On a generalization of sandwich type theorems

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Abstract. We introduce affine and convex functions with a control function and present some sandwich type theorems for them. Also, Hyers–Ulam stability type results for affine and convex functions with a control function are given.

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1. Introduction

Let $I \subset \mathbb{R}$ be an open interval. In paper [3] the authors proved that two functions $f, g : I \rightarrow \mathbb{R}$ can be separated by a convex function if and only if

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y), \quad x, y \in I, \quad t \in [0, 1].$$

A counterpart of this result for strongly convex functions, i.e. functions satisfying the inequality

$$h(tx + (1-t)y) \leq th(x) + (1-t)h(y) - ct(1-t)|x-y|^2, \quad x, y \in I, \quad t \in [0, 1],$$

where c is a fixed positive number, is presented in [7] and it appears that the necessary and sufficient condition for the separation of two functions $f, g : I \rightarrow \mathbb{R}$ by strongly convex function is the following

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y) - ct(1-t)|x-y|^2, \quad x, y \in I, \quad t \in [0, 1].$$

In [1] the author introduced a concept of strong convexity in a more general case, i.e. functions satisfying the inequality

$$h(tx + (1-t)y) \leq th(x) + (1-t)h(y) - t(1-t)F(x-y), \quad x, y \in I, \quad t \in [0, 1],$$

where F is a fixed positive function are considered, and called them F -strongly convex. It is a natural question in the context of the aforementioned separation results and F -strong convexity, whether the inequality

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y) - t(1-t)F(x-y), \quad x, y \in I, \quad t \in [0, 1],$$

guaranties the separation of the functions f and g by an F -strongly convex function. In general, the answer to this question is that it does not guarantee such a separation. It can be verified that for constant functions $f \equiv 0$, $g \equiv 1$ and $F \equiv 1$ the above inequality holds true, but we cannot separate them by a 1-strongly convex function, because a 1-strongly convex function does not exist.

The aim of this paper is to present a condition under which a separation result holds true and to show it in a more general case than F -strong convexity.

2. Main result

We start with the following two definitions.

Definition 1. Let $G : [0, 1] \times I^2 \rightarrow \mathbb{R}$ be a given function. A function $f : I \rightarrow \mathbb{R}$ we will call a convex function with a control function G if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + G(t, x, y)$$

for all $t \in [0, 1]$ and $x, y \in I$.

Definition 2. Let $G : [0, 1] \times I^2 \rightarrow \mathbb{R}$ be a given function. A function $f : I \rightarrow \mathbb{R}$ we will call an affine function with a control function G if

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) + G(t, x, y)$$

for all $t \in [0, 1]$ and $x, y \in I$.

Of course, if we take a function $G(t, x, y) = -ct(1-t)|x-y|^2$ we will obtain a well known strong convexity case, and if we take a function $G(t, x, y) = -t(1-t)F(x-y)$ then we will get F -strong convexity. It appears, that between convex functions and strongly convex functions we have some connections (see [2, 6, 9, 12–14]). In particular, under some assumptions, a function f is strongly convex if and only if the function $f - |\cdot|^2$ is convex and also a function f is F -strongly convex if and only if the function $f - F$ is convex (see [1]). Now we present an obvious counterpart of these results.

Observation. Assume that a function $\phi : I \rightarrow \mathbb{R}$ is an affine function with a control function G . Then a function $f : I \rightarrow \mathbb{R}$ is a convex function with a control function G (affine function with a control function G , resp.) if and only if the function $f - \phi$ is a convex function (affine function, resp.) on I .

At this moment we will focus our attention on affine functions with a control function G and we will try to describe a structure of the family

$$\mathcal{F} := \{\phi : I \rightarrow \mathbb{R} \mid \phi \text{ is affine with a control function } G\}.$$

Lemma 1. *Let $x_1, x_2 \in I$, $x_1 \neq x_2$ and $\mathcal{F} \neq \emptyset$. If $\phi, \mu \in \mathcal{F}$ and $\phi(x_1) = \mu(x_1)$ and $\phi(x_2) = \mu(x_2)$, then $\phi = \mu$.*

Proof. Functions $\phi, \mu \in \mathcal{F}$ satisfy the following equalities

$$\begin{aligned}\phi(tx + (1-t)y) &= t\phi(x) + (1-t)\phi(y) + G(t, x, y), \\ \mu(tx + (1-t)y) &= t\mu(x) + (1-t)\mu(y) + G(t, x, y),\end{aligned}$$

for all $x, y \in I$ and $t \in [0, 1]$. Subtracting these equations side by side we get

$$(\phi - \mu)(tx + (1-t)y) = t(\phi - \mu)(x) + (1-t)(\phi - \mu)(y).$$

Thus $\phi - \mu$ is an affine function such that $(\phi - \mu)(x_1) = 0 = (\phi - \mu)(x_2)$ and in consequence $\phi = \mu$. The proof is finished. $\square$

Taking into consideration the definitions of affine functions with a control function G and affine functions, we have the next lemma.

Lemma 2. *$\phi \in \mathcal{F}$ if and only if $\phi + a \in \mathcal{F}$ and a is an affine function on I .*

And finally Theorem 1 gives a full description of the family $\mathcal{F}$.

Theorem 1. *If $\mathcal{F} \neq \emptyset$ and $\phi_0 \in \mathcal{F}$ then $\phi \in \mathcal{F}$ if and only if $\phi = \phi_0 + a$ and a is an affine function on I .*

Proof. If $\phi = \phi_0 + a$ then from Lemma 2 also $\phi \in \mathcal{F}$. Assume now that $\phi \in \mathcal{F}$. Let's fix different points $x_1, x_2 \in I$ and adjust an affine function a such that the function $\mu = \phi_0 + a$ satisfies the conditions

$$\mu(x_1) = \phi(x_1) \text{ and } \mu(x_2) = \phi(x_2).$$

In view of Lemma 1 and Lemma 2 we get that $\phi = \phi_0 + a$. The proof ends. $\square$

From the above theorem we immediately have the following corollary.

Corollary 1. *If the family $\mathcal{F} \neq \emptyset$ and $\phi_0 \in \mathcal{F}$, then*

$$\mathcal{F} = \{\phi : I \rightarrow \mathbb{R} \mid \phi(x) = \phi_0(x) + ax + b \text{ and } a, b \in \mathbb{R}\}.$$

Now we go back to sandwich problems. The next theorem is a counterpart of the classical Baron–Matkowski–Nikodem result [3] and of the Merentes–Nikodem result [7].

Theorem 2. *Let $G : [0, 1] \times I^2 \rightarrow \mathbb{R}$ be a given function and $\mathcal{F} \neq \emptyset$. Then functions $f, g : I \rightarrow \mathbb{R}$ satisfy the inequality*

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y) + G(t, x, y),$$

for all $t \in [0, 1]$ and $x, y \in I$ if and only if there exists a function $h : I \rightarrow \mathbb{R}$ convex with a control function G such that $f \leq h \leq g$ on I .

Proof. The "only if" part is evident. To prove the "if" assume that functions f, g satisfy the inequality

$$f(tx + (1-t)y) \leq tg(x) + (1-t)g(y) + G(t, x, y)$$

and ϕ is a member of the family $\mathcal{F}$, i.e.

$$\phi(tx + (1-t)y) = t\phi(x) + (1-t)\phi(y) + G(t, x, y),$$

for all $t \in [0, 1]$ and $x, y \in I$. Subtracting from the inequality this equation side by side we get

$$(f - \phi)(tx + (1-t)y) \leq t(g - \phi)(x) + (1-t)(g - \phi)(y),$$

for all $t \in [0, 1]$ and $x, y \in I$. It means that the functions $f - \phi$ and $g - \phi$ satisfy the sufficient conditions of the Baron–Matkowski–Nikodem theorem [3]. Thus, there exists a convex function $h^* : I \rightarrow \mathbb{R}$ such that

$$(f - \phi)(x) \leq h^*(x) \leq (g - \phi)(x), \quad x \in I.$$

It means that the function $h := h^* + \phi$ is between f and g and from the aforementioned observation it is convex with a control function G . The proof is complete. $\square$

Taking a function $g := f + \epsilon$, where ϵ is a fixed positive number, and substituting a function h by a function $h + \frac{\epsilon}{2}$ in the sandwich theorem above we get the following Hyers–Ulam type stability result for convex functions with a control function G (the classical Hyers–Ulam theorem we can find in [5]).

Corollary 2. *Assume that $\mathcal{F} \neq \emptyset$ and a function $f : I \rightarrow \mathbb{R}$ is approximately convex with a control function G i.e.*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + G(t, x, y) + \epsilon,$$

for all $t \in [0, 1]$, $x, y \in I$ and ϵ is a fixed positive number. Then there exists a function $h : I \rightarrow \mathbb{R}$ convex with a control function G such that

$$|f(x) - h(x)| \leq \frac{\epsilon}{2}, \quad x \in I.$$

Remark. Taking a control function $G(t, x, y) = -ct(1-t)^2$ it is easy to check that the function $\phi_0(x) = cx^2$ belongs to the family $\mathcal{F}$. Thus we get the results presented in [7].

Applying techniques and arguments as in proof of Theorem 1 but, instead of the Baron–Matkowski–Nikodem result, using the Nikodem–Wąsowicz result [11] we will obtain the following.

Theorem 3. *Let $G : [0, 1] \times I^2 \rightarrow \mathbb{R}$ be a given function and $\mathcal{F} \neq \emptyset$. Then functions $f, g : I \rightarrow \mathbb{R}$ satisfy the system of inequalities*

$$\begin{cases} f(tx + (1-t)y) \leq tg(x) + (1-t)g(y) + G(t, x, y) \\ g(tx + (1-t)y) \geq tf(x) + (1-t)f(y) + G(t, x, y) \end{cases}$$

for all $t \in [0, 1]$ and $x, y \in I$ if and only if there exists a function $h : I \rightarrow \mathbb{R}$ affine with a control function G such that $f \leq h \leq g$ on I .

Corollary 3. Assume that $\mathcal{F} \neq \emptyset$ and a function $f : I \rightarrow \mathbb{R}$ is approximately affine with a control function G i.e.

$$|f(tx + (1-t)y) - tf(x) - (1-t)f(y) - G(t, x, y)| < \epsilon,$$

for all $t \in [0, 1]$, $x, y \in I$ and ϵ is a fixed positive number. Then there exists a function $h : I \rightarrow \mathbb{R}$ affine with a control function G such that

$$|f(x) - h(x)| \leq \frac{\epsilon}{2}, \quad x \in I.$$

In [7, 8] the authors showed a connection between strong convexity and general convexity in the Beckenbach sense. In the concept of general convexity Beckenbach replaced straight lines (in fact affine functions) by some functions from a two-parameter family. Recall that a family of functions defined on I is a two-parameter family if for any two points $(x_1, x_2), (y_1, y_2) \in \mathbb{R}$, such that $x_1 \neq y_1$, there exists exactly one function from this family going through the points $(x_1, x_2), (y_1, y_2)$ (for more details see [4]). Of course, if the family $\mathcal{F}$ is nonempty, then $\mathcal{F}$ is a two-parameter family and we have the following.

Theorem 4. Let $G : [0, 1] \times I^2 \rightarrow \mathbb{R}$ be a given function and $\mathcal{F} \neq \emptyset$. Then

(1) A function $f : I \rightarrow \mathbb{R}$ is convex with a control function G if and only if

$$f(tx + (1-t)y) \leq \phi_{(x, f(x)), (y, f(y))}(tx + (1-t)y), \quad x, y \in I, t \in (0, 1),$$

where $\phi_{(x, f(x)), (y, f(y))}$ is the unique function from the family $\mathcal{F}$, going through the points $(x, f(x)), (y, f(y))$.

(2) A function $f : I \rightarrow \mathbb{R}$ is affine with a control function G if and only if

$$f(tx + (1-t)y) = \phi_{(x, f(x)), (y, f(y))}(tx + (1-t)y), \quad x, y \in I, t \in (0, 1),$$

where $\phi_{(x, f(x)), (y, f(y))}$ is the unique function from the family $\mathcal{F}$, going through the points $(x, f(x)), (y, f(y))$ and in fact $\phi_{(x, f(x)), (y, f(y))} = f$.

Proof. Fix different points $x, y \in I$ and the unique member $\phi_{(x, f(x)), (y, f(y))} \in \mathcal{F}$. Assume that f is convex with a control function G and observe the following:

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) + G(t, x, y) \\ &= t\phi_{(x, f(x)), (y, f(y))}(x) + (1-t)\phi_{(x, f(x)), (y, f(y))}(y) + G(t, x, y) \\ &= \phi_{(x, f(x)), (y, f(y))}(tx + (1-t)y). \end{aligned}$$

Thus, f satisfies the inequality from point (1). Now assume that the inequality from point (1) holds true. And we have the following:

$$\begin{aligned} f(tx + (1-t)y) &\leq \phi_{(x, f(x)), (y, f(y))}(tx + (1-t)y) \\ &= t\phi_{(x, f(x)), (y, f(y))}(x) + (1-t)\phi_{(x, f(x)), (y, f(y))}(y) + G(t, x, y) \\ &= tf(x) + (1-t)f(y) + G(t, x, y), \end{aligned}$$

which shows that f is convex with a control function G . Replacing in the above calculation inequalities by equalities we get a proof of point (2). $\square$

Notice that Nikodem and Páles in [10] obtained a sandwich type theorem for general convexity in the Beckenbach sense. We do not assume the continuity of the functions from the family $\mathcal{F}$. Thus, we cannot obtain the presented results from the results presented in [10].

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