On a modification of some conditional Cauchy equation

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Abstract. We show connections between generalized versions of some conditional Cauchy equation and its basic form. As a consequence we obtain the solution of the generalized equations in some classes of regular functions.

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The main subject of this paper is the functional equation

$$f(x + f(y)) = g(f^k(x)) + h(f^l(y)) \quad (1)$$

and its special case

$$f(x + f(y)) = f^k(x) + f^l(y) \quad (2)$$

where k and l are fixed natural numbers. In some particular cases Eq. (2) takes the form of well known equations. For instance, if $k = 1$ and $l = 2$, then Eq. (2) becomes the following conditional Cauchy equation

$$f(x + f(y)) = f(x) + f^2(y) \quad (3)$$

and for $k = l = 1$ Eq. (2) has the form

$$f(x + f(y)) = f(x) + f(y). \quad (4)$$

The general solution of Eq. (3), as well as (4), in the class of self-mappings of an abelian group was given by J. Dhombres in [2]. We present some properties of Eqs. (1) and (2); in particular, we show a connection between (1) and (3). This lets us solve Eqs. (1) and (2) in some classes of regular functions.

1. General case

In this section we work with Eq. (1). Theorem 1 below gives basic properties of this equation.

Theorem 1. *Let $(S, +)$ be an abelian cancellative semigroup with a zero and let $f, g, h: S \rightarrow S$. The triple (f, g, h) satisfies equation (1) if, and only if there exist $x_0, y_0 \in S$ such that for every $x, y \in S$ the following conditions hold:*

$$f(x + f(y)) + f^2(0) = f(x + f(0)) + f^2(y) \quad (5)$$

$$f(x + f(0)) = g(f^k(x)) + x_0 \quad (6)$$

$$f^2(x) = h(f^l(x)) + y_0 \quad (7)$$

$$x_0 + y_0 = f^2(0). \quad (8)$$

Proof. Assume that the triple (f, g, h) satisfies Eq. (1). Applying (1) with $y = 0$ we get

$$f(x + f(0)) = g(f^k(x)) + h(f^l(0)) \quad (9)$$

for every $x \in S$. In particular,

$$f^2(0) = g(f^k(0)) + h(f^l(0)). \quad (10)$$

Using (1) with $x = 0$ we obtain

$$f^2(y) = g(f^k(0)) + h(f^l(y)) \quad (11)$$

for every $y \in S$. Put $x_0 = h(f^l(0))$ and $y_0 = g(f^k(0))$. It follows from conditions (9)–(11) that equalities (6)–(8) are satisfied. Using already proved conditions and equality (1) we get

$$f(x + f(0)) + f^2(y) = g(f^k(x)) + x_0 + y_0 + h(f^l(y)) = f(x + f(y)) + f^2(0)$$

for every $x, y \in S$. This proves (5).

Conversely, assume that there exist $x_0, y_0 \in S$ such that conditions (5)–(8) are satisfied for every $x, y \in S$. Then

$$\begin{aligned} f(x + f(y)) + f^2(0) &= f(x + f(0)) + f^2(y) = g(f^k(x)) + x_0 + f^2(y) \\ &= g(f^k(x)) + x_0 + y_0 + h(f^l(y)) \\ &= g(f^k(x)) + h(f^l(y)) + f^2(0) \end{aligned}$$

for every $x, y \in S$. This in view of the cancellativity of S means that the triple (f, g, h) satisfies Eq. (1). □

Corollary 1. *Let $(G, +)$ be an abelian group and let $f, g, h: G \rightarrow G$. If the triple (f, g, h) satisfies Eq. (1), then the function $-f + f(0)$ is a solution of equation (3).*

Proof. Assume that the triple (f, g, h) satisfies Eq. (1). Applying Theorem 1 we deduce that f satisfies Eq. (5). Inserting $x - f(y)$ in place of x in (5) we get

$$f(x) + f^2(0) = f(x - f(y) + f(0)) + f^2(y) \quad (12)$$

for every $x, y \in G$. In particular,

$$f(-f(y) + f(0)) + f^2(y) = f(0) + f^2(0) \quad (13)$$

for every $y \in G$. Using (12) and (13) we obtain

$$f(x - f(y) + f(0)) = f(x) + f^2(0) - f^2(y) = f(x) + f(-f(y) + f(0)) - f(0)$$

for every $x, y \in G$. Hence

$$-f(x - f(y) + f(0)) = -f(x) + f(0) - f(-f(y) + f(0))$$

for every $x, y \in G$. Adding $f(0)$ to both sides of the last equality we see that the function $-f + f(0)$ satisfies Eq. (3). This completes the proof. $\square$

Now we consider regular solutions of Eq. (1).

Theorem 2. *Let $(G, +)$ be an abelian topological group with the T_0 property and let $f, g, h: G \rightarrow G$. Assume that the group $\langle (f - f(0))(G) \rangle$ is dense in G and f has a limit at a point. Then the triple (f, g, h) satisfies Eq. (1) if, and only if there exist $x_0, z_0 \in G$ and a continuous additive function $a: G \rightarrow G$ such that $f = a + z_0$ and for every $x \in G$ the following conditions hold*

$$g((a + z_0)^k(x)) = a(x) + a(z_0) + z_0 - x_0 \quad (14)$$

$$h((a + z_0)^l(x)) = a^2(x) + x_0. \quad (15)$$

Proof. Assume that the triple (f, g, h) satisfies Eq. (1). By Corollary 1 the function $-f + f(0)$ is a solution of Eq. (3). Applying Theorem 9 and Lemma 4 from [1] we deduce that the function $-f + f(0)$ is additive and continuous. Putting $z_0 = f(0)$ and $a = f - f(0)$ we see that $f = a + z_0$. It follows from Theorem 1 that there exist $x_0, y_0 \in G$ such that conditions (5)–(8) are satisfied for every $x, y \in G$. In particular, $y_0 = f^2(0) - x_0$. Using equalities (6) and (7) we deduce that conditions (14) and (15) hold for every $x \in G$.

Now assume that the right side of the implication in the Theorem is satisfied. The function f is affine, and therefore equality (5) holds for every $y \in S$. Put $y_0 = f^2(0) - x_0$. Then (8) holds and from (15) we obtain (7). Using (14) we get (6). This completes the proof. $\square$

From Theorem 2 we derive the following result.

Corollary 2. *Let $f, g, h: \mathbb{R} \rightarrow \mathbb{R}$ be functions. Assume that the group $\langle (f - f(0))(\mathbb{R}) \rangle$ is dense in $\mathbb{R}$ and f has a finite limit at a point. Then the triple (f, g, h) satisfies Eq. (1) if, and only if there exists an $a \in \mathbb{R} \setminus \{0\}$ and $x_0, z_0 \in \mathbb{R}$ such that $f(x) = ax + z_0$ and the following conditions hold:*

$$g(x) = a^{1-k}(x - z_0 \sum_{i=0}^{k-1} a^i) + az_0 + z_0 - x_0 \quad (16)$$

and

$$h(x) = a^{2-l}(x - z_0 \sum_{i=0}^{l-1} a^i) + x_0 \quad (17)$$

for every $x \in \mathbb{R}$.

Proof. Assume that the triple (f, g, h) satisfies Eq. (1). By Theorem 2 there exist $x_0, z_0 \in \mathbb{R}$ and an $a \in \mathbb{R}$ such that the equality $f(x) = ax + z_0$ and conditions (14) and (15) are satisfied for every $x \in \mathbb{R}$. Since the set $\langle (f - f(0))(\mathbb{R}) \rangle$ is dense in $\mathbb{R}$, we have $a \neq 0$. By induction it is easy to prove that $f^n(x) = a^n x + z_0 \sum_{i=0}^{n-1} a^i$ for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Using (14) and (15) we obtain

$$g(a^k x + z_0 \sum_{i=0}^{k-1} a^i) = ax + az_0 + z_0 - x_0 \quad (18)$$

and

$$h(a^l x + z_0 \sum_{i=0}^{l-1} a^i) = a^2 x + x_0 \quad (19)$$

for every $x \in \mathbb{R}$. Hence $g(x + z_0 \sum_{i=0}^{k-1} a^i) = a^{1-k}x + az_0 + z_0 - x_0$ and $h(x + z_0 \sum_{i=0}^{l-1} a^i) = a^{2-l}x + x_0$ for every $x \in \mathbb{R}$. Inserting here $x - z_0 \sum_{i=0}^{k-1} a^i$ and, respectively, $x - z_0 \sum_{i=0}^{l-1} a^i$ we see that conditions (16) and (17) hold for every $x \in \mathbb{R}$. This completes the proof. $\square$

2. Case $g = h = \text{id}$

Now we consider Eq. (2). We start with some auxiliary results which present general properties of this equation.

Lemma 1. *Let $(G, +)$ be a set with a binary operation and let $f: G \rightarrow G$ be a solution of Eq. (2). Then for every $n \in \mathbb{N}$ and $x, y \in G$ the following condition holds:*

$$f^n(x + f(y)) = f^{kn}(x) + f^{(l-1)n+1}(y). \quad (20)$$

Moreover, for every $n \in \mathbb{N}$ the function f^n satisfies Eq. (2).

Proof. It follows from (2) that if $n = 1$, then condition (20) is satisfied for every $x, y \in G$. Fix an $n \in \mathbb{N}$ and assume that (20) is satisfied for every $x, y \in G$. Then for every $x, y \in G$ we have

$$\begin{aligned} f^{n+1}(x + f(y)) &= f(f^n(x + f(y))) = f(f^{kn}(x) + f^{(l-1)n+1}(y)) \\ &= f(f^{kn}(x) + f(f^{(l-1)n}(y))) = f^k(f^{kn}(x)) + f^l(f^{nl-n}(y)) \\ &= f^{k(n+1)}(x) + f^{(l-1)(n+1)+1}(y). \end{aligned}$$

This finishes the inductive proof of (20).

For the proof of the second part fix any $n \in \mathbb{N}$. Inserting $f^{n-1}(y)$ in place of y in (20) we get

$$f^n(x + f^n(y)) = f^{kn}(x) + f^{ln}(y)$$

for every $x, y \in G$. Therefore f^n satisfies Eq. (2). □

Lemma 2. *Let $(S, +)$ be an abelian cancellative semigroup and let $f: S \rightarrow S$ be a solution of Eq. (2). Then*

$$f^{k^2}(x) + f^{2l-1}(y) = f^k(x) + f^{l^2-l+1}(y) \quad (21)$$

for every $x, y \in S$. In particular, if $l \in \{1, 2\}$, then

$$f^{k^2} = f^k \quad (22)$$

and f satisfies the equation

$$f^k(x + f^k(y)) = f^k(x) + f^{lk}(y). \quad (23)$$

Proof. Using Lemma 1 we deduce that f satisfies condition (20) for every $x, y \in S$ and for every $n \in \mathbb{N}$. Hence

$$f(x + f(y) + f(z)) = f^k(x + f(y)) + f^l(z) = f^{k^2}(x) + f^{lk-k+1}(y) + f^l(z)$$

for every $x, y, z \in S$. Inserting here $f^{k-1}(y)$ in place of y and $f^{l-1}(z)$ in place of z we obtain

$$f(x + f^k(y) + f^l(z)) = f^{k^2}(x) + f^{kl}(y) + f^{2l-1}(z) \quad (24)$$

for every $x, y, z \in S$. On the other hand, making use of (2) and 20 we get

$$\begin{aligned} f(x + f^k(y) + f^l(z)) &= f(x + f(y + f(z))) = f^k(x) + f^l(y + f(z)) \\ &= f^k(x) + f^{kl}(y) + f^{l^2-l+1}(z) \end{aligned} \quad (25)$$

for every $x, y, z \in S$. The semigroup S is abelian and cancellative and therefore, comparing the right sides of equalities (24) and (25), we obtain the first statement of the Lemma.

Now assume that $l \in \{1, 2\}$. Then $l^2 - l + 1 = 2l - 1$ and, using (21) and the commutativity and cancellativity of S again, we get (22). By virtue of Lemma 1 the function f^k satisfies Eq. (2). This in view of (22) means that f satisfies Eq. (23). □

Lemma 3. *Let $(S, +)$ be an abelian cancellative semigroup with a zero. If $f: S \rightarrow S$ satisfies Eq. (2), then*

$$f^{k+l-2}(0) = 0. \quad (26)$$

Proof. We may assume that $\max\{k, l\} \geq 2$. First we consider the case $k \geq 2$. Inserting $f^{k-2}(y)$ in place of y in (2) we obtain the condition

$$f(x + f^{k-1}(y)) = f^k(x) + f^{k+l-2}(y)$$

for every $x, y \in S$. Applying them for $x = y = 0$ we get $f^k(0) = f^k(0) + f^{k+l-2}(0)$, which, by virtue of the cancellativity of S , implies $f^{k+l-2}(0) = 0$.

Now assume that $k = 1$. Then $l \geq 2$. It follows from Lemma 2 that f satisfies Eq. (21). Hence

$$f^{l^2-l+1} = f^{2l-1}. \quad (27)$$

It follows from Lemma 1 that for every $n \in \mathbb{N}$ and for every $x, y \in S$ condition (20) is satisfied. Applying them for $n = l - 1$ we get

$$f^{l-1}(x + f(y)) = f^{l-1}(x) + f^{l^2-2l+2}(y) \quad (28)$$

for every $x, y \in S$. Inserting $f^{l-2}(y)$ in place of y in (28) we obtain

$$f^{l-1}(x + f^{l-1}(y)) = f^{l-1}(x) + f^{l^2-l}(y) \quad (29)$$

for every $x, y \in S$. Condition (29) with $f^{l-1}(x)$ in place of x gives

$$f^{l-1}(f^{l-1}(x) + f^{l-1}(y)) = f^{2l-2}(x) + f^{l^2-l}(y)$$

for every $x, y \in S$. The left-hand side of the last equality is symmetric with respect to replacing x and y and thus its right-hand side also has this property. Therefore

$$f^{2l-2}(x) + f^{l^2-l}(y) = f^{2l-2}(y) + f^{l^2-l}(x)$$

for every $x, y \in S$. Inserting $f(y)$ in place of y in the last condition we get

$$f^{2l-2}(x) + f^{l^2-l+1}(y) = f^{2l-1}(y) + f^{l^2-l}(x) \quad (30)$$

for every $x, y \in S$. Since S is abelian and cancellative, using (27) and (30) we obtain $f^{l^2-l}(x) = f^{2l-2}(x)$ for every $x \in S$. Hence, in view of equality (29), we deduce that f^{l-1} satisfies Eq. (3). In particular, $f^{2l-2}(0) = f^{l-1}(0 + f^{l-1}(0)) = f^{l-1}(0) + f^{2l-2}(0)$, which in view of the cancellativity of S gives $f^{l-1}(0) = 0$. Therefore condition (26) follows with $k = 1$. This completes the proof. $\square$

Lemma 4. *Let $(S, +)$ be an abelian cancellative semigroup with a zero. Assume that*

$$(k + l - 2)x \neq 0 \quad \text{for every } x \in S \setminus \{0\}. \quad (31)$$

If $f: S \rightarrow S$ is a solution of Eq. (2), then $f(0) = 0$.

Proof. Assume that $f: S \rightarrow S$ is a solution of Eq. (2). We may assume that $S \neq \{0\}$. It follows from Lemma 3 that f satisfies condition (26). In particular,

$$f^{k+l-1}(0) = f(0). \quad (32)$$

We will show that

$$f^{k+n}(0) = f^k(0) + f^n(0) \quad (33)$$

for every $n \in \mathbb{N} \cup \{0\}$. If $n = 0$, then the equality above is obvious. Fix an $n \in \mathbb{N} \cup \{0\}$ and assume that equality (33) is satisfied. Using equalities (2) and (32) we obtain

$$\begin{aligned} f^{k+n+1}(0) &= f(f^{k+n}(0)) = f^k(0) + f^{l+k+n-1}(0) = f^k(0) + f^n(f(0)) \\ &= f^k(0) + f^{n+1}(0). \end{aligned}$$

This finishes the inductive proof.

Applying condition (33) with $n = l - 1$ and making use of (32) we get

$$f^k(0) + f^{l-1}(0) = f(0). \quad (34)$$

Now we show that

$$f^n(0) = nf(0) \quad (35)$$

for every $n \in \mathbb{N} \cup \{0\}$. For $n = 0$ equality (35) holds. Fix an $n \in \mathbb{N} \cup \{0\}$ and assume that (35) is satisfied. Since $S \neq \{0\}$, it follows from (31) that $\max\{k, l\} \geq 2$. Therefore $k + l \geq 3$ and using equalities (26), (2), (33) and (34) we obtain

$$\begin{aligned} f^{n+1}(0) &= f(f^n(0)) = f(f^n(0) + f^{l+k-2}(0)) = f(f^n(0) + f(f^{l+k-3}(0))) \\ &= f^{k+n}(0) + f^{2l+k-3}(0) = f^k(0) + f^n(0) + f^{l-1}(f^{l+k-2}(0)) \\ &= f^k(0) + f^n(0) + f^{l-1}(0) = f^n(0) + f(0) = nf(0) + f(0) \\ &= (n+1)f(0). \end{aligned}$$

This finishes the inductive proof of (35). Making use of (35) with $n = k + l - 2$ and applying (26) we get $(k + l - 2)f(0) = 0$. Using (31) we obtain the assertion. $\square$

Making use of Theorem 1 and Lemma 4 we will prove the following result.

Theorem 3. *Let $(S, +)$ be an abelian cancellative semigroup with a zero. Assume that S satisfies condition (31). Let $f: S \rightarrow S$ be a function. Then f satisfies Eq. (2) if, and only if it satisfies Eq. (3) and the condition*

$$f^k = f \text{ and } f^l = f^2. \quad (36)$$

Proof. Assume that f satisfies Eq. (2). Applying Theorem 1 with $g = h = \text{id}$ we deduce that there exist $x_0, y_0 \in S$ such that f satisfies conditions (5)–(8) for every $x, y \in S$. It follows from Lemma 4 that $f(0) = 0$. Therefore, making use of (5), we see that f satisfies Eq. (3). Using (6) with $x = 0$ we obtain $x_0 = 0$. Hence, by virtue of (8), we get $y_0 = 0$. Making use of equalities (6) and (7) we deduce that condition (36) is satisfied. The converse is obvious. $\square$

From Theorem 3 we derive results concerning regular solutions of Eq. (2).

Theorem 4. *Let $(G, +)$ be an abelian topological group [real linear topological space] with the T_0 property. Assume that G satisfies condition (31). Let $f: G \rightarrow G$ be a function such that f has a limit at a point and the group $\langle f(G) \rangle$ is dense in G . Then f satisfies Eq. (2) if, and only if f is a continuous additive [linear] mapping satisfying the condition*

$$f^{k-1} = \text{id} \text{ and } f^{|l-2|} = \text{id}. \quad (37)$$

Proof. Assume that f satisfies Eq. (2). It follows from Lemma 4 that $f(0) = 0$. Applying Theorem 2 with $g = h = \text{id}$ we deduce that f is a continuous affine function. Therefore f is additive. In particular, the set $f(G)$ is a subgroup of G . Hence in view of the assumption we infer that $f(G)$ is dense in G . By Theorem 3 the function f satisfies equalities (36). This means that f^{k-1} and $f^{|l-2|}$ are equal to the identity function on $f(G)$. Making use of the density of $f(G)$ in G and the continuity of f we deduce that f satisfies condition (37). The converse follows from Theorem 3. □

Corollary 3. *Let $\max\{k, l\} \geq 2$ and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function. Assume that f has a finite limit at a point and the group $\langle f(\mathbb{R}) \rangle$ is dense in $\mathbb{R}$. Then f satisfies Eq. (2) if, and only if one of the following conditions holds:*

(p₁) $k = 1$, $l = 2$ and there exists an $a \in \mathbb{R} \setminus \{0\}$ such that $f(x) = ax$ for every $x \in \mathbb{R}$;

(p₂) k is odd and l is even, $k \neq 1$ or $l \neq 2$, and there exists an $a \in \{-1, 1\}$ such that $f(x) = ax$ for every $x \in \mathbb{R}$;

(p₃) k is even or l is odd, and $f = \text{id}$.

Proof. It is easy to check that the functions presented in cases (p₁)–(p₃) are solutions of Eq. (2). So assume that f satisfies Eq. (2). By Theorem 4 the function f is a linear mapping and it satisfies equalities (37). Therefore there exists an $a \in \mathbb{R}$ such that

$$f(x) = ax \quad (38)$$

for every $x \in \mathbb{R}$. Since $\langle f(\mathbb{R}) \rangle$ is dense in $\mathbb{R}$, we have $a \neq 0$.

If $k = 1$ and $l = 2$, then, using (38), we complete the proof. Assume that $k \neq 1$ or $l \neq 2$. It follows from (37) that

$$a^{k-1} = 1 \quad \text{and} \quad a^{|l-2|} = 1. \quad (39)$$

If k is even or l is odd, then condition (39) yields $a = 1$, and thus $f = \text{id}$. If k is odd and l is even, then (39) implies that $a \in \{-1, 1\}$ and therefore either $f = -\text{id}$, or $f = \text{id}$. This completes the proof. □

Proposition 3.2 [2] presents the general solution of Eq. (3) in the class of continuous functions mapping $\mathbb{R}$ into itself. In the next Corollary we extend that result to Eq. (2).

Corollary 4. *Let $\max\{k, l\} \geq 2$ and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. The function f satisfies Eq. (2) if, and only if one of the following conditions holds:*

- (w₁) $k = 1$, $l = 2$ and there exists an $a \in \mathbb{R}$ such that $f(x) = ax$ for every $x \in \mathbb{R}$;
- (w₂) k is odd and l is even, $k \neq 1$ or $l \neq 2$, and there exists an $a \in \{-1, 0, 1\}$ such that $f(x) = ax$ for every $x \in \mathbb{R}$;
- (w₃) k is even or l is odd, and there exists an $a \in \{0, 1\}$ such that $f(x) = ax$ for every $x \in \mathbb{R}$.

Proof. Assume that f satisfies Eq. (2). If f is constant, then it follows from (2) that $f = 0$. Otherwise the function f is non-constant and continuous and therefore the set $f(\mathbb{R})$ is an interval with non-empty interior. Hence $\langle f(\mathbb{R}) \rangle = \mathbb{R}$ and making use of Corollary 3 we get the suitable form of f . To prove the converse it is enough to apply Corollary 3 and to observe that the zero function satisfies Eq. (2). $\square$

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