the invariance equation $K \circ (M, N) = K$ is either a Gini mean, or a Stolarsky mean. There are six such equations and two of them, that is (4.13) and (5.9), have been already discussed. Then it remains to study the following four:

$$G^{p,q} \circ \left(E^{a,b}, G^{c,d}\right) = G^{p,q}, \tag{5.11}$$

$$G^{p,q} \circ \left(E^{a,b}, E^{c,d}\right) = G^{p,q}, \tag{5.12}$$

$$E^{p,q} \circ \left(G^{a,b}, G^{c,d}\right) = E^{p,q}, \tag{5.13}$$

$$E^{p,q} \circ \left(G^{a,b}, E^{c,d}\right) = E^{p,q}. \tag{5.14}$$

All of them were solved in [9] by Bájak and Páles. We do not provide the formulation of the main results (see [9, Theorems 1-4]) listing all 6-tuples (p, q, a, b, c, d) which determine solutions of each of the above four equations, but would rather focus on a very interesting common attempt to these equations. To solve them the *Maple Release 9* machinery was extensively used to compute some derivatives of order 12. However, this is a rather standard although really tedious task. In our opinion the following reasoning seems to be much more interesting.

Given a Borel measure μ on $[0,1]$ and numbers $r, s \in \mathbb{R}$ we define the function $M_{r,s,\mu}\colon\ (0,+\infty)^2 \to (0,+\infty)$ putting

$$M_{r,s,\mu}(x,y) = \begin{cases} \left(\dfrac{\int_0^1 \left(x^t y^{1-t}\right)^r d\mu(t)}{\int_0^1 (x^t y^{1-t})^s d\mu(t)} \right)^{\frac{1}{r-s}}, & \text{if } r \neq s, \\ \exp\left(\dfrac{\int_0^1 \left(x^t y^{1-t}\right)^r \log\left(x^t y^{1-t}\right) d\mu(t)}{\int_0^1 (x^t y^{1-t})^r d\mu(t)} \right), & \text{if } r = s, \end{cases}$$

for all $x, y \in (0,+\infty)$. A standard computation shows that $M_{r,s,\mu}$ is a mean on $(0,+\infty)$ which is homogeneous, that is

$$M_{r,s,\mu}(ux, uy) = uM_{r,s,\mu}(x,y), \qquad x, y, u \in (0,+\infty),$$

and symmetric provided the measure μ is symmetric with respect to $1/2$, i.e. $\mu(1-A) = \mu(A)$ for all Borel sets $A \subset [0,1]$. It is not hard to check that if μ is the arithmetic mean of the *Dirac measures* δ_0 and δ_1 *concentrated at* 0 *and* 1, respectively, then $M_{r,s,\mu}$ is the Gini mean $G^{r,s}$, and in the case when μ is the Lebesgue measure on $[0,1]$, the mean $M_{r,s,\mu}$ is the Stolarski mean $E^{r,s}$. Thus we have just defined a common generalization of the Gini and Stolarsky means. Therefore, each of the equations (4.13), (5.9) and (5.11)–(5.14) can be written as the equation

$$M_{r,s,\kappa} \circ (M_{a,b,\mu}, M_{c,d,\nu}) = M_{r,s,\kappa}, \tag{5.15}$$

where each of κ, μ and ν is either $(\delta_0 + \delta_1)/2$, or the Lebesgue measure on $[0,1]$. The following lemmas play an important role while preparing equations (5.11)–(5.14) for using the computational machinery of *Maple V*.

Lemma 5.18. *Let $M, N\colon (0,+\infty)^2 \to (0,+\infty)$ be homogeneous strict means. Then*

(i) *the Gauss composition $M \otimes N$ is homogeneous;*
(ii) *if M, N are symmetric, then so is $M \otimes N$;*
(iii) *if $K\colon (0,+\infty)^2 \to (0,+\infty)$ is a homogeneous strict mean, then $K = M \otimes N$ if and only if the function $F_{K,M,N}\colon \mathbb{R} \to \mathbb{R}$, defined by*

$$F_{K,M,N}(u) = \log \frac{K\left(M\left(\mathrm{e}^u, \mathrm{e}^{-u}\right), N\left(\mathrm{e}^u, \mathrm{e}^{-u}\right)\right)}{K\left(\mathrm{e}^u, \mathrm{e}^{-u}\right)},$$

vanishes everywhere;

(iv) *if, in addition, the functions* K, M, N *are analytic, then so is* $F_{K,M,N}$ *and it vanishes on* $\mathbb{R}$ *if and only if* $F^{(i)}_{K,M,N}(0) = 0$ *for all* $i \in \mathbb{N}$*;*

(v) *if, in addition, the functions* K, M, N *are symmetric, then the function* $F_{K,M,N}$ *is even and vanishes on* $\mathbb{R}$ *if and only if* $F^{(i)}_{K,M,N}(0) = 0$ *for all even* $i \in \mathbb{N}$.

According to Lemma 5.18, Eq. (5.15) is satisfied if and only if

$$F^{(i)}_{M_{r,s,\kappa},M_{a,b,\mu},M_{c,d,\nu}}(0) = 0, \qquad i \in \mathbb{N}.$$

Introduce the function $L_\mu \colon \mathbb{R} \to (0,\infty)$ by

$$L_\mu(z) = \log \sum_{i=0}^{\infty} \frac{z^i}{i!}\mu_i,$$

where

$$\mu_i := \int_0^1 \left(t - \frac{1}{2}\right)^i d\mu(t), \qquad i \in \mathbb{N}_0,$$

are consecutive, *central moments of the measure* μ (observe that since μ is symmetric with respect to $1/2$ we have $\mu_{2i-1} = 0$, $i \in \mathbb{N}$, and thus $\sum_{i=0}^{\infty} \frac{z^i}{i!}\mu_i$ is positive for all $z \in \mathbb{R}$!). The next lemma provides a more useful representation of the mean $M_{r,s,\mu}$.

Lemma 5.19. *Let* μ *be a Borel probability measure on* $[0,1]$*, symmetric with respect to* $1/2$*, and let* $r, s \in \mathbb{R}$*. Then*

$$M_{r,s,\mu}(x,y) = \exp\left(M^*_{r,s,\mu}(\log x, \log y)\right), \qquad x, y \in (0,+\infty),$$

where $M^*_{r,s,\mu} \colon \mathbb{R}^2 \to \mathbb{R}$ *is defined by*

$$M^*_{r,s,\mu}(u,v) = \begin{cases} \frac{u+v}{2} + \frac{L_\mu(r(u-v)) - L_\mu(s(u-v))}{r-s}, & \text{if } r \neq s, \\ \frac{u+v}{2} + (u-v)L'_\mu\left(r(u-v)\right), & \text{if } r = s. \end{cases}$$

The final lemma deals with approximating functions defined as follows. Given a probability Borel measure μ on $[0,1]$, symmetric with respect to $1/2$, and a number $m \in \mathbb{N}$ we put

$$L_{\mu;m}(z) = \log \sum_{i=0}^{m} \frac{z^i}{i!}\mu_i, \qquad z \in \mathbb{R},$$

and, for all $r, s \in \mathbb{R}$,

$$M^*_{r,s,\mu;m}(u,v) = \begin{cases} \frac{u+v}{2} + \frac{L_{\mu;m}(r(u-v)) - L_{\mu;m}(s(u-v))}{r-s}, & \text{if } r \neq s, \\ \frac{u+v}{2} + (u-v)L'_{\mu;m}\left(r(u-v)\right), & \text{if } r = s, \end{cases}$$

and, for all $u, v \in \mathbb{R}$, and

$$M_{r,s,\mu;m}(x, y) = \exp\left(M^*_{r,s,\mu;m}\left(\log x, \log y\right)\right), \qquad x, y \in (0, +\infty).$$

Lemma 5.20. *Let μ be a Borel probability measure on $[0, 1]$, symmetric with respect to $1/2$. Then, for all $m \in \mathbb{N}_0$, we have*

$$L^{(i)}_{\mu}(0) = L^{(i)}_{\mu;m}(0), \qquad i = 1, \ldots, m,$$

and

$$\partial_1^i \partial_2^j M_{r,s,\mu}(1, 1) = \partial_1^i \partial_2^j M_{r,s,\mu;m}(1, 1), \quad i, j \in \mathbb{N}_0,\ i + j \le m,$$

for all $r, s \in \mathbb{R}$.

6. None of the above, but on invariance

6.1. Makó–Páles means

These means were introduced in [104] as a common generalization of weighted quasi-arithmetic and Lagrangian means. Given an interval I, a function $\varphi \in \mathcal{CM}(I)$ and a probability Borel measure μ on $[0, 1]$ we define the *Makó–Páles* mean $M_{\varphi,\mu} \colon I^2 \to I$ by

$$M_{\varphi,\mu}(x, y) = \varphi^{-1}\left(\int_0^1 \varphi\left(tx + (1 - t)y\right) d\mu(t)\right).$$

For any $\tau \in \mathbb{R}$ denote by δ_τ the *Dirac measure concentrated at* τ. Observe that if $p \in (0, 1)$ and $\mu = (1 - p)\delta_0 + p\delta_1$, then $M_{\varphi,\mu}$ is the weighted quasi-arithmetic mean A^{φ}_p. On the other hand, if μ is the Lebesgue measure on $[0, 1]$, then taking any primitive function f of φ, for all different $x, y \in I$ we have

$$\begin{aligned} M_{\varphi,\mu}(x, y) &= \varphi^{-1}\left(\int_0^1 \varphi\left(tx + (1 - t)y\right) dt\right) \\ &= \varphi^{-1}\left(\frac{1}{x - y}\int_y^x \varphi\left(u\right) du\right) = \left(f'\right)^{-1}\left(\frac{f(x) - f(y)}{x - y}\right), \end{aligned}$$

that is $M_{\varphi,\mu}$ is the Lagrangian mean L^f.

The paper [104] is devoted to the equality problem for the means $M_{\varphi,\mu}$ whereas the paper [105] deals with the equation

$$M_{\varphi,\mu}(x, y) + M_{\psi,\nu}(x, y) = x + y \tag{6.1}$$

expressing the invariance of the arithmetic mean with respect to a pair $(M_{\varphi,\mu}, M_{\psi,\nu})$. While studying Eq. (6.1) moments and central moments of the involved measures are especially important. Given a probability Borel measure μ on

$[0,1]$ and a nonnegative integer k the *kth moment* $\widehat{\mu}_k$ and the *kth central moment* μ_k are defined by

$$\widehat{\mu}_k := \int_0^1 t^k d\mu(t) \quad \text{and} \quad \mu_k := \int_0^1 (t-\widehat{\mu}_1)^k \, d\mu(t),$$

respectively. Moments of measures are also crucial while studying some properties of an individual Makó–Páles mean. The paper [25] provides characterizations of their conditional homogeneity and conditional translativity, whereas the note [26] is devoted to the symmetry of these means. Both papers are by Burai and the first author.

The plurality of probability Borel measures implies that the class of Makó–Páles means is pretty large and, consequently, the invariance problem studied there has various solutions which strongly depend on some parameters of the used measures, on their moments, in particular. From the numerous results proved in [105] we choose the following two describing the form of pairs (φ,ψ) satisfying Eq. (6.1). The first one (see [105, Theorem 6]) considers the case $\mu_2\nu_2 = 0$, while the second one (see [105, Theorem 8]) deals with the complementary case $\mu_2\nu_2 \neq 0$.

Theorem 6.1. *Let μ,ν be probability Borel measures on $[0,1]$ such that $\mu_2\nu_2 = 0$ and let $\varphi,\psi \in \mathcal{CM}(I)$ be twice continuously differentiable functions with nonvanishing first derivatives. The pair (φ,ψ) satisfies Eq. (6.1) if and only if one of the following conditions holds:*

(i) $\mu = \delta_\tau$ *and* $\nu = \delta_{1-\tau}$ *with some* $\tau \in [0,1]$;

(ii) $\mu = \delta_\tau$ *and* $\widehat{\nu}_1 = 1-\tau$ *with some* $\tau \in [0,1]$, $\nu_2 \neq 0$ *and*

$$\psi(x) \sim x, \qquad x \in I;$$

(iii) $\nu = \delta_{1-\tau}$ *and* $\hat{\mu}_1 = \tau$ *with some* $\tau \in [0,1]$, $\mu_2 \neq 0$ *and*

$$\varphi(x) \sim x, \qquad x \in I.$$

To determine the form of solutions in the case when $\mu_2\nu_2 \neq 0$ we need a higher regularity of functions φ,ψ and a nondegeneracy condition imposed on the second and third central moments of μ,ν.

Theorem 6.2. *Let μ,ν be probability Borel measures on $[0,1]$ such that $\mu_2\nu_2 \neq 0$ and let $\varphi,\psi \in \mathcal{CM}(I)$ be three times continuously differentiable functions with nonvanishing first derivatives. Assume that*

$$(\mu_3,\nu_3) \neq \frac{3\,(\hat{\mu}_1 - \hat{\nu}_1)}{\mu_2+\nu_3}\left(-\mu_2^2, \nu_2^2\right).$$

The pair (φ,ψ) satisfies Eq. (6.1) if and only if $\widehat{\mu}_1 + \widehat{\nu}_1 = 1$ and one of the following conditions holds:

(i)

$$\varphi(x) \sim x, \quad x \in I, \quad \textit{and} \quad \psi(x) \sim x, \quad x \in I;$$

(ii)

$$\varphi(x) \sim \mathrm{e}^{ax}, \quad x \in I, \quad \textit{and} \quad \psi(x) \sim \mathrm{e}^{bx}, \quad x \in I,$$

with some $a, b \in \mathbb{R}$ *such that* $ab < 0$ *and*

$$\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \left(\mu_{i+1}\nu_{n-i} + \mu_i \nu_{n+1-i}\right) = 0$$

for all $n \in \mathbb{N}$*;*

(iii) *there exist* $a, b \in \mathbb{R}$ *such that* $(a-1)(b-1) < 0$ *and*

$$\varphi(x) \sim \begin{cases} |x - x_0|^a, & \textit{if } a \neq 0, \\ \log|x - x_0|, & \textit{if } a \neq 0, \end{cases} \quad \psi(x) \sim \begin{cases} |x - x_0|^b, & \textit{if } b \neq 0, \\ \log|x - x_0|, & \textit{if } b \neq 0, \end{cases}$$

with some $x_0 \in \mathbb{R} \setminus I$ *and for all* $x \in I$*, and*

$$F_{a,\mu}(z) + F_{b,\nu}(z) = 2 + z, \qquad z \in (-1, +\infty),$$

where

$$F_{c,\lambda}(z) := \begin{cases} \left(\int\limits_0^1 (1+tz)^c \, d\lambda(t) \right)^{\frac{1}{c}}, & \textit{if } c \neq 0, \\ \exp\left(\int\limits_0^1 \log(1+tz) \, d\lambda(t) \right), & \textit{if } c = 0, \end{cases}$$

for all $z \in (-1, +\infty)$*,* $c \in \mathbb{R}$ *and all probability Borel measures* λ *on* $[0, 1]$*.*

The proof of Theorem 6.1 is quite elementary and immediate. However, unlike this one, the proofs of Theorem 6.2 and some other results of [105] need more sophisticated tools such as, e.g., the differential equations

$$\left(\frac{3\hat{\mu}_1 \mu_2 + \mu_3}{\mu_2} - \frac{3\hat{\nu}_1 \nu_2 + \nu_3}{\nu_2} \right) \phi' + \left(\frac{\mu_3}{\mu_2^2} + \frac{\nu_3}{\nu_2^2} \right) \phi^2 = 0$$

and

$$\left(\frac{6\hat{\mu}_1^2 \mu_2 + 4\hat{\mu}_1 \mu_3 + \mu_4}{\mu_2} - \frac{6\hat{\nu}_1^2 \nu_2 + 4\hat{\nu}_1 \nu_3 + \nu_4}{\nu_2} \right) \phi'' + \left(\frac{8\hat{\mu}_1 \mu_3 + 3\mu_4}{\mu_2^2} + \frac{8\hat{\nu}_1 \nu_3 + 3\nu_4}{\nu_2^2} \right) \phi\phi' + \left(\frac{\mu_4 - 3\mu_2^2}{\mu_2^3} - \frac{\nu_4 - 3\nu_2^2}{\nu_2^3} \right) \phi^3 = 0$$

(cf. [105, Theorem 9]).

In [105] the reader can also find results dealing with the case when $\mu_2\nu_2 \neq 0$, $\mu_3 + \nu_3 = 0$, and $\hat{\mu}_1 + \hat{\nu}_1 = 1$ ([105, Theorem 11]), or $\hat{\mu}_1 = \hat{\nu}_1 = 1/2$ and $\mu_4 = \nu_4$ ([105, Theorem 12]). In a particular case of the second one the following can be proved (see [105, Corollary 13]).

Theorem 6.3. *Let μ be a probability Borel measure on $[0,1]$, having $\mu_2 \neq 0$ and symmetric with respect to $1/2$, and let $\varphi, \psi \in \mathcal{CM}(I)$ be four times continuously differentiable functions with nonvanishing first derivatives. Then the pair (φ, ψ) satisfies the equation*

$$M_{\varphi,\mu}(x,y) + M_{\psi,\mu}(x,y) = x + y \tag{6.2}$$

if and only if either

$$\varphi(x) \sim x, \quad x \in I, \quad \textit{and} \quad \psi(x) \sim x, \quad x \in I,$$

or

$$\varphi(x) \sim \mathrm{e}^{ax}, \quad x \in I, \quad \textit{and} \quad \psi(x) \sim \mathrm{e}^{-ax}, \quad x \in I,$$

with some $a \in \mathbb{R} \setminus \{0\}$.

Eq. (6.2) considered here is a special case of Eq. (6.1) where we take $\nu = \mu$. Observe also that both the measures $(\delta_0 + \delta_1)/2$ and $l_1|_{[0,1]}$ satisfy the assumptions of Theorem 6.3, so it can be applied to equations (3.1) and (5.3) dealing with the original Matkowski-Sutô problem and the invariance of the arithmetic mean with respect to a pair of Lagrangian means, respectively. Remember, however, that the assertion of Theorem 6.3 need a higher regularity than Theorems 3.16 and 5.7. By the way the problem of reducing regularity assumptions in results of [105] is open. Of course, among equations of the form (6.1), there are a great number of equations not discussed in the present article up to now. The equation

$$\varphi^{-1}\left(\frac{2\varphi(x)+\varphi(y)}{3}\right) + \psi^{-1}\left(\frac{\psi(x)+4\psi\left(\frac{x+y}{2}\right)+4\psi(y)}{9}\right) = x + y$$

serves as an example. This is Eq. (6.1) with the measures μ and ν given by

$$\mu = \frac{\delta_0 + 2\delta_1}{3} \quad \text{and} \quad \nu = \frac{4\delta_0 + 4\delta_{1/2} + \delta_1}{9}$$

(cf. [105, Example 4]).

A year later a similar research was made by Qian Zhang and Bing Xu for the equation

$$M_{\varphi,\mu}(x,y) M_{\psi,\nu}(x,y) = xy \tag{6.3}$$

describing the invariance of the geometric mean G with respect to a pair $(M_{\varphi,\mu}, M_{\psi,\nu})$ of Makó–Páles means. An attempt presented by them in [156] is similar to that from [105] and relies on reducing the main problem again to some differential equations. However, in this case these are more complicated and harder to solve than those occurring in [105], and thus the obtained results are longer and less satisfactory. In particular, in the crucial case when $\mu_2\nu_2 \neq 0$ only necessary conditions for pairs (φ, ψ) to satisfy Eq. (6.3) (see [156, Theorem 3]) are given. These conditions are formulated mainly in terms of derivatives of φ and ψ, and do not allow us to express the means $M_{\varphi,\mu}$ and $M_{\psi,\nu}$ by elementary functions. For this reason it is impossible to verify if we have really come to solutions of (6.3).

At the very end of this section we recall the paper [148] once more (cf. Theorem 3.39), mainly because Eq. (3.26) is studied there. It is another example of a Makó–Páles mean, since putting $\mu = q\delta_0 + p\delta_1 + (1-p-q)\delta_t$ we have

$$\begin{aligned} M_{\varphi,\mu}(x,y) &= \varphi^{-1}\left(\int_0^1 \varphi\left(sx+(1-s)y\right)d\mu(s)\right) \\ &= \varphi^{-1}\left(p\varphi(x)+q\varphi(y)+(1-p-q)\varphi(tx+(1-t)y)\right), \end{aligned}$$

for all $x,y \in I$ and for all $\varphi \in \mathcal{CM}(I)$, that is $M_{\varphi,\mu}$ is the A_t-conjugate mean generated by φ and weighted by p and q.

6.2. Quotient means

These means were introduced by Matkowski in [121] in 2011. Given an interval $I \subset \mathbb{R}$ and positive functions $\varphi, \psi \in \mathcal{CM}(I)$ of different types of monotonicity the formula

$$Q_{\varphi,\psi}(x,y) = \left(\frac{\varphi}{\psi}\right)^{-1}\left(\frac{\varphi(x)}{\psi(y)}\right)$$

defines a mean on I; we call it a *quotient mean.* The main result of [121] provides a complete solution of the problem of invariance of a quotient mean $Q_{\varphi,\psi}$ with respect to a pair $\left(A_p^{\varphi}, A_q^{\psi}\right)$ of weighted quasi-arithmetic means. Notice that no regularity assumptions are imposed here.

Theorem 6.4. *Let $\varphi, \psi \in \mathcal{CM}(I)$ be positive functions of different types of monotonicity and let $p, q \in (0,1)$. Then the following statements are pairwise equivalent:*

(i) *the pair (φ,ψ) satisfies the equation*

$$Q_{\varphi,\psi} \circ \left(A_p^{\varphi}, A_q^{\psi}\right) = Q_{\varphi,\psi}; \tag{6.4}$$

(ii) *the product $\varphi\psi$ is a constant function and $p+q=1$;*

(iii)

$$Q_{\varphi,\psi}(x,y) = \varphi^{-1}\left(\sqrt{\varphi(x)\varphi(y)}\right) \text{ and } A_q^{\psi}(x,y) = \varphi^{-1}\left(\frac{\varphi(x)\varphi(y)}{p\varphi(x)+(1-p)\varphi(y)}\right)$$

for all $x, y \in I$.

Its proof is quite immediate and completely self-contained.

Just recently Qian Zhang and Bing Xu in [157] have studied the more general equation

$$Q_{\varphi,\psi} \circ \left(M_{\varphi,\mu}, M_{\psi,\nu}\right) = Q_{\varphi,\psi} \tag{6.5}$$

expressing the invariance of a quotient mean $Q_{\varphi,\psi}$ with respect to a pair $(M_{\varphi,\mu}, M_{\psi,\nu})$ of Makó–Páles means. They solved (6.5) under some regularity conditions. Their answer to this invariance problem is contained in the following three results (see [157, Theorems 2,4,5]).

Theorem 6.5. *Let μ,ν be probability Borel measures on $[0,1]$ such that $(1-\hat{\mu}_1)\,\hat{\nu}_1 = 0$ and let $\varphi,\psi \in \mathcal{CM}(I)$ be positive continuously differentiable functions of different types of monotonicity. Then the pair (φ,ψ) satisfies Eq. (6.5) if and only if $\mu = \delta_1$ and $\nu = \delta_0$.*

This result needs no advanced arguments and tedious calculations. But the next two theorems have been proved with the help of a differential equation (cf. [157, Theorem 3]).

Theorem 6.6. *Let μ,ν be probability Borel measures on $[0,1]$ such that $(1-\widehat{\mu}_1)\,\widehat{\nu}_1 \neq 0$ and*

$$\frac{1-\widehat{\mu}_2}{1-\widehat{\mu}_1} = \frac{\widehat{\nu}_2}{\widehat{\nu}_1},$$

and let $\varphi,\psi \in \mathcal{CM}(I)$ be positive twice continuously differentiable functions of different types of monotonicity. Then

$$\mu = p\delta_1 + (1-p)\delta_0 \quad \text{and} \quad \nu = q\delta_1 + (1-q)\delta_0$$

with $p = \widehat{\mu}_1$ and $q = \widehat{\nu}_1$, respectively, and Eq. (6.5) takes the form (6.4).

Now the form of φ and ψ can be learned from Theorem 6.4. In the last result of this subsection, given a real number t and an integer $i \in \mathbb{N}$ the symbol $\binom{t}{i}$ denotes the *generalized binomial coefficient* given by

$$\binom{t}{i} = \frac{(t-i+1)\cdot\ldots\cdot(t-1)t}{i!}.$$

Theorem 6.7. *Let μ,ν be probability Borel measures on $[0,1]$ such that $(1-\widehat{\mu}_1)\,\widehat{\nu}_1 \neq 0$ and*

$$\frac{1-\widehat{\mu}_2}{1-\widehat{\mu}_1} \neq \frac{\widehat{\nu}_2}{\widehat{\nu}_1},$$

and let $\varphi,\psi \in \mathcal{CM}(I)$ be positive twice continuously differentiable functions of different types of monotonicity. Put

$$p = -\frac{\frac{1-\widehat{\mu}_2}{(1-\widehat{\mu}_1)^2} - \frac{2}{1-\widehat{\mu}_1} + \frac{\widehat{\nu}_2}{\widehat{\nu}_1^2}}{\frac{1-\widehat{\mu}_2}{1-\widehat{\mu}_1} - \frac{\widehat{\nu}_2}{\widehat{\nu}_1}}.$$

(i) *In the case when $p = 0$ the pair (φ,ψ) satisfies Eq. (6.5) if and only if*

$$\varphi(x) = a\mathrm{e}^{\frac{c}{1-\widehat{\mu}_1}x} \quad \text{and} \quad \psi(x) = b\mathrm{e}^{-\frac{c}{\widehat{\nu}_1}x}, \quad x \in I,$$

with some $a, b \in (0, +\infty)$ *and* $c \in \mathbb{R} \setminus \{0\}$ *and*

$$(1-\widehat{\mu}_n)\left(\frac{c}{1-\widehat{\mu}_1}\right)^n + \sum_{i=1}^{n}\binom{n}{i}\widehat{\nu}_i\left(\frac{c}{1-\widehat{\mu}_1}\right)^{n-i}\left(-\frac{c}{\widehat{\nu}_1}\right)^i = 0$$

for all $n \in \mathbb{N}$*.*

(ii) *In the case when* $p \neq 0$ *the pair* (φ, ψ) *satisfies Eq.* (6.5) *if and only if*

$$\varphi(x) = a\,|x-x_0|^{-\frac{1}{p(1-\widehat{\mu}_1)}} \quad \text{and} \quad \psi(x) = b\,|x-x_0|^{\frac{1}{p\widehat{\nu}_1}}, \quad x \in I,$$

with some $a, b \in (0, +\infty)$ *and* $x_0 \in \mathbb{R} \setminus I$ *and*

$$(1-\widehat{\mu}_n)\binom{-\frac{1}{p(1-\widehat{\mu}_1)}}{n} + \sum_{i=1}^{n}\widehat{\nu}_i\binom{-\frac{1}{p(1-\widehat{\mu}_1)}}{n-i}\binom{\frac{1}{p1-\widehat{\nu}_1}}{i} = 0$$

for all $n \in \mathbb{N}$*.*

6.3. Means of power growth

The *Heinz mean* was defined by Bhatia [17] in 2006. Given a number $p \in [0,1]$ the mean $H^{[p]}\colon (0,+\infty)^2 \to (0,+\infty)$ is given by

$$H^{[p]}(x,y) = \frac{x^p y^{1-p} + x^{1-p} y^p}{2}.$$

It is named after Heinz who considered its matrix version to prove several inequalities in the perturbation theory of operators (see [75]). In fact, the mean $H^{[p]}$ occurred already in [74], called there *symmetric*. Observe that it is positively homogeneous and symmetric. Since $H^{[1-p]} = H^{[p]}$ for all $p \in [0,1]$, it is enough to consider Heinz means only for $p \in [0,1/2]$. Notice that $H^{[1/2]} = G \leq H^{[p]} \leq A = H^{[0]}$ for all $p \in [0,1/2]$. Moreover,

$$H^{[p]} = A \circ (G_p, G_{1-p}), \qquad p \in [0,1/2],$$

where G_p stands for the p-weighted geometric mean.

The invariance equation in the class of Heinz means, that is

$$H^{[p]} \circ \left(H^{[q]}, H^{[r]}\right) = H^{[p]}, \tag{6.6}$$

was solved by Besenyei [18]. His main result (see [18, Theorem 4]) reads as follows.

Theorem 6.8. *Let* $p, q, r \in [0,1/2]$*. Then the triple* (p,q,r) *satisfies Eq.* (6.6) *if and only if* $p = q = r$*.*

To prove this result Besenyei used the Taylor expansion of the Heinz mean up to order 6.

Theorem 6.8 was extended to a much broader class of means by Witkowski [155]. He introduced the notion of mean of power growth. A positively homogeneous and symmetric mean M on $(0,+\infty)$ is said to be a *mean of power growth* if the limit

$$\lim_{x\to 0}\frac{M(x,1)}{x^{\alpha}}$$

exists and is positive and finite for some $\alpha\in\mathbb{R}$. Clearly, if such an α exists at all, it is unique. We call it the *order of* M and denote by $\mathrm{ord}(M)$. It is easy to observe that the order of any mean of power growth lies in $[0,1]$. Among means of power growth are, for instance, the weighted arithmetic restricted to $(0,+\infty)$, geometric, and harmonic means $A_p|_{(0,+\infty)}$, G_p and H_p, respectively, the power means H^p and the Heinz means $H^{[p]}$. Standard calculations show that $\mathrm{ord}\left(A_p|_{(0,+\infty)}\right)=0$, $\mathrm{ord}\left(G_p\right)=p$ and $\mathrm{ord}\left(H_p\right)=1$ for all $p\in(0,1)$,

$$\mathrm{ord}\left(H^p\right)=\begin{cases}0, & \text{if } p\in(0,+\infty),\\ 1/2, & \text{if } p=0,\\ 1, & \text{if } p\in(-\infty,0),\end{cases}$$

and

$$\mathrm{ord}\left(H^{[p]}\right)=\begin{cases}1/2, & \text{if } p\in[0,1/2],\\ 1, & \text{if } p=1/2.\end{cases}$$

In [155] Witkowski proposed the following definition. Given a symmetric homogeneous mean M on $(0,+\infty)$, of power growth, and a number $p\in[0,1/2]$ we define the mean M_p on $(0,+\infty)$ by

$$M_p=M\circ\left(G_p,G_{1-p}\right).$$

If $M=A$ then $M_p=H^{[p]}$ for all $p\in[0,1/2]$. It is easy to verify that M_p is again of power growth and

$$\mathrm{ord}\left(M_p\right)=p+(1-2p)\mathrm{ord}(M)$$

for all $p\in[0,1/2]$. Invariance in the class of such functions was examined in the following result (see [155, Theorem 2]).

Theorem 6.9. *Let $p,q,r\in[0,1/2]$ and M be a symmetric homogeneous mean on $(0,+\infty)$, of power growth. Assume that* $\mathrm{ord}(M)\neq 1/2$ *and*

$$\lim_{x\to 0}\frac{M(x,1)}{x^{\mathrm{ord}(M)}}\neq 1.$$

Then the triple (p,q,r) satisfies the equation

$$M_p\circ\left(M_q,M_r\right)=M_p \tag{6.7}$$

if and only if $p=q=r$.

As an immediate corollary, taking here $M=A$, we obtain Theorem 6.8 of Besenyei. The proof of Theorem 6.9 is quite elementary and relies on controlling the asymptotic behaviour at 0 of the functions involved in equality (6.7).

Acknowledgements

The authors are especially indebted to Krzysztof Ciepliński for his strong encouragement to write this survey.

References

[1] Aczél, J.: Un problème de M.L. Fejér sur la construction de Leibniz. Bull. Sci. Math. (2) **72**, 39–45 (1948)
[2] Aczél, J.: Lectures on Functional Equations and their Applications. Academic Press, New York (1966)
[3] Aczél, J., Daróczy, Z.: Über verallgemeinerte quasilineare Mittelwerte, die mit Gewischts-funktionen gebildet sind. Publ. Math. Debrecen **10**, 171–190 (1963)
[4] Aczél, J., Dhombres, J.: Functional Equations in Several Variables with Applications to Mathematics, Information Theory and to the Natural and Social Sciences. Cambridge University Press, Cambridge (1989)
[5] Aczél, J., Fenyő, I., Horváth, J.: Sur certaines classes de fonctionnelles. Portugaliae Math. **8**, 1–11 (1949)
[6] Baják, S., Páles, Z.: Invariance equation for generalized quasi-arithmetic means. Aequationes Math. **77**, 133–145 (2009)
[7] Baják, S., Páles, Z.: Computer aided solution of the invariance equation for two-variable Gini means. Comput. Math. Appl. **58**, 334–340 (2009)
[8] Baják, S., Páles, Z.: Computer aided solution of the invariance equation for two-variable Stolarsky means. Appl. Math. Comput. **216**, 3219–3227 (2010)
[9] Baják, S., Páles, Z.: Solving invariance equations involving homogeneous means with the help of computer. Appl. Math. Comput. **219**, 6297–6315 (2013)
[10] Bajraktarević, M.: Sur une équation fonctionnelle aux valeurs moyennes. Glasnik Mat.-Fiz. Astr. (2) **13**, 243–248 (1958)
[11] Bajraktarević, M.: Sur une généralisation des moyennes quasilinéraires. Publ. Inst. Math. (Beograd) (N.S.) **3**, 69–76 (1963)
[12] Bajraktarević, M.: Über die Vergleichbarkeit der mit Gewichtsfunktionen gebildeten Mittelwerte. Studia Sci. Math. Hungar. **4**, 3–8 (1969)
[13] Bakula, M.K., Páles, Z., Pečarić, J.: On weighted L-conjugate means. Commun. Appl. Anal. **11**, 95–110 (2007)
[14] Baron, K., Kuczma, M.: Iteration of random-valued functions on the unit interval. Colloq. Math. **37**, 263–269 (1977)
[15] Beckenbach, E.F.: A class of mean value functions. Am. Math. Monthly **57**, 1–6 (1950)
[16] Berrone, L.R., Moro, J.: Lagrangian means. Aequationes Math. **55**, 217–226 (1998)
[17] Bhatia, R.: Interpolating the arithmetic-geometric mean inequality and its operator version. Linear Algebra Appl. **413**, 355–363 (2006)
[18] Besenyei, A.: On the invariance equation for Heinz means. Math. Inequal. Appl. **15**, 973–979 (2012)
[19] Błasińska-Lesk, J., Głazowska, D., Matkowski, J.: An invariance of the geometric mean with respect to Stolarsky mean-type mappings. Results Math. **43**, 42–55 (2003)

[20] Borwein, J.M., Borwein, P.B.: Pi and the AGM. A study in analytical number theory and computational complexity, Canadian Mathematical Society Series of Monographs and Advanced Texts, A Wiley-Interscience Publication. Wiley, New York (1987)
[21] Bullen, P.S.: Handbook of Means and their Inequalities. Kluwer, Dordrecht (2003)
[22] Bullen, P.S., Mitrinović, D.S., Vasić, P.M.: Means and their inequalities. Mathematics and its Applications (East European Series) 31, D. Reidel Publishing Co., Dordrecht (1988)
[23] Burai, P.: A Matkowski-Sutô type equation. Publ. Math. Debrecen **70**, 233–247 (2007)
[24] Burai, P.: Matkowski-Sutô type equation on symmetrized weighted quasi-arithmetic means. Results Math. **63**, 397–408 (2013)
[25] Burai, P., Jarczyk, J.: Conditional homogeneity and translativity of Makó–Páles means. Ann. Univ. Sci. Budapest. Sect. Comp. **40**, 159–172 (2013)
[26] Burai, P., Jarczyk, J.: On symmetry of Makó–Páles means (submitted)
[27] Carlson, B.C.: Algorithms involving arithmetic and geometric means. Am. Math. Monthly (1971)
[28] Cauchy, A.L.: Cours d'analyse de l'École Royale Polytechnique, I^{re} partie: Analyse Algébrique. Imprimérie Royale, Paris (1821)
[29] Chisini, O.: Sur concetto di media. Period. Mat. **9**, 106–122 (1929)
[30] Chisini, O.: Numeros indice et medias. Schola at Vita **5**, 181–190 (1930)
[31] Chisini, O.: Alcuni theoremi sulle medie. Scritt. Math. in onore di F. Sibirani, Bologna (1957)
[32] Costin, I., Toader, G.: Invariance in the class of weighted Lehmer means, JIPAM J. Inequal. Pure. Appl. Math 9, Article 54, 7 pp (2008)
[33] Costin, I., Toader, G.: Invariance of a weighted power mean in the class of weighted Gini means. Autom. Comput. Appl. Math. **21**, 35–43 (2012)
[34] Costin, I., Toader, Gh.: Invariance in the family of weighted Gini means, Handbook of functional equations, 105–127, Springer Optim. Appl., 95, Spinger, New York (2014)
[35] Daróczy, Z.: On a class of means of two variables. Publ. Math. Debrecen **55**, 177–197 (1999)
[36] Daróczy, Z.: Matkowski-Sutô type problem for conjugate arithmetic means. Rocznik Nauk. –Dydakt. Prace Mat. **17**, 89–100 (2000)
[37] Daróczy, Z.: Gaussian iteration of mean values and the existence of $\sqrt{2}$. Teach. Math. Comput. Sci. **1**, 35–45 (2003)
[38] Daróczy, Z.: Mean values and functional equations. Differ. Equ. Dyn. Syst. **17**, 105–113 (2009)
[39] Daróczy, Z.: On the equality and comparison problem of a class of mean values. Aequationes Math. **81**, 201–208 (2011)
[40] Daróczy, Z., Dascăl, J.: On the general solution of a family of functional equations with two parameters and its application. Math. Pannon. **20**, 27–36 (2009)
[41] Daróczy, Z., Dascăl, J.: On the equality problem of conjugate means. Results Math. **58**, 69–79 (2010)
[42] Daróczy, Z., Hajdu, G.: On linear combinations of weighted quasi-arithmetic means. Aequationes Math. **69**, 58–67 (2005)
[43] Daróczy, Z., Hajdu, G., Ng, C.T.: An extension theorem for a Matkowski-Sutô problem. Colloq. Math. **95**, 153–161 (2003)
[44] Daróczy, Z., Losonczi, L.: Über der Vergleich von Mittelverter. Publ. Math. Debrecen **17**, 289–297 (1970)
[45] Daróczy, Z., Maksa, G.: Functional equations on convex sets. Acta Math. Hungar. **68**, 187–195 (1995)
[46] Daróczy, Z., Maksa, G.: On a problem of Matkowski. Colloq. Math. **82**, 117–123 (1999)
[47] Daróczy, Z., Maksa, G., Páles, Z.: Extension theorems for the Matkowski-Sutô problem. Demonstratio Math. **23**, 544–556 (2000)

[48] Daróczy, Z., Maksa, G., Páles, Z.: On two variable means with variable weights. Aequat. Math. **67**, 154–159 (2004)

[49] Daróczy, Z., Páles, Z.: On means that are both quasi-arithmetic and conjugate arithmetic. Acta Math. Hungar. **90**, 271–282 (2001)

[50] Daróczy, Z., Páles, Z.: On a class of means on several variables. Math. Inequal. Appl. **4**, 331–341 (2001)

[51] Daróczy, Z., Páles, Z.: A Matkowski-Sutô Type Problem for quasi-arithmetic Means of Order α, In: Daróczy, Z., Páles, Zs. (eds.) Functional Equations Results and Advances, Adv. Math. (Dordr.), vol. 3, Kluwer, Dordrecht, pp. 189–200 (2002)

[52] Daróczy, Z., Páles, Z.: Gauss-composition of means and the solution of the Matkowski-Sutô problem. Publ. Math. Debrecen **61**, 157–218 (2002)

[53] Daróczy, Z., Páles, Z.: The Matkowski-Sutô problem for weighted quasi-arithmetic means. Acta Math. Hungar. **100**, 237–243 (2003)

[54] Daróczy, Z., Páles, Z.: A Matkowski-Sutô-type problem for weighted quasi-arithmetic means. Ann. Univ. Sci. Budapest **22**, 69–81 (2003)

[55] Daróczy, Z., Páles, Z.: On functional equations involving means. Publ. Math. Debrecen **62**, 363–377 (2003)

[56] Daróczy, Z., Páles, Z.: Generalized convexity and comparison of mean values. Acta Sci. Math. (Szeged) **71**, 105–116 (2005)

[57] Daróczy, Z., Páles, Z.: On the equality of means, Report of Meeting. The sixth Debrecen-Katowice Winter Seminar on Functional Equations and Inequalities. http://riesz.math.klte.hu/~debkat/2006/2006.html (2006)

[58] Daróczy, Z., Páles, Z.: On symmetrized weighted quasi-arithmetic means, Report of Meeting. The Fourty-third International Symposium on Functional Equations. Aequat. Math. **71**, 178–179 (2006)

[59] Dascăl, J.: On a functional equation with a symmetric component. Publ. Math. Debrecen **75**, 33–39 (2009)

[60] Diamond, P.: A stochastic functional equation. Aequat. Math. **15**, 225–233 (1977)

[61] Domsta, J., Matkowski, J.: Invariance of the arithmetic mean with respect to special mean-type mappings. Aequat. Math. **71**, 70–85 (2006)

[62] de Finetti, B.: Sur concetto di media. Giornale Dell Istitutio Italiano degli Attuari **2**, 369–396 (1931)

[63] Foster, D.M.E., Phillips, G.M.: The arithmetic-harmonic mean. Math. Comput. **42**, 183–191 (1984)

[64] Gauss, C.F.: Werke. Göttingen (1876)

[65] Gauss, C.F., Geppert, H.: Bestimmung der Anziehung eines elliptischen Ringes: Nachlass zur Teorie des arithmetisch-geometrischen Mittels und der Modulfunktion. Engelmann, Leipzig (1927)

[66] Gini, C.: Di una formula comprensiva delle medie. Metron **13**, 3–22 (1938)

[67] Głazowska, D.: An invariance of the geometric with respect to Lagrangian conditionally homogeneous mean-type mappings. Demonstratio Math. **40**(2), 289–302 (2007)

[68] Głazowska, D.: A solution of an open problem concerning Lagrangian mean-type mappings. Cent. Eur. J. Math. **9**, 1067–1073 (2011)

[69] Głazowska, D.: Some Cauchy mean-type mappings for which the geometric mean is invariant. J. Math. Anal. Appl. **375**, 418–430 (2011)

[70] Głazowska, D., Jarczyk, J., Jarczyk, W.: Square iterative roots of some mean-type mappings J. Differ. Equ. Appl. https://doi.org/10.1080/10236198.2017.1326913

[71] Głazowska, D., Jarczyk, W., Matkowski, J.: Arithmetic mean as a linear combination of two quasi-arithmetic means. Publ. Math. Debrecen **61**, 455–467 (2002)

[72] Głazowska, D., Matkowski, J.: An invariance of the geometric mean with respect to Lagrangian means. J. Math. Anal. Appl. **331**, 1187–1199 (2007)

[73] Hajdu, G.: An extension theorem for Matkowski-Sutô problem for conjugate arithmetic means. In: Daróczy, Z., Páles, Zs (eds.) Adv. Math. (Dordr.), vol. 3, pp. 201–208. Kluwer, Dordrecht (2002)

[74] Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1934) (1st edn), 1952 (2nd edn)

[75] Heinz, E.: Beiträge zur Störungstheorie der Spektalzerlegung. Math. Ann. **123**, 415–438 (1951)

[76] Horváth, J.: Note sur un probléme de L. Fejér. Bull. École Polytech. Jassy **3**, 164–168 (1948)

[77] Járai, A.: Regularity properties of functional equations in several variables. Advances in Mathematics 8, Springer, New York (2005)

[78] Jarczyk, J.: Invariance in the class of weighted quasi-arithmetic means with continuous generators. Publ. Math. Debrecen **71**, 279–294 (2007)

[79] Jarczyk, J.: Invariance of quasi-arithmetic means with function weights. J. Math. Anal. Appl. **353**, 134–140 (2009)

[80] Jarczyk, J.: Regularity theorem for a functional equation involving means. Publ. Math. Debrecen **75**, 123–135 (2009)

[81] Jarczyk, J.: On an equation involving weighted quasi-arithmetic means. Acta Math. Hungar. **129**, 96–111 (2010)

[82] Jarczyk, J.: Invariance in a class of Bajraktarević means. Nonlinear Anal. Theory Methods Appl. **72**, 2608–2619 (2010)

[83] Jarczyk, J.: Determination of conjugate means by reducing to the generalized Matkowski-Sutô equation. Acta Math. Hungar. **139**(1), 1–10 (2013)

[84] Jarczyk, J.: Parametrized means and limit properties of their Gaussian iterations. Appl. Math. Comput. **261**, 81–89 (2015)

[85] Jarczyk, J., Matkowski, J.: Invariance in the class of weighted quasi-arithmetic means. Ann. Polon. Math. **88**, 39–51 (2006)

[86] Jecklin, H.: Qusiarithmetische Mittelwerte I. Elem. Math. **4**, 112–115 (1949)

[87] Jecklin, H.: Qusiarithmetische Mittelwerte II. Elem. Math. **4**, 128–133 (1949)

[88] Jecklin, H.: La notion de moyenne. Metron **15**, 3–11 (1949)

[89] Jessen, B.: Bemærkinger om konvekse Funktioner og Uligheder imellem Middelværdier. II (Danish), Matematisk Tidsskrift B, pp. 84–95 (1931)

[90] Jessen, B.: Über die Verallgemeinerungen des aritmetischen Mittels. Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Sectio Scientarum Mathematicarum, Szeged **5**, 108–116 (1931)

[91] Kahlig, P., Matkowski, J.: On the composition of homogeneous quasi-arithmetic means. J. Math. Anal. Appl. **216**, 69–85 (1997)

[92] Kahlig, P., Matkowski, J.: Invariant means related to classical weighted means. Publ. Math. Debrecen **216**, 373–387 (2016)

[93] Knaster, B.: Sur une équivalence pour les fonctions. Colloquium Math. **2**, 1–4 (1949)

[94] Knopp, K.: Über Reihen mit positiven Gliedern. J. Lond. Math. Soc. **3**, 205–211 (1928)

[95] Knopp, K.: Neuere Sätze über Reihen mit positiven Gliedern. Math. Zeitschr. **30**, 387–413 (1929)

[96] Kolesárová, A.: Limit properties of quasi-arithmetic means. Fuzzy Sets Syst. **124**, 65–71 (2001)

[97] Kolmogoroff, A.: Sur la notion de la moyenne. Rend. Accad. Lincei **6**, 388–391 (1930)

[98] Kuczma, M.: An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality, Uniwersytet Śląski, Katowice, and Państwowe Wydawnictwo Naukowe, Kraków–Warszawa, 1985 (1st edn); Birkhäuser Verlag, Basel, 2009 (2nd edn, edited by A. Gillanyi)

[99] Lagrange, J.L.: Sur une nouvelle Méthode de Calcul Intégrale pour différentielles affectées d'un radical carre, Mem. Acad. R. Sci. Turin II, 2, 252–312 (1784–1785)

[100] Lehmer, D.H.: On the compounding of certain means. J. Math. Anal. Appl. **36**, 183–200 (1971)

[101] Losonczi, L.: Über den Vergleich von Mittelverten die mit Gewichtsfunktionen gebildet sind. Publ. Math. Debrecen **17**, 203–208 (1970)

[102] Losonczi, L.: Equality of two variable weighted means: reduction to differential equations. Aequat. Math. **58**, 223–241 (1999)
[103] Losonczi, L.: Equality of two variable Cauchy mean values. Aequat. Math. **65**, 61–81 (2003)
[104] Makó, Z., Páles, Z.: On the equality of generalized quasi-arithmetic means. Publ. Math. Debrecen **72**, 407–440 (2008)
[105] Makó, Z., Páles, Z.: The invariance of the arithmetic mean with respect to generalized quasi-arithmetic means. J. Math. Anal. Appl. **353**, 8–23 (2009)
[106] Maksa, G., Páles, Z.: Remarks on the comparison of weighted quasiarithmetic. Colloq. Math. **120**, 77–84 (2010)
[107] Marichal, J.L.: On an axiomatization of the quasi-arithmetic mean values without the symmetry axiom. Aequat. Math. **59**, 74–83 (2000)
[108] Matkowski, J.: Complementary quasi-arithmetic means. In: Leaflets in Mathematics, Proc. Numbers, Functions, Equations '98 Internat. Conf. Noszvaj (Hungary), Pecs, 123–124 (1998)
[109] Matkowski, J.: Invariant and complementary quasi-arithmetic means. Aequat. Math. **57**, 87–107 (1999)
[110] Matkowski, J.: Mean value property and associated functional equations. Aequat. Math. **58**, 46–59 (1999)
[111] Matkowski, J.: Iterations of mean-type mappings and invariant means. In: European Conference on Iteration Theory (Muszyna-Złockie, 1998). Ann. Math. Sil. vol. 13, pp. 211–226 (1999)
[112] Matkowski, J.: On iteration semigroups of mean-type mappings and invariant means. Aequat. Math. **64**, 297–303 (2002)
[113] Matkowski, J.: On invariant generalized Beckenbach-Gini means. In: Daróczy, Z., Páles, Zs. (eds), Functional Equations Results and Advances. in: Advances in Mathematics, vol. 3, Kluwer, Dordrecht, pp. 219–230 (2002)
[114] Matkowski, J.: Remark 1 (at the Second Debrecen-Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszlo, 2002). Ann. Math. Silesianae **16**, 93 (2003)
[115] Matkowski, J.: Solution of a regularity problem in equality of Cauchy means. Publ. Math. Debrecen **64**, 391–400 (2004)
[116] Matkowski, J.: Lagrangian mean-type mappings for which the arithmetic mean is invariant. J. Math. Anal. Appl. **309**, 15–24 (2005)
[117] Matkowski, J.: On iterations of means and functional equations, Iteration Theory (ECIT'04) Grazer Math. Ber. **350**, 184–197 (2006)
[118] Matkowski, J.: Iterations of the mean-type mappings, In: Iteration theory (ECIT'08), A. N. Sharkovsky, I. M. Susko (eds.), Grazer Math. Ber., Bericht Nr., 354, 158–179 (2009)
[119] Matkowski, J.: Generalized weighted quasi-arithmetic means. Aequat. Math. **79**, 203–212 (2010)
[120] Matkowski, J.: On weighted extensions of Bajraktarevic means. Sarajevo J. Math. **6**, 169–188 (2010)
[121] Matkowski, J.: Quotient mean, its invariance with respect to a quasi-arithmetic mean-type mapping, and some applications. Aequat. Math. **82**, 247–253 (2011)
[122] Matkowski, J.: Invariance of the Cauchy means with respect to Lagrange mean-type mappings of the same generators. Acta Math. Hungar. **132**, 92–106 (2011)
[123] Matkowski, J.: Invariance of a quasi-arithmetic mean with respect to a system of generalized Bajraktarević means. Appl. Math. Lett. **25**, 1651–1655 (2012)
[124] Matkowski, J.: Generalized weighted arithmetic means. In: Rassias, T., Brzdęk, J. (eds.) Functional Equations in Mathematical Analysis, pp. 563–582. Springer, New York (2012)
[125] Matkowski, J.: Invariance of Bajraktarevic mean with respect to quasi-arithmetic means. Publ. Math. Debrecen **80**, 441–455 (2012)

[126] Matkowski, J.: Halfconvex functions and equality of Daróczy-Páles conjugate means. Aequat. Math. **86**, 171–185 (2013)

[127] Matkowski, J.: Iterations of the mean-type mappings and uniqueness of invariant means. Ann. Univ. Sci. Budapest. Sect. Comput. **41**, 145–158 (2013)

[128] Matkowski, J.: Generalized weighted quasi-arithmetic means and the Kolmogorov–Nagumo theorem. Colloq. Math. **133**, 35–49 (2013)

[129] Matkowski, J.: Invariance of the Bajraktarević means with respect to Beckenbach-Gini means. Math. Slovaca **63**, 493–502 (2013)

[130] Matkowski, J.: Invariance identity in the class of generalized quasi-arithmetic means. Colloq. Math. **137**, 221–228 (2014)

[131] Matkowski, J., Páles, Z.: Characterization of generalized quasi-arithmetic means. Acta Sci. Math. (Szeged) **81**, 447–456 (2015)

[132] Matkowski, J., Volkmann, P.: A functional equation with two unknown functions, http://www.mathematik.uni-karlsruhe.de/~semlv, Seminar LV 30, 6 pp. (28.04.2008)

[133] Matkowski, J., Wróbel, M.: A functional equation with two unknown functions. Ann. Math. Sil. **10**, 7–12 (1996)

[134] Matkowski, J., Nowicka, M., Witkowski, A.: Explicit solutions of the invariance equation for means. Results Math. **71**, 397–410 (2017)

[135] Mitrinović, D.S., Bullen, P.S., Vasić, P.M.: Sredine i sa njima povezane nejednakosti. I (Serbo-Croatian), Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz. 600, 232 pp (1977)

[136] Mitrinović, D.S., Vasić, P.M.: Uvodenje mladih u naučni rad. V: Sredine (Serbo-Croatian), Mathematička Biblioteka 40, Beograd (1969)

[137] Nagumo, M.: Uber eine Klasse der Mittelwerte. Jpn. J. Math. **7**, 71–79 (1930)

[138] Oxtoby, J.C.: Measure and category, Graduate Texts in Mathematics, vol. 2. Springer, New York (1971)

[139] Páles, Z.: On the characterization of quasi-arithmetic means with weight function. Aequat. Math. **32**, 171–194 (1987)

[140] Páles, Z.: On two-variable functional inequalities. C.R. Math. Rep. Acad. Sci. Canada **10**, 25–28 (1988)

[141] Páles, Z., Zakaria, A.: On the invariance equation for two-variable weighted nonsymmetric Bajraktarević means (submitted)

[142] Pasteczka, P.: Scale of quasi-arithmetic means determined by an invariance property. J. Differ. Equ. Appl. **21**, 1321–1327 (2015)

[143] Pasteczka, P.: A new estimate of the difference among quasi-arithmetic means. Math. Inequal. Appl. **18**, 1321–1327 (2015)

[144] Pasteczka, P.: Limit properties in a family of quasi-arithmetic means. Aequat. Math. **90**, 773–785 (2016)

[145] Pasteczka, P.: Iterated quasi-arithmetic mean-type mappings. Colloq. Math. **144**, 215–228 (2016)

[146] Ryll-Nardzewski, C.: Sur les moyennes. Studia Math. **11**, 31–37 (1949)

[147] Schoenberg, I.J.: Mathemaical Time Exposures. Mathematical Association of America, Washington (1982)

[148] Sonubon, A., Orankitjaroen, S.: A functional equation with conjugate means derived from a weighted arithmetic mean. Malays. J. Math Sci. **9**, 21–31 (2015)

[149] Sutô, O.: Studies on some functional equations I. Tôhoku Math. J. **6**, 1–15 (1914)

[150] Sutô, O.: Studies on some functional equations II. Tôhoku Math. J. **6**, 82–101 (1914)

[151] Stolarsky, K.B.: Generalizations of the logarithmic mean. Math. Mag. **48**, 87–92 (1975)

[152] Thielman, H.P.: On generalized means. Proc. Iowa Acad. Sci. **56**, 241–247 (1949)

[153] Toader, Gh., and Costin, I.: Means in mathematical analysis. Bivariate means, Mathematical Analysis and its Application Series, Academic Press, An imprint of Elsevier, London (2018)

[154] Toader, G., Costin, I., Toader, S.: Invariance in some families of means, Functional equations in mathematical analysis, 709–717. Springer Optim. Appl., 52, Springer, New York (2012)
[155] Witkowski, A.: On invariance equation for means of power growth. Math. Inequal. Appl. **17**, 1091–1094 (2014)
[156] Zhang, Q., Bing, X.: An invariance of geometric mean with respect to generalized quasi-arithmetic means. J. Math. Anal. Appl. **379**, 65–74 (2011)
[157] Zhang, Q., Bing, X.: On some invariance of the quotient mean with respect to Makó–Páles means. Aequat. Math. **91**, 1147–1156 (2017)

Justyna Jarczyk and Witold Jarczyk
Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
Szafrana 4a
65-516 Zielona Góra
Poland
e-mail: j.jarczyk@wmie.uz.zgora.pl

Witold Jarczyk
e-mail: w.jarczyk@wmie.uz.zgora.pl

Received: December 19, 2017
Revised: April 7, 2018