Geometry on the lines of spine spaces

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Abstract. Spine spaces can be considered as fragments of a projective Grassmann space. We prove that the structure of lines together with a binary coplanarity relation, as well as with the binary relation of being in one pencil of lines, is a sufficient system of primitive notions for these geometries. It is also shown that, over a spine space, the geometry of pencils of lines can be reconstructed in terms of the two binary relations.

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Introduction

It was proved in [8] that the structure of lines together with a binary coplanarity relation π is a sufficient system of primitive notions for projective and polar Grassmann spaces. Following the ideas, motivations, and objectives of that paper we use similar methods to prove the same for spine spaces. This is also the reason why our terminology and notation is heavily based on [8]. In consequence many definitions and concepts may sound familiar for the reader.

A relation close in its nature to the relation π is the relation ρ of being in one pencil of lines. This closeness is revealed simply by the fact that $\rho \subseteq \pi$ and causes that some of the reasonings are common for π and ρ . There are differences though, mainly at the technical level at the beginning up to the point where bundles of lines are constructed. From that point onwards the reasoning is unified and we get that π as well as ρ is a sufficient primitive notion for the geometry of a spine space $\mathfrak{M}$. Generally, as in the case of Grassmann spaces, the key role is played by maximal cliques of π and ρ , as well as maximal strong subspaces containing them. The structure of strong subspaces in spine spaces is much more complex than in Grassmann spaces, but pretty well known if we take a look into [11–14]. The major difference is that we have to deal

with three types of lines and four types of strong subspaces as stars and tops can be projective or semiaffine (in particular affine) spaces.

The two relations π and ρ are mainly used to reconstruct the pointset of a spine space $\mathfrak{M}$, but they also appear to be sufficient primitive notions for the geometry of pencils of lines over $\mathfrak{M}$. Both of them are trivial on planes so, it is natural to require that all maximal strong subspaces of $\mathfrak{M}$ are at least 3-dimensional. This is all we need to express the geometry of pencils in terms of π and ρ .

Our approach to the reconstruction of $\mathfrak{M}$, based on bundles of its lines, requires a bit different assumption though, namely that stars or tops are at least 4-dimensional. In some cases excluded by this assumption, which we point out in the last section, the geometry $\mathfrak{M}$ can be recovered. The most interesting case is the neighbourhood of a point in the underlying Grassmann space. It is impossible to recover this spine space from the structure of lines equipped with either π or ρ , nonetheless the geometry of its lines and pencils can be reconstructed. This shows that the geometry of lines and pencils over $\mathfrak{M}$ is intrinsically weaker than the geometry of $\mathfrak{M}$.

In the vein of Chow's Theorem one would want to continue the procedure of creating pencils over pencils. Sadly, the geometry of pencils of planes, obtained in the first step, turns out to be disconnected which makes the whole idea pointless.

The "Appendix" at the end of this paper fixes a gap in the proof of Lemma 1.3 in [8].

1. Generalities

A point-line structure $\mathfrak{A} = \langle S, \mathcal{L} \rangle$, where the elements of S are called *points*, the elements of $\mathcal{L}$ are called *lines*, and where $\mathcal{L} \subset 2^S$, is said to be a *partial linear space*, or a *point-line space*, if two distinct lines share at most one point and every line is of size (cardinality) at least 2 (cf. [3]).

A *subspace* of $\mathfrak{A}$ is any set $X \subseteq S$ with the property that every line which shares two or more points with X is entirely contained in X . We say that a subspace X of $\mathfrak{A}$ is *strong* if any two points in X are collinear. A *plane* in $\mathfrak{A}$ is a strong subspace E of $\mathfrak{A}$ with the property that the restriction of $\mathfrak{A}$ to E is a projective plane. If S is strong, then $\mathfrak{A}$ is said to be a *linear space*.

Let us fix a nonempty subset $\mathcal{H} \subset S$ and consider two sets

$$S_{\mathcal{H}} := S \setminus \mathcal{H} \quad \text{and} \quad \mathcal{L}_{\mathcal{H}} := \{k \cap S_{\mathcal{H}} : k \in \mathcal{L} \text{ and } |k \cap S_{\mathcal{H}}| \geq 2\}.$$

The structure

$$\mathfrak{M} := \langle S_{\mathcal{H}}, \mathcal{L}_{\mathcal{H}} \rangle$$

is a *fragment* of $\mathfrak{A}$ and it is a partial linear space itself. The incidence relation in $\mathfrak{M}$ is again $\in$, inherited from $\mathfrak{A}$, but limited to the new pointset and lineset.

Following a standard convention we call the points of $\mathfrak{M}$ *proper*, and the points in $\mathcal{H}$ *improper*. The set $\mathcal{H}$ will be called the *horizon of $\mathfrak{M}$* . To every line $L \in \mathcal{L}_{\mathcal{H}}$ we can uniquely assign the line $\bar{L} \in \mathcal{L}$, the *closure* of L , so that $L \subseteq \bar{L}$. For a subspace $X \subseteq S_{\mathcal{H}}$ the closure of X is the minimal subspace $\bar{X}$ of $\mathfrak{A}$ containing X . A line $L \in \mathcal{L}_{\mathcal{H}}$ is said to be a *projective line* if $L = \bar{L}$, and it is said to be an *affine line* if $|\bar{L} \setminus L| = 1$. In case $\mathcal{L}_{\mathcal{H}}$ contains projective or affine lines only, then $\mathfrak{M}$ is a *semiaffine geometry* (for details on terminology and axiom systems see [15, 16]). In this approach an affine space is a particular case of a semiaffine space. For affine lines $L_1, L_2 \in \mathcal{L}_{\mathcal{H}}$ we can define parallelism in a natural way:

$$L_1, L_2 \text{ are parallel iff } \bar{L}_1 \cap \bar{L}_2 \cap \mathcal{H} \neq \emptyset.$$

From now on we assume that $\mathfrak{M}$ is a semiaffine geometry, that is, $\mathcal{H}$ is a subspace of $\mathfrak{A}$. We say that E is a plane in $\mathfrak{M}$ if $\bar{E}$ is a plane in $\mathfrak{A}$. Observe that there are two types of planes in $\mathfrak{M}$: projective and semiaffine. A semiaffine plane E arises from $\bar{E}$ by removing a point or a line. In result we get a punctured plane or an affine plane, respectively. For lines $L_1, L_2 \in \mathcal{L}_{\mathcal{H}}$ we say that they are *coplanar* and write

$$L_1 \pi L_2 \quad \text{iff} \quad \text{there is a plane } E \text{ such that } L_1, L_2 \subset E \quad (1)$$

Let E be a plane in $\mathfrak{M}$ and $U \in \bar{E}$. A set

$$\mathbf{p}(U, E) := \{L \in \mathcal{L}_{\mathcal{H}} : U \in \bar{L} \subseteq \bar{E}\} \quad (2)$$

will be called a *pencil of lines* if U is a proper point, or a *parallel pencil* otherwise. The point U is said to be the *vertex* and the plane E is said to be the *base plane* of that pencil. We write

$$L_1 \rho L_2 \quad \text{iff} \quad \text{there is a pencil } p \text{ such that } L_1, L_2 \in p. \quad (3)$$

Recall that for any binary relation ρ a set is said to be ρ -*clique* iff every two elements of this set are ρ -related. If $L_1 \rho L_2$, then clearly $L_1 \pi L_2$. This means that every ρ -clique is a π -clique.

For a subspace X of $\mathfrak{M}$ we write

$$\mathbf{L}(X) = \{L \in \mathcal{L}_{\mathcal{H}} : L \subset X\}. \quad (4)$$

If E is a plane in $\mathfrak{M}$, then the set $\mathbf{L}(E)$ will be called a *flat*. The set of all projective lines on E augmented with a maximal set of affine lines on E such that no two are parallel will be called a *semiflat*. As the plane E can be projective, punctured or affine, we have projective, punctured or affine semiflat (flat), respectively. Semiflats that are not projective will be called semiaffine semiflats.

Note that a projective semiflat is a projective flat. On an affine plane E , where parallelism partitions the lineset into directions, a semiflat is a selector of $\mathcal{L}_{\mathcal{H}}/\parallel$ (cf. [9]).

For a subspace X of $\mathfrak{M}$, if $U \in \bar{X}$ we write

$$L_U(X) = \{L \in \mathcal{L}_{\mathcal{H}} : U \in \bar{L} \text{ and } L \subseteq X\}. \quad (5)$$

If X is a strong subspace of $\mathfrak{M}$, then $L_U(X)$ is said to be a *semibundle*. As the vertex U of a semibundle $L_U(X)$ can be proper or improper, we call the semibundle *proper* or *improper* accordingly. We omit the adjective when we mean a semibundle in general.

1.1. Grassmann spaces

Let V be a vector space of dimension n with $3 \leq n < \infty$. The set of all subspaces of V will be written as $\text{Sub}(V)$ and the set of all k -dimensional subspaces (or k -subspaces in short) as $\text{Sub}_k(V)$. By a *k-pencil* we mean the set of the form

$$\mathbf{p}(H, B) := \{U \in \text{Sub}_k(V) : H \subset U \subset B\},$$

where $H \in \text{Sub}_{k-1}(V)$, $B \in \text{Sub}_{k+1}(V)$, and $H \subset B$. The family of all such k -pencils will be denoted by $\mathcal{P}_k(V)$. A *Grassmann space* (also known as a *space of pencils* or a *projective Grassmannian*) is a point–line space

$$\mathbf{P}_k(V) = \langle \text{Sub}_k(V), \mathcal{P}_k(V) \rangle,$$

with k -subspaces of V as points and k -pencils as lines (see [7, 17] for a more general definition, see also [6]). For $0 < k < n$ it is a partial linear space. For $k = 1$ and $k = n - 1$ it is a projective space. So we assume that

$$1 < k < n - 1.$$

It is known that there are two classes of maximal strong subspaces in $\mathbf{P}_k(V)$: *stars* of the form

$$S(H) = [H]_k = \{U \in \text{Sub}_k(V) : H \subset U\},$$

where $H \in \text{Sub}_{k-1}(V)$, and *tops* of the form

$$T(B) = (B)_k = \{U \in \text{Sub}_k(V) : U \subset B\},$$

where $B \in \text{Sub}_{k+1}(V)$. Although non-maximal stars $[H, Y]_k$ and non-maximal tops $[Z, B]_k$, for some $Y, Z \in \text{Sub}(V)$, make sense but in this paper when we say ‘a star’ or ‘a top’ we mean a maximal strong subspace. It is trivial that every line, a k -pencil $p = \mathbf{p}(H, B)$, of $\mathfrak{M}$ can be uniquely extended to the star $S(p) := S(H)$ and to the top $T(p) := T(B)$.

TABLE 1. The classification of lines in a spine space $\mathbf{A}_{k,m}(V, W)$

Class	Representative line $g = \mathbf{p}(H, B) \cap \mathcal{F}_{k,m}(W)$	g^∞
$\mathcal{A}ffines_{k,m}(W)$	$H \in \mathcal{F}_{k-1,m}(W), B \in \mathcal{F}_{k+1,m+1}(W)$	$H + (B \cap W)$
$\mathcal{L}_{k,m}^\alpha(W)$	$H \in \mathcal{F}_{k-1,m}(W), B \in \mathcal{F}_{k+1,m}(W)$	–
$\mathcal{L}_{k,m}^\omega(W)$	$H \in \mathcal{F}_{k-1,m-1}(W), B \in \mathcal{F}_{k+1,m+1}(W)$	–

1.2. Spine spaces

A *spine space* is a fragment of a Grassmann space chosen so that it consists of subspaces of V which meet a fixed subspace in a specified way. The concept of spine spaces was introduced in [10] and developed in [11–14].

Let W be a fixed subspace of V and let m be an integer with

$$k - \text{codim}(W) \leq m \leq k, \dim(W). \quad (6)$$

From the points of the Grassmann space $\mathbf{P}_k(V)$ we take those which as subspaces of V meet W in dimension m , that is:

$$\mathcal{F}_{k,m}(W) := \{U \in \text{Sub}_k(V) : \dim(U \cap W) = m\}.$$

As new lines we take those lines of $\mathbf{P}_k(V)$ which have at least two new points:

$$\mathcal{G}_{k,m}(W) := \{L \cap \mathcal{F}_{k,m}(W) : L \in \mathcal{P}_k(V) \text{ and } |L \cap \mathcal{F}_{k,m}(W)| \geq 2\}.$$

The point–line structure:

$$\mathfrak{M} = \mathbf{A}_{k,m}(V, W) := \langle \mathcal{F}_{k,m}(W), \mathcal{G}_{k,m}(W) \rangle$$

will be called a *spine space*. This is a Gamma space. Specifically, depending on k, m and $\dim(W)$ it can be: a projective space, a slit space (cf. [4, 5]), an affine space or the *space of linear complements* (cf. [1, 14]). As $\mathfrak{M}$ is a fragment of the Grassmann space $\mathbf{P}_k(V)$ we can distinguish a set $\mathcal{H}$ of improper points in $\mathfrak{M}$, i.e. a horizon. Consequently, a line of $\mathfrak{M}$ is either affine or projective, that is, $\mathfrak{M}$ is a semiaffine space.

The class of affine lines is denoted by $\mathcal{A}$. Projective lines fall into two disjoint classes $\mathcal{L}^\alpha$ and $\mathcal{L}^\omega$. For brevity $\mathcal{L} := \mathcal{A} \cup \mathcal{L}^\alpha \cup \mathcal{L}^\omega$. Details can be found in Table 1.

The geometry of a spine space is complex in that there is an overwhelming variety of types of subspaces. As usual most important are strong subspaces. A star from $\mathbf{P}_k(V)$ restricted to $\mathfrak{M}$ either contains affine lines or not. In the first case it is called an α -star which is a semiaffine space, in the other case it is called an ω -star which is a projective space. A top from $\mathbf{P}_k(V)$ restricted to $\mathfrak{M}$ also contains affine lines or not and is called an ω -top or an α -top, respectively. On the other hand, each strong subspace X of a spine space is a slit space, that is a projective space $\mathbf{P}$ with a subspace $\mathcal{D}$ removed. The form of a maximal

TABLE 2. The classification of stars and tops in a spine space $\mathbf{A}_{k,m}(V, W)$

Class	Representative subspace	$\dim(\mathbf{P})$	$\dim(\mathcal{D})$
ω -stars	$[H, H + W]_k : H \in \mathcal{F}_{k-1, m-1}(W)$	$\dim(W) - m$	-1
α -stars	$[H, V]_k \cap \mathcal{F}_{k,m}(W) : H \in \mathcal{F}_{k-1, m}(W)$	$\dim(V) - k$	$\dim(W) - m - 1$
α -tops	$[B \cap W, B]_k : B \in \mathcal{F}_{k+1, m}(W)$	$k - m$	-1
ω -tops	$[\Theta, B]_k \cap \mathcal{F}_{k,m}(W) : B \in \mathcal{F}_{k+1, m+1}(W)$	k	$k - m - 1$

strong subspace from each class and the dimension of the corresponding spaces $\mathbf{P}$ and $\mathcal{D}$ are presented in Table 2. In the extremes $\mathcal{D}$ can be void, then X is basically a projective space, or a hyperplane, then X is an affine space. One should be aware that for specific values of $\dim(V)$, $\dim(W)$, k , and m some classes of maximal strong subspaces are void.

A line of $\mathfrak{M}$ can be in at most two maximal strong subspaces of different types: a star and a top.

Fact 1.1. *A projective star and a projective top are either disjoint or they share a point. In the remaining cases, a star and a top are either disjoint or they share a line. Two stars (or two tops) are either disjoint or they share a point.*

Lemma 1.2. *Three pairwise coplanar and concurrent, or parallel, lines not all on a plane span a star or a top.*

Proof. There are three planes and it suffices to note by 1.1 that no two of them can be of distinct types, i.e. they all are of type star or top. Consequently, all these lines lie in one maximal strong subspace. $\square$

2. Maximal cliques

Let $\mathfrak{M}$ be a spine space. The goal now is to show that the set of lines equipped with either the coplanarity relation π or relation ρ of being in one pencil, is a sufficient system of primitive notions for $\mathfrak{M}$. The key tool to achieve that are maximal cliques.

Lemma 2.1. (i) *Flats and semibundles are π -cliques.*

(ii) *Semiflats and proper semibundles are ρ -cliques.*

Proof. (i) It is clear that flats are π -cliques. Let X be a strong subspace of $\mathfrak{M}$ and $U \in \overline{X}$. Note that X is, up to isomorphism, a slit space. Therefore, any two lines in the semibundle $L_U(X)$ are coplanar.

(ii) Let K be a semiflat. If K is a projective semiflat, then every two lines of K are concurrent. In case K is a punctured semiflat, there is a single affine line in K which intersects all the other lines in K . If K is an

affine semiflat, then all the lines in K are affine and any two of them are concurrent. This justifies that K is a ρ -clique. The fact that proper semibundles are ρ -cliques is evident. $\square$

Proposition 2.2. *Every maximal π -clique is either a flat or a semibundle.*

Proof. Let K be a maximal π -clique which is not a flat. So, there are three pairwise distinct lines in K not all on a plane. They all meet in a point U , possibly improper. Moreover, for any $L \in K$ we have $U \in \bar{L}$.

Let U_1, U_2 be distinct points in $\bigcup K$. There are lines $M_1, M_2 \in K$ such that $U_1 \in M_1, U_2 \in M_2$. As $M_1 \pi M_2$ the points U_1, U_2 are collinear. This means that $\bigcup K$ is a collinearity clique.

Now, let $\bar{\mathfrak{M}}$ be the Grassmann space embracing $\mathfrak{M}$, and let $\bar{X}$ be a maximal collinearity clique in $\bar{\mathfrak{M}}$ containing $\bigcup K$. Set $X := \bar{X} \cap \mathcal{F}_{k,m}(W)$. It is clear that $\bigcup K \subseteq X$. As $\bar{\mathfrak{M}}$ is a Gamma space $\bar{X}$ is a maximal strong subspace of $\bar{\mathfrak{M}}$. Therefore, X is a maximal strong subspace of $\mathfrak{M}$. Note that $U \in \bar{X}$. Take a point $W \in X$ distinct from U . There is a, possibly affine, line $M = \overline{U, W}$ contained in $\bar{X}$. Since $\bar{X}$ is, up to isomorphism, a projective space, all the lines through U in $\bar{X}$ are pairwise coplanar, so $M \in K$. Hence $W \in \bigcup K$ and consequently $\bigcup K = X$. We have actually shown that $K = L_U(X)$ which means that K is a semibundle. $\square$

Proposition 2.3. *Every maximal ρ -clique is either a semiflat or a proper semibundle.*

Proof. Let K be a maximal ρ -clique which is not a semiflat. Every ρ -clique is a π -clique, so K is a π -clique though not necessarily maximal. Let K' be a maximal π -clique containing K . By 2.2 $K' = L_U(X)$ for some maximal strong subspace X and a proper point U . As X is a projective space and $K \subseteq K'$ we get $K = K'$ and the claim follows. $\square$

Lemma 2.4. *A maximal ρ -clique K satisfies the following condition:*

$$\begin{aligned} &\text{there are lines } L_1 \in K, L_2 \in \mathcal{L} \setminus K \text{ such that} \\ &(K \setminus \{L_1\}) \cup \{L_2\} \text{ is a maximal } \rho\text{-clique} \end{aligned} \quad (7)$$

iff K is a semiaffine semiflat.

Proof. $\Rightarrow$: Let K be a maximal ρ -clique that satisfies (7) and is not a semiaffine semiflat. By 2.3, K is either a projective flat or a proper semibundle. In both cases the unique line that complements the set $K \setminus \{L_1\}$ to a maximal ρ -clique is L_1 . This contradicts (7).

$\Leftarrow$: Let E be a semiaffine plane and K be a semiflat on E . Take two affine lines: $L_1 \in K$ and $L_2 \subseteq E$ such that $L_2 \parallel L_1, L_2 \neq L_1$. Note that $L_2 \notin K$. Then the set of affine lines in K with L_1 replaced by L_2 is a maximal set of affine lines in E such that no two are parallel. Consequently, $(K \setminus \{L_1\}) \cup \{L_2\}$ is a maximal ρ -clique. $\square$

The criterion (7) from 2.4 could be used to distinguish semiaffine semiflats in the family of all ρ -cliques.

Let $\delta \in \{\pi, \rho\}$. As indispensable as the property of the family of maximal δ -cliques provided by 2.2 and 2.3 is the characterization of this family in terms of lines and the relation δ , that is an elementary definition of maximal δ -cliques within $\langle \mathcal{L}, \delta \rangle$. For lines $L_1, L_2, \dots, L_n \in \mathcal{L}$ we define

$$\begin{aligned} \Delta_n^\delta(L_1, L_2, \dots, L_n) \quad &\text{iff} \\ &\neq (L_1, L_2, \dots, L_n) \quad \text{and} \quad L_i \delta L_j \quad \text{for all } i, j = 1, \dots, n, \\ &\text{and for all } M_1, M_2 \in \mathcal{L} \quad \text{if} \quad M_1, M_2 \delta L_1, L_2, \dots, L_n \text{ then } M_1 \delta M_2. \end{aligned} \quad (8)$$

Note that the relation Δ_2^δ is empty. In spine spaces 3 lines satisfying Δ_3^δ determine a δ -clique. From this point of view, Δ_n^δ could be defined as a ternary relation in (8). The reason why we introduce such a general formula will be explained later in “Appendix”.

As the relation δ , and in consequence Δ_3^δ , makes little sense on a plane we assume that every plane in $\mathfrak{M}$ is contained in a star or top of dimension at least 3. In view of Table 2 it reads as follows:

$$3 \leq n - k \quad \text{and} \quad 3 \leq k - m. \quad (9)$$

Lemma 2.5. *Let $L_1, L_2, L_3 \in \mathcal{L}$.*

- (i) $\Delta_3^\pi(L_1, L_2, L_3)$ iff $\overline{L_1}, \overline{L_2}, \overline{L_3}$ form a tripod or a triangle.
- (ii) $\Delta_3^\rho(L_1, L_2, L_3)$ iff L_1, L_2, L_3 form a ρ -clique, they are not in a pencil of lines, they are not on an affine plane, and in case they are on a punctured plane, one of L_1, L_2, L_3 is an affine line.

Proof. (i) Observe that L_1, L_2, L_3 are pairwise coplanar iff $\overline{L_1}, \overline{L_2}, \overline{L_3}$ form a tripod, a triangle or a pencil of lines. If the latter, a line M_1 through the vertex of the pencil but not on its base plane is not coplanar with a line M_2 that lies on the base plane but misses the vertex.

- (ii) A pencil of lines is singled out taking the same lines M_1, M_2 as in (i). On an affine plane, as well as on a punctured plane if none of L_1, L_2, L_3 goes through the point at infinity, the lines M_1, M_2 from (8) can be parallel. $\square$

Provided that $\Delta_3^\delta(L_1, L_2, L_3)$, the maximal δ -clique spanned by the lines L_1, L_2, L_3 is the set

$$[L_1, L_2, L_3]_\delta := \{L \in \mathcal{L} : L \delta L_1, L_2, L_3\}. \quad (10)$$

Note that all maximal δ -cliques except affine semiflats can be spanned in this way. Now, let us define

$$\mathcal{K}_\delta = \{[L_1, L_2, L_3]_\delta : L_1, L_2, L_3 \in \mathcal{L} \text{ and } \Delta_3^\delta(L_1, L_2, L_3)\}. \quad (11)$$

Lemmas 2.2 and 2.3 explain which maximal δ -cliques fall into $\mathcal{K}_\delta$.

- Lemma 2.6.** (i) $\mathcal{K}_\pi$ is the family of all maximal π -cliques, i.e. flats and semi-bundles.
(ii) $\mathcal{K}_\rho$ is the family of all projective flats, punctured semiflats and proper semibundles.

In consequence we get the following.

Proposition 2.7. The family of maximal δ -cliques is definable in $\langle \mathcal{L}, \delta \rangle$.

3. Proper semibundles

Let us recall that our goal is to reconstruct the point universe of a spine space given a line universe equipped with the relation δ . The idea is to use vertices of semibundles to do that. This means that only proper semibundles are of concern here. The problem is we need to distinguish them in the family of all δ -cliques, which we are going to do now using pencils of lines as an essential tool.

3.1. Pencils of lines

The geometry induced by pencils of lines is interesting in itself and, we believe, it is worth giving it a little more attention here. For three lines $L_1, L_2, L_3 \in \mathcal{L}$ we give the following two definitions:

$$\mathbf{p}_\pi(L_1, L_2, L_3) \quad \text{iff} \quad L_i \pi L_j \text{ for all } i, j = 1, 2, 3 \text{ and } \neg \Delta_3^\pi(L_1, L_2, L_3), \quad (12)$$

$$\mathbf{p}_\rho(L_1, L_2, L_3) \quad \text{iff there are } M_1, M_2, M_3 \in \mathcal{L} \text{ such that}$$

$$\Delta_3^\rho(M_1, M_2, M_3) \quad \text{and} \quad [M_1, M_2, M_3]_\rho \text{ does not satisfy (7)}$$

$$\text{and } L_1, L_2, L_3 \in [M_1, M_2, M_3]_\rho \quad \text{and} \quad \neg \Delta_3^\rho(L_1, L_2, L_3). \quad (13)$$

Lemma 3.1. Let $L_1, L_2, L_3 \in \mathcal{L}$.

- (i) $\mathbf{p}_\pi(L_1, L_2, L_3)$ iff L_1, L_2, L_3 form a pencil of lines or a parallel pencil.
(ii) $\mathbf{p}_\rho(L_1, L_2, L_3)$ iff L_1, L_2, L_3 form a pencil of lines.

Proof. (i): Immediate by 2.5(i).

(ii) $\Rightarrow$: For M_1, M_2, M_3 in (13) let $K := [M_1, M_2, M_3]_\rho$. Note that $K \in \mathcal{K}_\rho$, but in view of 2.4, it is not a punctured semiflat. So, by 2.6 K is a projective flat or a proper semibundle that is a maximal ρ -clique. If $L_1, L_2, L_3 \in K$ and $\neg \Delta_3^\rho(L_1, L_2, L_3)$, then by 2.5(ii) the lines L_1, L_2, L_3 form a pencil of lines.

$\Leftarrow$: Assume that L_1, L_2, L_3 form a pencil of lines, U is its vertex and E is its base plane. If E is projective, then as M_1, M_2, M_3 any triangle on E can be taken. If E is semiaffine, then by (9) the plane E is extendable to some star or top X of dimension at least 3 and as M_1, M_2, M_3 a tripod in $L_U(X)$ should be taken. $\square$

Pencils can be defined in a standard way using ternary concurrency provided by (12) or (13) so, we claim that:

- Lemma 3.2.** (i) *The family $\mathcal{P}_\pi$ of all pencils of lines and parallel pencils is definable in $\langle \mathcal{L}, \pi \rangle$.*
(ii) *The family $\mathcal{P}_\rho$ of all pencils of lines is definable in $\langle \mathcal{L}, \rho \rangle$.*

3.2. Parallel pencils

As proper semibundles contain no parallel pencils we need to get rid of them in $\mathcal{P}_\pi$. It is a bit tricky however to express that the vertex of a pencil is improper in terms of $\langle \mathcal{L}, \mathcal{P}_\pi \rangle$. Affine planes and punctured planes, base planes of pencils in $\mathcal{P}_\pi$, have to be treated separately.

Two pencils $p_1, p_2 \in \mathcal{P}_\pi$ are *coplanar* iff every two lines $l_1 \in p_1, l_2 \in p_2$ are coplanar. That is: let $p_1, p_2 \in \mathcal{P}_\pi$

$$p_1 \amalg p_2 \iff \text{for all } l_1 \in p_1, l_2 \in p_2 \text{ we have } l_1 \pi l_2. \quad (14)$$

On an affine plane parallel pencils are the pencils that contain the line at infinity. So, formally, a pencil $p_1 \in \mathcal{P}_\pi$ is a parallel pencil if there is another pencil $p_2 \in \mathcal{P}_\pi$ such that $p_1 \amalg p_2$ and $p_1 \cap p_2 = \emptyset$. Hence, a plane in $\langle \mathcal{L}, \mathcal{P}_\pi \rangle$ is affine iff it contains two disjoint pencils. A pencil $p \in \mathcal{P}_\pi$ lies on an affine plane iff there are two distinct pencils $p_1, p_2 \in \mathcal{P}_\pi$ such that $p_1 \amalg p, p_2 \amalg p$ and $p_1 \cap p_2 = \emptyset$. We say that a line l lies on an affine plane iff there is a pencil $p \in \mathcal{P}_\pi$ such that $l \in p$ and p lies on an affine plane.

If the base plane of a pencil $p \in \mathcal{P}_\pi$ is not affine but every line $l \in p$ lies on some affine plane, then the vertex of p is an improper point. If that is the case the pencil p is a parallel pencil and its base plane is a punctured plane.

Finally, we have proved that:

Lemma 3.3. *The family $\mathcal{P}_\parallel$ of all parallel pencils is definable in $\langle \mathcal{L}, \pi \rangle$.*

Let $\mathcal{P}$ be the family of all pencils of lines in $\mathfrak{M}$. By 3.2(i) and 3.3 the family $\mathcal{P} = \mathcal{P}_\pi \setminus \mathcal{P}_\parallel$ is definable for π . By 3.2(ii) the family $\mathcal{P} = \mathcal{P}_\rho$ is definable for ρ . Note that two pencils from $\mathcal{P}$ are either disjoint or share a line. This means that $\langle \mathcal{L}, \mathcal{P} \rangle$ is a partial linear space. Interestingly enough, we are able to reconstruct this geometry, induced by pencils of lines on $\mathfrak{M}$, using nothing but its lines and the relation δ .

Proposition 3.4. *If $\mathfrak{M}$ satisfies (9), then $\langle \mathcal{L}, \mathcal{P} \rangle$ is definable in $\langle \mathcal{L}, \delta \rangle$.*

3.3. Geometry induced by pencils of lines

A π -clique is proper if it contains no parallel pencils while all ρ -cliques are proper. The family of all proper maximal δ -cliques consists of projective flats,

punctured semiflats and proper semibundles. Every proper maximal δ -clique together with pencils of lines it contains carries some geometry. A projective flat determines a projective plane, a punctured semiflat determines a projective plane with all but one points on some line removed, and a proper semibundle determines a projective space. The geometrical dimension of a proper flat is always 2 whereas a proper semibundle $L_U(X)$ has dimension one less than the dimension of X . This lets us distinguish proper flats from proper semibundles if we assume that

stars or tops in $\mathfrak{M}$ are at least 4-dimensional projective or semiaffine spaces. (15)

Let us write $\mathcal{P}_0$ for the family of all pencils of lines definable in $\langle \mathcal{L}, \delta \rangle$ and set

$$\mathcal{K}_\delta^0 := \{K \in \mathcal{K}_\delta : \text{there is } q \in \mathcal{P}_0 \text{ such that } q \subset K\}. \quad (16)$$

In case $\mathfrak{M}$ satisfies (9) we have $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{K}_\delta^0 = \mathcal{K}_\delta$. Otherwise, $\mathcal{P}_0$ contains only those pencils of lines and $\mathcal{K}_\delta^0$ only those maximal δ -cliques which, according to (15), lie in 4-dimensional stars or tops. Any way, for a δ -clique $K \in \mathcal{K}_\delta^0$ we can define its geometrical dimension $\dim(K)$. This allows us to give the following definition

$$\mathcal{B} := \{K \in \mathcal{K}_\delta^0 : \dim(K) \geq 3\}. \quad (17)$$

Under assumptions (15) it is the family of all proper semibundles regardless which of the two relations π or ρ we take. In the following lemma we state that more precisely.

Lemma 3.5. *The family $\mathcal{B}$ defined in $\langle \mathcal{L}, \delta \rangle$ coincides with the family of all proper top semibundles, the family of all proper star semibundles or the union of these two families depending on whether tops, stars or all of them as projective or semiaffine spaces are at least 4-dimensional.*

4. Bundles

On the family $\mathcal{B}$ of proper semibundles we define

$$\Upsilon(K_1, K_2) \iff (\exists L_1, L_2 \in K_1)(\exists M_1, M_2 \in K_2) \\ [L_1 \neq L_2 \wedge L_1 \delta M_1 \wedge L_2 \delta M_2]. \quad (18)$$

Lemma 4.1. *Let $K_i := L_{U_i}(X_i) \in \mathcal{B}$, $i = 1, 2$. If $\Upsilon(K_1, K_2)$ and $K_1 \cap K_2 = \emptyset$, then X_1, X_2 are both stars or tops and $U_1 = U_2$.*

Proof. By (18) there are lines $L_1 \in K_1$ and $M_1 \in K_2$, which are coplanar. Note that $X_1 \neq X_2$ and $L_1 \neq M_1$ as we assume that $K_1 \cap K_2 = \emptyset$. Let E_1 be a plane containing L_1, M_1 . Then $L_1 \subseteq X_1, E_1$ and $M_1 \subseteq X_2, E_1$. In view of 1.1 it means that X_1, X_2 are both of the same type and E_1 is of a different type.

There is another pair of coplanar lines L_2, M_2 such that $L_2 \in K_1$ and $M_2 \in K_2$, since $\Upsilon(K_1, K_2)$. Let E_2 be the plane spanned by L_2, M_2 . Note that E_1, E_2 are planes of the same type. As $U_1, U_2 \in E_1, E_2$, by 1.1 we get that $E_1 = E_2$ or $U_1 = U_2$. If $E_1 = E_2$, then $L_1, L_2, M_1, M_2 \subseteq X_1, X_2$ which yields a contradiction as X_1, X_2 are of the same type. $\square$

The inverse of 4.1 is not true in general, which is manifested in the following example. Let X_1 be a semiaffine, but not affine, star. Then X_1 is an α -star (cf. [10–12]). Take a projective line $L \in K_1$. The line L is an α -line and the unique top-extension of L is an α -top Y , that is a projective space. In case X_2 is an ω -star, that is also a projective space, there is no line in $Y \cap X_2$, provided by 1.1. Therefore we cannot find a line in X_2 , which is coplanar with L . However this problem ceases to exist when we have at least two affine lines in K_1 . So X_1 cannot be a punctured projective space as then there would be only one affine line through a given proper point.

In view of Table 2 punctured projective spaces arise in a spine space as stars when $\dim(W) - m - 1 = 0$ or as tops when $k - m - 1 = 0$. Note that either all or none of the stars, and respectively, all or none of the tops, are punctured projective spaces. For this reason we assume that no star or no top is a punctured projective space, more precisely, considering (15), we assume that stars or tops in $\mathfrak{M}$ are at least 4-dimensional projective or semiaffine but not punctured projective spaces. In view of (6) and Table 2, our assumptions read as follows

$$4 \leq n - k \quad \text{and} \quad \dim(W) \neq m + 1 \quad \text{or} \quad 4 \leq k - m \quad \text{and} \quad k \neq m + 1. \quad (19)$$

Lemma 4.2. *Let $K_i := L_{U_i}(X_i) \in \mathcal{B}$, $i = 1, 2$. If X_1, X_2 are both stars or tops and $U_1 = U_2$, then $\Upsilon(K_1, K_2)$ and $K_1 \cap K_2 = \emptyset$.*

Proof. By 1.1 we get $K_1 \cap K_2 = \emptyset$. Without loss of generality assume that X_1, X_2 are stars. If X_1 is a projective space (i.e. it is an ω -star), then we take two distinct projective ω -lines $L_1, L_2 \in K_1$. Each of L_1, L_2 can be extended to the semiaffine ω -top Y_1, Y_2 , respectively. We have $U_1 \in Y_i \cap X_2$ for $i = 1, 2$, so X_2 and Y_i share a line M_i . Moreover $U_1 \in L_i, M_i$, hence $L_i \rho M_i$ and thus $L_i \pi M_i$, so $\Upsilon(K_1, K_2)$ anyway.

Assume that X_1 is a semiaffine space (i.e. it is an α -star). There are two distinct affine lines $L_1, L_2 \in K_1$ as X_1 is not a punctured projective space. As in the first case we extend L_1, L_2 to semiaffine tops and we get our claim. $\square$

We need an equivalence relation to form the bundle of all lines through a given point. For proper semibundles $K_1, K_2 \in \mathcal{B}$ we write

$$\begin{aligned} \Upsilon_\emptyset(K_1, K_2) \quad \text{iff} \quad & \Upsilon(K_1, K_2), \Upsilon(K_2, K_1), \\ & \text{and either } K_1 \cap K_2 = \emptyset \text{ or } K_1 = K_2. \end{aligned} \quad (20)$$

To give an idea what $\Upsilon_\emptyset$ stands for let us summarize 4.1 and 4.2.

Lemma 4.3. *Let $K_i := L_{U_i}(X_i) \in \mathcal{B}$. Then $\Upsilon_\emptyset(K_1, K_2)$ iff X_1, X_2 are both stars or tops and $U_1 = U_2$.*

For a proper semibundle $K \in \mathcal{B}$ we write

$$\Lambda_{\Upsilon_\emptyset}(K) := \bigcup \{K' \in \mathcal{B} : \Upsilon_\emptyset(K, K')\}. \quad (21)$$

We will show that it is the bundle of all lines through the point determined by the semibundle K . Thanks to (19) all stars or all tops, no matter if they are α or ω , are at least 4-dimensional and are not punctured. This is essential here.

Lemma 4.4. *Let U be a point and X be a maximal strong subspace. If $U \in X$, then*

$$\Lambda_{\Upsilon_\emptyset}(L_U(X)) = \{L \in \mathcal{L} : U \in L\}. \quad (22)$$

Proof. The left-to-right inclusion is immediate by 4.3. To show the right-to-left inclusion let L be a line through U . By (19) there is a maximal strong subspace Y of the same type as X which contains L and is not a punctured projective space. Then $L \in L_U(Y)$. Again by 4.3 we get $\Upsilon_\emptyset(L_U(X), L_U(Y))$ which makes the proof complete. $\square$

In fact 4.4 says that $\Lambda_{\Upsilon_\emptyset}(L_U(X))$ is the bundle of all lines through U . We can partition the lineset of $\mathfrak{M}$ by $\Upsilon_\emptyset$, so that the equivalence classes will be the points of $\mathfrak{M}$. Note that points $U_1, U_2, \dots, U_t$ are collinear iff $\Lambda_{\Upsilon_\emptyset}(L_{U_1}(X)) \cap \Lambda_{\Upsilon_\emptyset}(L_{U_2}(X)) \cap \dots \cap \Lambda_{\Upsilon_\emptyset}(L_{U_t}(X)) \neq \emptyset$. This suffices to state the following theorem.

Theorem 4.5. *Let $\mathfrak{M}$ be a spine space and let $\mathcal{L}$ be its lineset. If $\mathfrak{M}$ satisfies (19), then*

$$\begin{aligned} &\text{the spine space } \mathfrak{M} \text{ and the structure } \langle \mathcal{L}, \delta \rangle \text{ of its lines} \\ &\text{equipped with relation } \delta \in \{\pi, \rho\} \text{ are equivalent by definition.} \end{aligned} \quad (*)$$

Admittedly, our main theorem is proved but it is worth making some comments regarding geometries on lines in $\mathfrak{M}$. There are several geometries on lines that appear throughout this paper: with pencils of lines, with bundles of lines, and with the binary relation δ . Comparing 3.4 with 4.5, note that significantly weaker assumptions are required to reconstruct pencils of lines than bundles of lines in $\langle \mathcal{L}, \delta \rangle$. It is seen that recovering the geometry of bundles from the geometry of pencils is shorter and easier than recovering it from the geometry of δ what we actually did. However, our goal was to prove some variant of Chow's Theorem (cf. [2]) which says that the underlying geometry of a spine space can be defined in terms of binary relations (adjacencies) π and ρ on lines.

5. Excluded cases

The question now is what about cases excluded by the assumptions (19). The bundle method does not work in these spine spaces, however, for some of them we are able to say if $(*)$ holds or not. Let us write $w = \dim(W)$.

- $w = n$

In this case $\mathfrak{M}$ is a Grassmann space which was treated in [8] so, $(*)$ holds true.

- $w = m = k$

This is a trivial case, the entire $\mathfrak{M}$ is a single point so, $(*)$ holds true.

- $w = m = k - 1$

In this case $\mathfrak{M}$ is a star in $\mathbf{P}_k(V)$, i.e. it is a projective space. Hence, the condition $(*)$ holds true.

- $w = k + 1, m = k$

Dual to the previous case. $\mathfrak{M}$ is a top in $\mathbf{P}_k(V)$, i.e. it is a projective space. Again $(*)$ holds true.

- $w = k, m = k - 1$

This time $\mathfrak{M}$ is the neighbourhood of a point W in $\mathbf{P}_k(V)$, i.e. the set of all points that are collinear with W . Maximal strong subspaces of $\mathfrak{M}$ are punctured projective spaces arising from maximal strong subspaces of $\mathbf{P}_k(V)$ containing W . If both X_1, X_2 are stars or they both are tops, then $\overline{X_1} \cap \overline{X_2} = \{W\}$ and $X_1 \cap X_2 = \emptyset$. If X_1 is a star and X_2 is a top, then $X_1 \cap X_2$ is a line L with $W \in \overline{L}$.

Let $\mathcal{S}$ be the family of all stars and $\mathcal{T}$ be the family of all tops in $\mathfrak{M}$. Consider a star X in $\mathfrak{M}$, a homology $\varphi \neq \text{id}$ on $\overline{X}$ with the center W , and a map $f: \text{Sub}_k(V) \rightarrow \text{Sub}_k(V)$ given as follows

$$f(U) = \begin{cases} \varphi(U), & U \in X, \\ U, & U \notin X. \end{cases}$$

We will write $\mathcal{L}_{\mathcal{S}}$ for the set of all lines contained in stars from $\mathcal{S}$ and $\mathcal{L}_{\mathcal{T}}$ for the set of all lines contained in tops from $\mathcal{T}$. Let $F_{\mathcal{S}}: \mathcal{L}_{\mathcal{S}} \rightarrow \mathcal{L}_{\mathcal{S}}$ be the map induced by f and $F_{\mathcal{T}}: \mathcal{L}_{\mathcal{T}} \rightarrow \mathcal{L}_{\mathcal{T}}$ such that $F_{\mathcal{T}}(L) = L$.

Consider $F := F_{\mathcal{S}} \cup F_{\mathcal{T}}$. It is seen that F is an automorphism of $\langle \mathcal{L}, \delta \rangle$. Note that $F(L) \neq L$ iff $L \subset X$ and $W \notin \overline{L}$. Take $Y \in \mathcal{T}$, the line $L := X \cap Y$, and a point $U \in L$. We have $W \in \overline{L}$, $F(L_U(Y)) = L_U(Y)$, and $F(L_U(X)) = L_{U'}(X)$ for some $U' \neq U$, $U' \in L$. This means that F does not preserve bundles of lines, so F is not a collineation of $\mathfrak{M}$. Thus, $(*)$ cannot be proved in this case.

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Appendix. Erratum to: Coplanarity of lines in projective and polar Grassmann spaces [8]

There is a gap in the proof of Lemma 1.3 in [8], where projective and polar Grassmann spaces are considered in view of the coplanarity relation π . We assume implicitly in this lemma that every π -clique is spanned by 3 lines, which turns out to be false in polar Grassmann spaces. Let us show that by the following example.

Example A. 1. Let $\mathfrak{M}$ be a polar Grassmann space. Two star semibundles in $\mathfrak{M}$ can have a lot in common (cf. [8]). Assume that all stars in $\mathfrak{M}$ as projective spaces have dimension $d > 3$ and all tops in $\mathfrak{M}$ have dimension d' . Without loss of generality we can assume that $d' \leq d$. Let us take t lines $L_1, L_2, \dots, L_t$, which are pairwise coplanar, but not all lie on a plane, and $t < d$. Assume that these lines are contained in a star S . There is another star $S' \neq S$ in $\mathfrak{M}$ which contains all the lines $L_1, L_2, \dots, L_t$. We can find two lines M_1, M_2 such that $M_1 \subseteq S$, $M_2 \subseteq S'$, $M_1, M_2 \not\subseteq S \cap S'$, and $M_1, M_2 \pi L_1, L_2, \dots, L_t$ but $M_1 \not\pi M_2$. It means that t lines are not enough to span a π -clique in $\mathfrak{M}$, and d lines are required.

In the current paper we also deal with the coplanarity relation, but in the context of spine spaces. However, Proposition 2.2 here and Lemma 1.3 in [8] have the same meaning. Note that, the proof of Proposition 2.2 is universal in that it refers to the ambient Grassmann space. As polar Grassmann spaces are fragments of Grassmann spaces, this proof is also a proof of Lemma 1.3 in [8]. This way we eliminate the problematic assumption and fill the gap.

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