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Separation by Jensen and affine stochastic processes

Dawid Kotrys and Kazimierz Nikodem

Abstract. Characterizations of pairs of stochastic processes that can be separated by Jensen and by affine stochastic processes are presented. As a consequence, some stability results of the Hyers–Ulam-type are obtained.

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1. Introduction

It is known from the classical Hahn–Banach separation theorem, that if a function f is concave, g is convex and $f \leqslant g$, then there exists an affine function h such that $f \leqslant h \leqslant g$. This separation (sandwich) theorem plays an important role in many fields of mathematics and has numerous applications in convex analysis, optimization theory and economics. Many other results providing conditions under which two given functions can be separated by a function from some special class can be found in the literature (see, for instance [1–3, 5, 6, 10, 11, 15–19, 23] and the references therein).

In this note we investigate the separation problem for stochastic processes. We characterize pairs of processes that can be separated by Jensen and affine processes. As a consequence of our main theorems we obtain Hyers–Ulam-type stability results for Jensen and affine stochastic processes.

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space and $I \subset \mathbb{R}$ be an interval. A function $A : \Omega \to \mathbb{R}$ is called a *random variable*, if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$ is called a *stochastic process*, if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Recall that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is called *midconvex* (cf. [12]) if

$$X\Big(\frac{t_1+t_2}{2}, \cdot\Big) \leqslant \frac{X(t_1,\cdot)+X(t_2,\cdot)}{2} \quad \text{(a.e.)},$$

for all $t_1, t_2 \in I$. It is called *midconcave* if it satisfies the reverse inequalities. We say that X is a *Jensen stochastic process* if it is both midconvex and midconcave simultaneously, that is the following condition holds

$$X\Big(\frac{t_1+t_2}{2}, \cdot\Big) = \frac{X(t_1,\cdot)+X(t_2,\cdot)}{2} \quad \text{(a.e.)},$$

for all $t_1, t_2 \in I$. Let D denote the set of all dyadic real numbers. It is known (see [12]) that if X is a Jensen process then

$$X\Big(\sum_{i=1}^{n} p_i t_i, \cdot\Big) = \sum_{i=1}^{n} p_i X(t_i,\cdot) \quad \text{(a.e.)},$$

for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in I$ and $p_1, \dots, p_n \in [0,1] \cap D$ such that $p_1 + \dots + p_n = 1$.

A stochastic process $X : I \times \Omega \to \mathbb{R}$ is said to be *convex*, if the inequality

$$X\big(\lambda t_1 + (1-\lambda)t_2, \cdot\big) \leqslant \lambda X(t_1,\cdot) + (1-\lambda)X(t_2,\cdot) \quad \text{(a.e.)},$$

is satisfied for all $t_1, t_2 \in I$ and $\lambda \in [0,1]$. It is called *concave* if it satisfies the reverse inequalities. We say that X is an *affine stochastic process* if it is both convex and concave simultaneously, that is

$$X\big(\lambda t_1 + (1-\lambda)t_2, \cdot\big) = \lambda X(t_1,\cdot) + (1-\lambda)X(t_2,\cdot) \quad \text{(a.e.)},$$

for all $t_1, t_2 \in I$ and $\lambda \in [0,1]$. Many properties of convex and midconvex stochastic processes can be found in [6–8, 12, 21, 22].

At the end of this section, let us recall the definitions of the essential infimum and essential supremum of a collection of functions. We will use these notions as a basic tool in the proof of our main theorems. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{S}$ be a collection of measurable functions $f : \Omega \to \mathbb{R}$. On $\mathbb{R}$ the Borel σ-algebra is used. If $\mathcal{S}$ is a countable set, then we may define the pointwise infimum of the functions from $\mathcal{S}$, which is measurable itself. If $\mathcal{S}$ is uncountable, then the pointwise infimum need not be measurable. In this case, the essential infimum can be used. The *essential infimum* of $\mathcal{S}$, written as $\operatorname{ess\,inf} \mathcal{S}$, if it exists, is a measurable function $f : \Omega \to \mathbb{R}$ satisfying the following two axioms:

- $f \leqslant g$ almost everywhere, for any $g \in \mathcal{S}$,
- if $h : \Omega \to \mathbb{R}$ is measurable and $h \leqslant g$ almost everywhere for every $g \in \mathcal{S}$, then $h \leqslant f$ almost everywhere.

Similarly, the *essential supremum* of $\mathcal{S}$, written as $\operatorname{ess\,sup} \mathcal{S}$, is a measurable function $f : \Omega \to \mathbb{R}$ satisfying the following two axioms:

- $f \geqslant g$ almost everywhere, for any $g \in \mathcal{S}$,

- if $h : \Omega \to \mathbb{R}$ is measurable and $h \geqslant g$ almost everywhere for every $g \in \mathcal{S}$, then $h \geqslant f$ almost everywhere.

It can be shown that for a σ-finite measure μ, the essential infimum (essential supremum) of $\mathcal{S}$ does exist, whenever $\mathcal{S}$ is a family of measurable functions jointly bounded from below (from above). For more details we refer the reader to [4].

2. Separation by Jensen processes

In this section we characterize pairs of stochastic processes that can be separated by a Jensen process. We start with a sandwich theorem of Rodé type (see [20]), which plays a crucial role in our investigations. The proof presented is based on the idea taken from [9] (cf also [14]).

Theorem 1. *Let $X, Y : I \times \Omega \to \mathbb{R}$ be midconcave and midconvex stochastic processes respectively. We additionally assume, that for all $t \in I$*

$$X(t,\cdot) \leqslant Y(t,\cdot) \quad (a.e.).$$

Then there exists a Jensen stochastic process $Z : I \times \Omega \to \mathbb{R}$, such that

$$X(t,\cdot) \leqslant Z(t,\cdot) \leqslant Y(t,\cdot) \quad (a.e.)$$

for all $t \in I$.

Proof. Consider the family of stochastic processes

$$\mathcal{M} = \Big\{\Phi : I \times \Omega \to \mathbb{R} : \Phi \text{ - midconcave and for every } t \in I \\ X(t,\cdot) \leqslant \Phi(t,\cdot) \leqslant Y(t,\cdot) \quad \text{(a.e.)}\Big\}$$

endowed with the partial order defined by:

$$\Phi_1 \leqslant \Phi_2 \quad \text{if for every} \quad t \in I \quad \Phi_1(t,\cdot) \leqslant \Phi_2(t,\cdot) \quad \text{(a.e.)}.$$

$\mathcal{M}$ is not empty, because $X \in \mathcal{M}$. If $\mathcal{L}$ is a chain of elements of $\mathcal{M}$, then the process $\Phi_0 : I \times \Omega \to \mathbb{R}$ defined by

$$\Phi_0(t,\cdot) = \text{ess sup}\big\{\Phi(t,\cdot) : \Phi \in \mathcal{L}\big\}, \quad t \in I,$$

is its upper bound. Therefore, by the Kuratowski–Zorn lemma, there exists a maximal element Z of $\mathcal{M}$. We will show that Z is a Jensen process. Since for any $s, t \in I$

$$Z\Big(\frac{s+t}{2},\cdot\Big) \leqslant Y\Big(\frac{s+t}{2},\cdot\Big) \leqslant \frac{1}{2}Y(s,\cdot) + \frac{1}{2}Y(t,\cdot) \quad \text{(a.e.)}, \tag{1}$$

we can define

$$H(s,\cdot) = \text{ess sup}\Big\{Z\Big(\frac{s+t}{2},\cdot\Big) - \frac{1}{2}Y(t,\cdot) : t \in I\Big\}, \quad s \in I.$$

Then, for every $s \in I$,

$$H(s,\cdot) \leqslant \frac{1}{2}Y(s,\cdot) \quad \text{(a.e.)}$$

and H is midconcave. Indeed, by the midconcavity of Z and the midconvexity of Y, for every $s,t,u,v \in I$, we have

$$\begin{aligned}\frac{1}{2}\Big[Z\Big(\frac{s+u}{2},\cdot\Big) - \frac{1}{2}Y(u,\cdot)\Big] + \frac{1}{2}\Big[Z\Big(\frac{t+v}{2},\cdot\Big) - \frac{1}{2}Y(v,\cdot)\Big] \\ \leqslant Z\Big(\frac{1}{2}\Big(\frac{s+t}{2} + \frac{u+v}{2}\Big),\cdot\Big) - \frac{1}{2}Y\Big(\frac{u+v}{2}\Big) \leqslant H\Big(\frac{s+t}{2}\Big) \quad \text{(a.e.)}.\end{aligned}$$

Hence, taking the essential supremum over u and next over v, we obtain

$$H\Big(\frac{s+t}{2},\cdot\Big) \geqslant \frac{1}{2}H(t\cdot) + \frac{1}{2}H(s\cdot) \quad \text{(a.e.)}.$$

By inequality (1), we have

$$H(s,\cdot) \leqslant \frac{1}{2}Y(s,\cdot) \quad \text{(a.e.)}, \quad s \in I. \tag{2}$$

Since

$$\begin{aligned}Z\Big(\frac{s+t}{2},\cdot\Big) - \frac{1}{2}Y(t,\cdot) \geqslant \frac{Z(s,\cdot)+Z(t,\cdot)}{2} - \frac{1}{2}Y(t,\cdot) = \\ = \frac{1}{2}Z(s,\cdot) + \frac{1}{2}\Big[Z(t,\cdot) - Y(t,\cdot)\Big] \quad \text{(a.e.)}\end{aligned}$$

and, in view of the maximality of (2)

$$\text{ess sup}\big\{Z(t,\cdot) - Y(t,\cdot))\big\} = \text{ess inf}\big\{Y(t,\cdot) - Z(t,\cdot))\big\} = 0,$$

we also have

$$H(s,\cdot) \geqslant \frac{1}{2}Z(s,\cdot) \quad \text{(a.e.)}, \quad s \in I. \tag{3}$$

By (2) and (3)

$$\frac{1}{2}Z(s,\cdot) \leqslant H(s,\cdot) \leqslant \frac{1}{2}Y(s,\cdot) \quad \text{(a.e.)}$$

for any $s \in I$. Thus, using once more the maximality of Z, we get

$$H(s,\cdot) = \frac{1}{2}Z(s.\cdot) \quad \text{(a.e.)}.$$

Hence

$$Z\Big(\frac{s+t}{2},\cdot\Big) - \frac{1}{2}Y(t,\cdot) \leqslant H(s,\cdot) = \frac{1}{2}Z(s,\cdot) \quad \text{(a.e.)}$$

so

$$Z\Big(\frac{s+t}{2},\cdot\Big) \leqslant \frac{1}{2}Z(s,\cdot) + \frac{1}{2}Y(t,\cdot) \quad \text{(a.e.)}. \tag{4}$$

Now, fix an arbitrary $t \in I$ and define

$$G(s,\cdot) = Z\Big(\frac{s+t}{2},\cdot\Big) - \frac{1}{2}Z(t,\cdot). \tag{5}$$

It is easy to check that G is midconcave, $G(s,\cdot) \geqslant \frac{1}{2}Z(s,\cdot)$, and because of (4) $G(s,\cdot) \leqslant \frac{1}{2}Y(s,\cdot)$. Therefore

$$G(s,\cdot) = \frac{1}{2}Z(s,\cdot) \quad \text{(a.e.)}.$$

Hence, using (5), we obtain

$$Z\Big(\frac{s+t}{2},\cdot\Big) = \frac{1}{2}Z(s,\cdot) + \frac{1}{2}Z(t,\cdot) \quad \text{(a.e.)},$$

which proves that Z is a Jensen process. □

The above theorem gives a sufficient, but not necessary condition for the existence of a Jensen separator. A full characterization of pairs of stochastic processes that can be separated by a Jensen process provides the following result.

Theorem 2. *Let $X, Y : I \times \Omega \to \mathbb{R}$ be stochastic processes such that $X(t,\cdot) \leqslant Y(t\cdot)$ for every $t \in I$. The following conditions are equivalent:*

(i) for all $m, n \in \mathbb{N}$, $s_1, \ldots, s_m \in I$, $t_1, \ldots, t_n \in I$ and $p_1, \ldots, p_m, q_1, \ldots, q_n \in [0,1] \cap D$ such that $p_1 + \cdots + p_m = q_1 + \cdots + q_n = 1$ and $p_1 s_1 + \cdots + p_m s_m = q_1 t_1 + \cdots + q_n t_n$, the following inequality holds

$$\sum_{i=1}^{m} p_i X(s_i,\cdot) \leqslant \sum_{j=1}^{n} q_j Y(t_j,\cdot) \quad \textit{(a.e.)}; \tag{6}$$

(ii) there exists a midconcave stochastic process $Z_1 : I \times \Omega \to \mathbb{R}$ and a midconvex stochastic process $Z_2 : I \times \Omega \to \mathbb{R}$ such that

$$X(t,\cdot) \leqslant Z_1(t,\cdot) \leqslant Z_2(t,\cdot) \leqslant Y(t,\cdot) \quad \textit{(a.e.)}$$

for every $t \in I$;

(iii) there exists an Jensen stochastic process $Z : I \times \Omega \to \mathbb{R}$ such that

$$X(t,\cdot) \leqslant Z(t,\cdot) \leqslant Y(t,\cdot) \quad \textit{(a.e.)}$$

for every $t \in I$.

Proof. We will prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

To prove that $(i) \Rightarrow (ii)$ define first the process $Z_2 : I \times \Omega \to \mathbb{R}$ by

$$Z_2(t,\cdot) = \operatorname{ess\,inf}\Big\{\sum_{j=1}^{n} q_j Y(t_j,\cdot) : \quad n \in \mathbb{N}, t_1, \ldots, t_n \in I, q_1, \ldots, q_n \in [0,1] \cap D$$
$$\text{such that } q_1 + \cdots + q_n = 1 \text{ and } t = q_1 t_1 + \cdots + q_n t_n\Big\}.$$

By (6) (for $m = 1$, $p_1 = 1$ and $s_1 = t$) and the definition of essential infimum we have

$$X(t,\cdot) \leqslant Z_2(t,\cdot) \quad \text{(a.e.)}, \quad t \in I.$$

By the definition of Z_2 (taking $n=1$, $q_1=1$ and $t_1=t$) we also get

$$Z_2(t,\cdot)\leqslant Y(t,\cdot)\quad \text{(a.e.)},\quad t\in I.$$

To prove that Z_2 is midconvex fix $s,t\in I$ and take arbitrary $s_1,\dots,s_m,t_1,\dots,t_n\in I$ and $p_1,\dots,p_m q_1,\dots,q_n\in[0,1]\cap D$ such that $p_1+\cdots+p_m=1$, $q_1+\cdots+q_n=1$ and $s=p_1s_1+\cdots+p_ms_m$, $t=q_1t_1+\cdots+q_nt_n$. Since

$$\frac{s+t}{2}=\sum_{i=1}^{m}\frac{p_i}{2}s_i+\sum_{j=1}^{n}\frac{q_j}{2}t_j,$$

by the definition of Z_2 we have

$$Z_2\Big(\frac{s+t}{2},\cdot\Big)\leqslant\frac{1}{2}\sum_{i=1}^{m}p_iY(s_i,\cdot)+\frac{1}{2}\sum_{j=1}^{n}q_jY(t_j,\cdot)\quad\text{(a.e.)}.\tag{7}$$

This inequality holds for every $n\in\mathbb{N}$, $s_1,\dots,s_m\in I$ and $p_1,\dots,p_m\in[0,1]\cap D$ such that $p_1+\cdots+p_m=1$ and $p_1s_1+\cdots+p_ms_m=s$, as well as for all $m\in\mathbb{N}$, $t_1,\dots,t_n\in I$ and $q_1,\dots,q_n\in[0,1]\cap D$ such that $q_1+\cdots+q_n=1$ and $q_1t_1+\cdots+q_nt_n=t$. Therefore taking the essential infimum in the first term of the right hand side of (7) and next in the second term and using the second axiom of the definition of essential infimum, we get

$$Z_2\Big(\frac{s+t}{2},\cdot\Big)\leqslant\frac{1}{2}Z_2(s,\cdot)+\frac{1}{2}Z_2(t,\cdot)\quad\text{(a.e.)}.$$

This shows that Z_2 is midconvex.

Now, define $Z_1: I\times\Omega\to\mathbb{R}$ by

$$Z_1(s,\cdot)=\operatorname{ess\,sup}\Bigg\{\sum_{i=1}^{m}p_iX(s_i,\cdot):\quad m\in\mathbb{N},s_1,\dots,s_m\in I,p_1,\dots,p_m\in[0,1]\cap D\\ \text{such that } p_1+\cdots+p_m=1 \text{ and } s=p_1s_1+\cdots+p_ms_m\Bigg\}.$$

Similarly as above, we can prove that Z_1 is midconcave and

$$X(s,\cdot)\leqslant Z_1(s,\cdot)\leqslant Y(s,\cdot)\quad\text{(a.e.)},$$

for every $s\in I$.

Finally, using (6) once more and taking the essential infimum of the term on the right hand side and next the essential supremum of the term on the left hand side, we get

$$Z_1(t,\cdot)\leqslant Z_2(t,\cdot)\quad\text{(a.e.)}$$

for every $t\in I$.

Implication $(ii)\Rightarrow(iii)$ follows by Theorem 1 applied for $X=Z_1$ and $Y=Z_2$.

To prove $(iii) \Rightarrow (i)$ fix $s_1, \ldots, s_m, t_1, \ldots, t_n \in I$ and $p_1, \ldots, p_m, q_1, \ldots, q_n \in [0,1] \cap D$ such that $p_1 + \cdots + p_m = q_1 + \cdots + q_n$ and $p_1 s_1 + \cdots + p_m s_m = q_1 t_1 + \cdots + q_n t_n$. By the definition of Z, we get

$$\sum_{i=1}^{m} p_i X(s_i, \cdot) \leqslant \sum_{i=1}^{m} p_i Z(s_i, \cdot) = Z\Big(\sum_{i=1}^{m} p_i s_i, \cdot\Big) = \\ = Z\Big(\sum_{j=1}^{n} q_j t_j, \cdot\Big) = \sum_{j=1}^{n} q_j Z(t_j, \cdot) \leqslant \sum_{j=1}^{n} q_j Y(t_j, \cdot) \quad \text{(a.e.)}.$$

This finishes the proof. □

3. Separation by affine processes

In this section we present the second main result of this paper. It gives a condition under which two given stochastic processes can be separated by an affine stochastic process. We start with the following counterpart of the Hahn–Banach separation theorem.

Theorem 3. *Let $X, Y : I \times \Omega \to \mathbb{R}$ be stochastic processes such that $X(t, \cdot) \leqslant Y(t, \cdot)$ (a.e.) for any $t \in I$. If X is concave and Y is convex, then there exists an affine stochastic process $Z : I \times \Omega \to \mathbb{R}$ such that*

$$X(t, \cdot) \leqslant Z(t, \cdot) \leqslant Y(t, \cdot) \quad \textit{(a.e.)} \tag{8}$$

for every $t \in I$.

Proof. By Theorem 1 there exists a Jensen process Z satisfying (8). Since Z is majorized by a convex (and hence continuous) process Y, it follows that it is continuous (see [12]). Consequently, Z is affine. □

Theorem 4. *Let $X, Y : I \times \Omega \to \mathbb{R}$ be stochastic processes such that $X(t, \cdot) \leqslant Y(t\cdot)$ (a.e.) for every $t \in I$. The following conditions are equivalent:*

(i) for all $m, n \in \mathbb{N}$, $s_1, \ldots, s_m \in I$, $t_1, \ldots, t_n \in I$ and $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \geqslant 0$ such that $\alpha_1 + \cdots + \alpha_m = \beta_1 + \cdots + \beta_n = 1$ and $\alpha_1 s_1 + \cdots + \alpha_m s_m = \beta_1 t_1 + \cdots + \beta_n t_n$ the following inequality holds

$$\sum_{i=1}^{m} \alpha_i X(s_i, \cdot) \leqslant \sum_{j=1}^{n} \beta_j Y(t_j, \cdot) \quad \textit{(a.e.)}; \tag{9}$$

(ii) there exists a concave stochastic process $Z_1 : I \times \Omega \to \mathbb{R}$ and a convex stochastic process $Z_2 : I \times \Omega \to \mathbb{R}$ such that

$$X(t, \cdot) \leqslant Z_1(t, \cdot) \leqslant Z_2(t, \cdot) \leqslant Y(t, \cdot) \quad \textit{(a.e.)}$$

for every $t \in I$;

(iii) there exists an affine stochastic process $Z : I \times \Omega \to \mathbb{R}$ *such that*

$$X(t, \cdot) \leqslant Z(t, \cdot) \leqslant Y(t, \cdot) \quad \textit{(a.e.)}$$

for every $t \in I$.

Proof. Similarly as in the proof of Theorem 2 we will show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

To prove that $(i) \Rightarrow (ii)$ define first the process $Z_2 : I \times \Omega \to \mathbb{R}$ by

$$Z_2(t, \cdot) = \text{ess inf}\Big\{\sum_{j=1}^{n} \beta_j Y(t_j, \cdot) : \quad n \in \mathbb{N}, t_1, \ldots, t_n \in I, \beta_1, \ldots, \beta_n \in [0, 1] \\ \text{such that } \beta_1 + \cdots + \beta_n = 1 \text{ and } t = \beta_1 t_1 + \cdots + \beta_n t_n\Big\}.$$

By (9) and the definition of essential infimum we get

$$X(t, \cdot) \leqslant Z_2(t, \cdot) \leqslant Y(t, \cdot) \quad \text{(a.e.)}, \quad t \in I.$$

To prove that Z_2 is convex fix $s, t \in I$ and $\lambda \in [0, 1]$. Take arbitrary $s_1, \ldots, s_m \in I$, $\alpha_1, \ldots, \alpha_m \in [0, 1]$ and $t_1, \ldots, t_n \in I$, $\beta_1, \ldots, \beta_n \in [0, 1]$ such that $\alpha_1 + \cdots + \alpha_m = 1$, $\beta_1 + \cdots + \beta_n = 1$ and $s = \alpha_1 s_1 + \cdots + \alpha_m s_m$, $t = \beta_1 t_1 + \cdots + \beta_n t_n$. Since

$$\sum_{i=1}^{m} \lambda \alpha_i + \sum_{j=1}^{n} \big(1 - \lambda\big) \beta_j = 1$$

the point $\lambda s + (1 - \lambda)t$ is a convex combination of $s_1, \ldots, s_m, t_1, \ldots, t_n$ and

$$\lambda s + (1 - \lambda)t = \lambda \sum_{i=1}^{m} \alpha_i s_i + \big(1 - \lambda\big) \sum_{j=1}^{n} \beta_j t_j.$$

Therefore, by the definition of Z_2 we have

$$Z_2\big(\lambda s + (1 - \lambda)t, \cdot\big) \leqslant \lambda \sum_{i=1}^{m} \alpha_i Y(s_i, \cdot) + (1 - \lambda) \sum_{j=1}^{n} \beta_j Y(t_j, \cdot) \quad \text{(a.e.)}. \tag{10}$$

This inequality holds for every $n \in \mathbb{N}$, $s_1, \ldots, s_m \in I$ and $\alpha_1, \ldots, \alpha_m \in [0, 1]$ such that $\alpha_1 + \cdots + \alpha_m = 1$ and $\alpha_1 s_1 + \cdots + \alpha_m s_m = s$, as well as for all $m \in \mathbb{N}$, $t_1, \ldots, t_n \in I$ and $\beta_1, \ldots, \beta_n \in [0, 1]$ such that $\beta_1 + \cdots + \beta_n = 1$ and $\beta_1 t_1 + \cdots + \beta_n t_n = t$. Therefore taking the essential infimum in the first term of the right hand side of (10) and next in the second term and using the second axiom of the definition of essential infimum, we get

$$Z_2\big(\lambda s + (1 - \lambda)t, \cdot\big) \leqslant \lambda Z_2(s, \cdot) + (1 - \lambda) Z_2(t, \cdot) \quad \text{(a.e.)}.$$

This shows that Z_2 is convex.

Now, define $Z_1 : I \times \Omega \to \mathbb{R}$ by

$$Z_1(s,\cdot) = \text{ess sup}\Bigg\{\sum_{i=1}^{m}\alpha_i X(s_i,\cdot) : \quad m \in \mathbb{N}, s_1,\dots,s_m \in I, \alpha_1,\dots,\alpha_m \in [0,1] \text{ such that } \alpha_1+\dots+\alpha_m = 1 \text{ and } s = \alpha_1 s_1 + \dots + \alpha_m s_m\Bigg\}.$$

Similarly as above, we can prove that Z_1 is concave and

$$X(s,\cdot) \leqslant Z_1(s,\cdot) \leqslant Y(s,\cdot) \quad \text{(a.e.)},$$

for every $s \in I$.

Finally, using (9) once more and taking the essential infimum of the term on the right hand side and next the essential supremum of the term on the left hand side, we get

$$Z_1(t,\cdot) \leqslant Z_2(t,\cdot) \quad \text{(a.e.)}$$

for every $t \in I$.

Implication $(ii) \Rightarrow (iii)$ follows by Theorem 3 applied for $X = Z_1$ and $Y = Z_2$.

To prove $(iii) \Rightarrow (i)$ fix $s_1,\dots,s_m,t_1,\dots,t_n \in I$ and $\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n \geqslant 0$ such that $\alpha_1+\dots+\alpha_m = \beta_1+\dots+\beta_n$ and $\alpha_1 s_1+\dots+\alpha_m s_m = \beta_1 t_1+\dots+\beta_n t_n$. By the definition of Z, we get

$$\sum_{i=1}^{m}\alpha_i X(s_i,\cdot) \leqslant \sum_{i=1}^{m}\alpha_i Z(s_i,\cdot) = Z\Big(\sum_{i=1}^{m}\alpha_i s_i,\cdot\Big) = = Z\Big(\sum_{j=1}^{n}\beta_j t_j,\cdot\Big) = \sum_{j=1}^{n}\beta_j Z(t_j,\cdot) \leqslant \sum_{j=1}^{n}\beta_j Y(t_j,\cdot) \quad \text{(a.e.)}.$$

This finishes the proof. □

4. Hyers–Ulam-type stability result

As an immediate consequence of Theorem 4 we obtain the following Hyers–Ulam-type stability results for affine stochastic processes.

Let ε be a positive constant. We say that a stochastic process $X : I\times\Omega \to \mathbb{R}$ is ε-*affine* if

$$\Big|X\Big(\sum_{i=1}^{n}\beta_i t_i,\cdot\Big) - \sum_{i=1}^{n}\beta_i X(t_i,\cdot)\Big| \leqslant \varepsilon \quad \text{(a.e.)} \tag{11}$$

for all $n \in \mathbb{N}$, $t_1,\dots,t_n \in I$ and $\beta_1,\dots,\beta_n \geqslant 0$ with $\beta_1+\dots+\beta_n = 1$.

Theorem 5. *If a stochastic process $X : I\times\Omega \to \mathbb{R}$ is ε-affine, then there exists an affine stochastic process Z such that*

$$|X(t,\cdot) - Z(t,\cdot)| \leqslant \varepsilon \quad \textit{(a.e.)} \tag{12}$$

for all $t \in I$.

Proof. Let us fix $n, m \in \mathbb{N}$ and $u = \alpha_1 s_1 + \cdots + \alpha_m s_m = \beta_1 t_1 + \cdots + \beta_n t_n$, where $s_1, \ldots, s_m, t_1, \ldots, t_n \in I$, $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in [0,1]$ and $\alpha_1 + \cdots + \alpha_m = 1$, $\beta_1 + \cdots + \beta_n = 1$. By (11) for $u = \alpha_1 s_1 + \cdots + \alpha_m s_m$ and $u = \beta_1 t_1 + \cdots + \beta_n t_n$ we get

$$\sum_{i=1}^{m} \alpha_i X(s_i,\cdot) - \varepsilon \leqslant X\big(u,\cdot\big) \leqslant \sum_{i=1}^{m} \alpha_i X(s_i,\cdot) + \varepsilon \quad \text{(a.e.)}$$

$$\sum_{j=1}^{n} \beta_j X(t_j,\cdot) - \varepsilon \leqslant X\big(u,\cdot\big) \leqslant \sum_{j=1}^{n} \beta_j X(t_j,\cdot) + \varepsilon \quad \text{(a.e.)}.$$

Hence

$$\sum_{i=1}^{m} \alpha_i X(s_i,\cdot) \leqslant \sum_{j=1}^{n} \beta_j X(t_j,\cdot) + 2\varepsilon \quad \text{(a.e.)}. \tag{13}$$

Define $Y(t,\cdot) = X(t,\cdot) + 2\varepsilon$ for every $t \in I$. In view of (13) the processes X and Y satisfy (9). Therefore, by Theorem 4, there exists an affine process $Z_1 : I \times \Omega \to \mathbb{R}$, such that $X(t,\cdot) \leqslant Z_1(t,\cdot) \leqslant X(t,\cdot) + 2\varepsilon$ for all $t \in I$. Putting $Z(t,\cdot) = Z_1(t,\cdot) - \varepsilon$ we get (12). This completes the proof. □

In the same way, applying Theorem 2 instead of Theorem 4, we can prove the stability result for Jensen stochastic processes. We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is ε-*Jensen* if

$$\Big| X\Big(\sum_{i=1}^{n} q_i t_i,\cdot\Big) - \sum_{i=1}^{n} q_i X(t_i,\cdot)\Big| \leqslant \varepsilon \quad \text{(a.e.)}$$

for all $n \in \mathbb{N}$, $t_1, \ldots, t_n \in I$ and $q_1, \ldots, q_n \in [0,1] \cap D$ with $q_1 + \cdots + q_n = 1$.

Theorem 6. *If a stochastic process* $X : I \times \Omega \to \mathbb{R}$ *is* ε*-Jensen, then there exists a Jensen stochastic process* Z *such that*

$$|X(t,\cdot) - Z(t,\cdot)| \leqslant \varepsilon \quad \textit{(a.e.)}$$

for all $t \in I$.

The stability problem for Jensen stochastic processes defined on the whole $\mathbb{R} \times \Omega$ (instead of $I \times \Omega$) was earlier investigated in [13]. Using the stability of additive stochastic processes (a counterpart of the classical Hyers theorem), the following result is proved there.

Theorem 7. *If a stochastic process* $X : \mathbb{R} \times \Omega \to \mathbb{R}$ *satisfies*

$$\Big| X\Big(\frac{s+t}{2},\cdot\Big) - \frac{X(s,\cdot) + X(t,\cdot)}{2}\Big| \leqslant \varepsilon \quad \textit{(a.e.)} \tag{14}$$

for all $s, t \in \mathbb{R}$*, then there exists a Jensen stochastic process* Z *such that*

$$|X(t,\cdot) - Z(t,\cdot)| \leqslant 4\varepsilon \quad \textit{(a.e.)}$$

for all $t \in \mathbb{R}$*.*

Proof. Define $X_1(t,\cdot) = X(t,\cdot) - X(0,\cdot), \quad t \in \mathbb{R}$. By (14)

$$\begin{aligned} |X_1(s+t,\cdot) - X_1(s,\cdot) - X_1(t,\cdot)| &= |X(s+t,\cdot) - X(s,\cdot) - X(t,\cdot) + X(0.\cdot)| \\ &\leqslant 2\Big|\frac{X(s+t,\cdot) + X(0,\cdot)}{2} - X\Big(\frac{s+t}{2},\cdot\Big)\Big| \\ &\quad +2\Big|X\Big(\frac{s+t}{2},\cdot\Big) - \frac{X(s,\cdot)+X(t,\cdot)}{2}\Big| \leqslant 4\varepsilon \quad \text{(a.e.)}, \end{aligned}$$

which means that X_1 is ε-additive. Therefore, because of the stability of additive stochastic processes (see [13]), there exists an additive process $Z_1 : \mathbb{R} \times \Omega \to \mathbb{R}$ such that

$$|X_1(t,\cdot) - Z_1(t,\cdot)| \leqslant 4\varepsilon \quad \text{(a.e.)}$$

for all $t \in \mathbb{R}$. Define $Z(t,\cdot) = Z_1(t,\cdot) + X(0,\cdot)$, $t \in \mathbb{R}$. Then Z is a Jensen stochastic process and

$$|X(t,\cdot) - Z(t,\cdot)| = |X_1(t,\cdot) - Z_1(t,\cdot)| \leqslant 4\varepsilon \quad \text{(a.e.)}$$

for all $t \in \mathbb{R}$. This finishes the proof. □

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Dawid Kotrys and Kazimierz Nikodem
Department of Mathematics
University of Bielsko-Biala
ul. Willowa 2
43-309 Bielsko-Biała
Poland
e-mail: dkotrys@gmail.com

Kazimierz Nikodem
e-mail: knikodem@ath.bielsko.pl

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