Beta-type functions and the harmonic mean

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Abstract. For arbitrary $f : (a, \infty) \rightarrow (0, \infty)$, $a \geq 0$, the bivariable function $B_f : (a, \infty)^2 \rightarrow (0, \infty)$, related to the Euler Beta function, is considered. It is proved that B_f is a mean iff it is the harmonic mean H . Some applications to the theory of iterative functional equations are given.

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1. Introduction

For $a > 0$ and $f : (a, \infty) \rightarrow (0, \infty)$, the two variable function $B_f : (a, \infty)^2 \rightarrow (0, \infty)$ defined by

$$B_f(x, y) = \frac{f(x)f(y)}{f(x+y)}, \quad x, y > a,$$

is called a *beta-type function*, and f is called its *generator* ([1]). The notion of beta-type functions arises from the well-known relation between the Beta function $\mathcal{B} : (0, \infty)^2 \rightarrow (0, \infty)$ and the Euler Gamma function $\Gamma : (0, \infty) \rightarrow (0, \infty)$,

$$\mathcal{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x, y > 0,$$

which can be written in the form $B_\Gamma = \mathcal{B}$.

In this paper we are interested in answering when a beta-type function is a bivariable mean in (a, ∞) . Our main result says that the beta-type function of a generator $f : (a, \infty) \rightarrow (0, \infty)$ is a mean iff $f(x) = 2xe^{\alpha(x)}$ where $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function, or equivalently, that B_f is the harmonic mean (Theorem 2). This substantially improves the result of [1] where the homogeneity of the beta-type function is assumed.

In the preliminary Sect. 2 we recall the notions of mean, premean, reflexivity of a function, and some of their properties. The increasingness of the beta-type function B_f is equivalent to the concavity of the function $\log \circ f$ in the sense of Wright (Proposition 1). In Sect. 3 we determine the class of all generators f for which B_f is reflexive, that is $B_f(x, x) = x$, or equivalently, f satisfies the iterative functional equation $[f(x)]^2 = xf(2x)$ (Theorem 1). Applying this, in Sect. 4, we prove Theorem 2, our main result. This allows us to conclude that every logarithmically convex function satisfying the above iterative functional equation must be of the form $f(x) = 2xc^x$ for some $c > 0$ (Theorem 3). The functional equation $f(2x) = x[f(x)]^2$ (related to beta-type functions) is also considered. We note that the Krull method [2, 3] allows us to figure out a unique solution of this iterative functional equation in a more specific class of functions (cf. Remark 7). At the end we propose a unique and natural extension of the harmonic bivariable mean to $\mathbb{R}^2$.

The case of k -variable beta-type functions, $k \geq 3$, will be considered in our next paper.

2. Preliminaries

We start with some definitions.

Definition 1. Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \rightarrow \mathbb{R}$ is called a *mean in an interval I* , if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I.$$

Remark 1. Let $I \subset \mathbb{R}$ be an interval and let $M : I^2 \rightarrow \mathbb{R}$. The following conditions are equivalent

- (i) M is a *mean in I* ;
- (ii) $M(J^2) \subset J$ for every subinterval $J \subset I$.

Remark 2. If M is a mean in an interval I , then it is *reflexive*, i.e.

$$M(x, x) = x, \quad x \in I.$$

Clearly, if M is a mean in I , then $M : I^2 \rightarrow I$.

Definition 2. Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \rightarrow I$ is called a *premean* if it is reflexive.

Let us note the following obvious

Remark 3. Let $I \subset \mathbb{R}$ be an interval. If a function $M : I^2 \rightarrow \mathbb{R}$ is reflexive and increasing with respect to each of the variables, then M is a mean in I .

Definition 3. Let $I = (a, \infty)$ if $a \geq 0$, or $I = [a, \infty)$ if $a > 0$. For a function $f : I \rightarrow (0, \infty)$ we call $B_f : I^2 \rightarrow (0, \infty)$ defined by

$$B_f(x, y) = \frac{f(x)f(y)}{f(x+y)}, \quad x, y \in I,$$

the beta-type function of generator f .

Note that the beta-type function B_f of a generator $f : (a, \infty) \rightarrow (0, \infty)$ is reflexive iff $[f(x)]^2 = xf(2x)$ for all $x > a$.

In this context one could also consider the functions of beta-type of the generators f defined on the intervals $(-\infty, a)$ with values in $(-\infty, 0)$.

Remark 4. Replacing f in this Definition 3 by $\frac{1}{f}$ we get

$$B_{\frac{1}{f}}(x, y) = \frac{f(x+y)}{f(x)f(y)}, \quad x, y \in I,$$

and $B_{\frac{1}{f}}$ is reflexive iff $f(2x) = x[f(x)]^2$ for all $x \in I$.

We shall prove the following

Proposition 1. Let $f : (0, \infty) \rightarrow (0, \infty)$ be a continuous function. The following two conditions are equivalent:

- (i) the beta-type function $B_f : (0, \infty)^2 \rightarrow (0, \infty)$ is an increasing mean;
- (ii) the function $\log \circ f$ is concave (that is f is logarithmically concave) and

$$[f(x)]^2 = xf(2x), \quad x > 0.$$

Proof. Assume (i). Then, as $\log$ is increasing in $(0, \infty)$, the function $\log \circ B_f$ is increasing in each variable. So, by the definition of B_f , for all $x, y, z \in (0, \infty)$, $x < z \implies \log f(x) + \log f(y) - \log f(x+y) \leq \log f(z) + \log f(y) - \log f(z+y)$, or, equivalently, for all $x, y, z \in (0, \infty)$,

$$x < z \implies \log f(x) + \log f(z+y) \leq \log f(z) + \log f(x+y).$$

Choosing arbitrary $u, v > 0$, $u < v$, and $t \in (0, 1)$, and taking

$$x := u, \quad y := (1-t)(v-u), \quad z := v - (1-t)(v-u),$$

we obtain that the above implication is equivalent to the following one: for all $u, v > 0$, and $t \in (0, 1)$,

$$u < v \implies \log f(u) + \log f(v) \leq \log f(tu + (1-t)v) + \log f((1-t)u + tv),$$

which shows that B_f is increasing if, and only if, $\log \circ f$ is concave in the sense of Wright. Since f is continuous, in view of a theorem of Ng [6], $\log \circ f$ is Wright concave if, and only if, $\log \circ f$ is concave. Since B_f is a mean, it is reflexive. Thus we have shown that (ii) holds true. The converse implication follows from Remark 3. $\square$

3. Beta-type functions and reflexivity

Applying the method of the theory of iterative functional equations by Kuczma [4], we prove the following

Theorem 1. *Assume that $a > 0$ is fixed.*

- (i) *Let $f : [a, \infty) \rightarrow (0, \infty)$ be arbitrary and let $f_0 := f|_{[a, 2a)}$. The function f satisfies the functional equation*

$$[f(x)]^2 = xf(2x), \quad x \geq a, \quad (3.1)$$

if, and only if,

$$f(x) = 2^{1+(n-1)2^n} x^{1-2^n} \left(f_0 \left(\frac{x}{2^n} \right) \right)^{2^n}, \quad x \in [2^n a, 2^{n+1} a), \quad n \in \mathbb{N}_0. \quad (3.2)$$

- (ii) *Let $f : (0, \infty) \rightarrow (0, \infty)$ be arbitrary and let $f_0 := f|_{[a, 2a)}$. The function f satisfies the functional equation*

$$[f(x)]^2 = xf(2x), \quad x > 0, \quad (3.3)$$

if, and only if,

$$f(x) = 2^{1+(n-1)2^n} x^{1-2^n} \left(f_0 \left(\frac{x}{2^n} \right) \right)^{2^n}, \quad x \in [2^n a, 2^{n+1} a), \quad n \in \mathbb{Z}. \quad (3.4)$$

- (iii) *Moreover, in each of the above cases, f is continuous if, and only if, f_0 is continuous and*

$$f(2a-) := \lim_{x \rightarrow 2a-} f(x) = (f_0(a))^2. \quad (3.5)$$

Proof. Without any loss of generality we can assume that $a = 1$. Indeed, a function $f : [a, \infty) \rightarrow (0, \infty)$ satisfies

$$[f(x)]^2 = xf(2x), \quad x \geq a,$$

if, and only if, the function $g : [1, \infty) \rightarrow (0, \infty)$ defined by

$$g(x) = \frac{f(ax)}{a}, \quad x \geq 1,$$

satisfies the equation

$$[g(x)]^2 = xg(2x), \quad x \geq 1.$$

(i) Let $f : [1, \infty) \rightarrow (0, \infty)$ be a solution to (3.1) and put $f_0 := f|_{[1, 2)}$. We shall show that, for every $n \in \mathbb{N}_0$,

$$x \in [2^n, 2^{n+1}) \Rightarrow f(x) = 2^{1+2^n(n-1)} x^{1-2^n} \left(f_0 \left(\frac{x}{2^n} \right) \right)^{2^n}. \quad (3.6)$$

Take $x \geq 1$. Then there exists a unique $n \in \mathbb{N}_0$ such that

$$2^n \leq x < 2^{n+1}.$$

If $n = 0$, then $1 \leq x < 2$ and, by the definition of f_0 ,

$$f(x) = f_0(x),$$

so (3.6) holds true for $n = 0$. Assume that (3.6) holds true for $x \in [2^n, 2^{n+1})$ and take arbitrary $x \in [2^{n+1}, 2^{n+2})$. Then $\frac{x}{2} \in [2^n, 2^{n+1})$ and, by (3.6),

$$f\left(\frac{x}{2}\right) = 2^{n2^n} x^{1-2^n} \left(f_0\left(\frac{x}{2^{n+1}}\right)\right)^{2^n}. \quad (3.7)$$

Replacing x by $\frac{x}{2}$ in (3.1), we obtain

$$f\left(\frac{x}{2}\right) = \left(\frac{x}{2} f(x)\right)^{\frac{1}{2}},$$

and thus

$$f(x) = \frac{2}{x} \left(f\left(\frac{x}{2}\right)\right)^2.$$

Using this and our assumption that f satisfies (3.1), we have

$$\begin{aligned} f(x) &= \frac{2}{x} \left(f\left(\frac{x}{2}\right)\right)^2 \\ &= \frac{2}{x} \left(2^{n2^n} x^{1-2^n} \left(f_0\left(\frac{x}{2^{n+1}}\right)\right)^{2^n}\right)^2 \\ &= 2^{1-2^{-(n+1)}} x^{1-2^{-(n+1)}} \left(f_0\left(\frac{x}{2^{n+1}}\right)\right)^{2^{n+1}}, \end{aligned}$$

giving us the validity of (3.6) for $x \in [2^{n+1}, 2^{n+2})$. By the induction principle, we have shown that formula (3.6) holds true for every $n \in \mathbb{N}_0$.

To prove the converse of (i), we show that, for arbitrary $f_0 : [1, 2) \rightarrow (0, \infty)$, the function $f : [1, \infty) \rightarrow (0, \infty)$ defined by (3.4) satisfies (3.3). Indeed, if $x \geq 1$, there exists a unique $n \in \mathbb{N}$ such that $x \in [2^n, 2^{n+1})$, whence $2x \in [2^{n+1}, 2^{n+2})$, and the left hand side of (3.3) reads

$$f(x) = 2^{1+(n-1)2^n} x^{1-2^n} \left(f_0\left(\frac{x}{2^n}\right)\right)^{2^n}; \quad (3.8)$$

the right hand side of (3.3) reads

$$\begin{aligned} (xf(2x))^{1/2} &= \left(x 2^{1+n2^{n+1}} (2x)^{1-2^{n+1}} \left(f_0\left(\frac{2x}{2^{n+1}}\right)\right)^{2^{n+1}}\right)^{1/2} \\ &= \left(2^{1+n2^{n+1}+1-2^{n+1}} x^{1-2^{n+1}+1} \left(f_0\left(\frac{x}{2^n}\right)\right)^{2^{n+1}}\right)^{1/2} \\ &= 2^{1+(n-1)2^n} x^{1-2^n} \left(f_0\left(\frac{x}{2^n}\right)\right)^{2^n}. \end{aligned} \quad (3.9)$$

Since the right hand sides of (3.8) and (3.9) are equal, the function f defined by (3.4) satisfies (3.3).

(ii) By induction, we shall prove that, for all $n \in \mathbb{N}_0$,

$$x \in \left[\left(\frac{1}{2} \right)^n, \left(\frac{1}{2} \right)^{n-1} \right) \Rightarrow f(x) = 2^{1-2^{-n}(n+1)} x^{1-2^{-n}} \left(f_0 \left(\frac{x}{2^{n+1}} \right) \right)^{2^n}, \quad (3.10)$$

which is (3.2) with n replaced by $-n$.

Take arbitrarily $x \in (0, 1)$. There exists a unique $n \in \mathbb{N}_0$ such that

$$\left(\frac{1}{2} \right)^n \leq x < \left(\frac{1}{2} \right)^{n-1}.$$

If $n = 0$, then $1 \leq x < 2$ and, by the definition of f_0 ,

$$f(x) = f_0(x),$$

so (3.10) holds true for $n = 0$. Assume that (3.10) holds true for some $n \in \mathbb{N}_0$.

Taking arbitrarily $x \in \left[\left(\frac{1}{2} \right)^{n+1}, \left(\frac{1}{2} \right)^n \right)$, we have $2x \in \left[\left(\frac{1}{2} \right)^n, \left(\frac{1}{2} \right)^{n-1} \right)$, thus, applying (3.10), we obtain

$$f(2x) = (2x)^{1-2^{-n}} 2^{1-2^{-n}(n+1)} (f_0(2^{n+1}x))^{1/2^n}.$$

Hence, by (3.3),

$$\begin{aligned} f(x) &= (xf(2x))^{1/2} \\ &= \left(x(2x)^{1-2^{-n}} 2^{1-2^{-n}(n+1)} (f_0(2^{n+1}x))^{1/2^n} \right)^{1/2} \\ &= 2^{1-2^{-(n+1)}(n+2)} x^{1-2^{-(n+1)}} (f_0(2^{n+1}x))^{1/2^{n+1}} \end{aligned}$$

so (3.10) holds true for $x \in \left[\left(\frac{1}{2} \right)^{n+1}, \left(\frac{1}{2} \right)^n \right)$, which means that (3.10) holds for $n + 1$. By the induction principle, formula (3.10) holds true for all $x \in (0, 1)$. Both reasonings prove the validity of (3.4), which is the second statement of our theorem.

Arguing similarly as in the previous case we can prove the converse of (ii).

In part (iii), since one implication is obvious, we must show that the continuity of f_0 and (3.5) imply the continuity of f . By (3.4), the continuity of f_0 on $(1, 2)$ implies that f is continuous on $\bigcup_{n \in \mathbb{Z}} (2^n, 2^{n+1})$. It remains to show the

continuity of f at the point 2^n for all $n \in \mathbb{Z}$.

Applying in turn (3.4), (3.5), we have

$$\begin{aligned} \lim_{x \rightarrow 2^n-} f(x) &= \lim_{x \rightarrow 2^n-} 2^{1+(n-2)2^n} x^{1-2^{n-1}} \left(f_0 \left(\frac{x}{2^{n-1}} \right) \right)^{2^{n-1}} \\ &= 2^{1+(n-2)2^n-n(1-2^{n-1})} \lim_{x \rightarrow 2-} (f_0(x))^{2^{n-1}} \\ &= 2^{1+(n-1)2^n-n(1-2^n)} \left((f_0(1))^2 \right)^{2^{n-1}} \\ &= f(2^n). \end{aligned}$$

The validity of

$$\lim_{x \rightarrow 2^n+} f(x) = f(2^n)$$

is obvious since, by (3.4), f is defined on the interval $[2^n, 2^{n+1})$ and the composition and multiplication of continuous functions are continuous.

This finishes the proof. $\square$

Remark 5. Since $(a, \infty) = \bigcup_{n \in \mathbb{N}} [a + \frac{1}{n}, \infty)$ the counterpart of part (i) holds true.

From Theorem 1, by Remark 2 and Definition 3, we obtain

Remark 6. A beta-type function B_f is a premean in an interval (a, ∞) , $a \geq 0$, if, and only if, the function $f : (a, \infty) \rightarrow (0, \infty)$ is of the form (3.2).

Remark 7. If $f : (0, \infty) \rightarrow (0, \infty)$ is such that $(\log \circ f)' \circ \exp$ is convex and the beta-type function B_f is a mean in $(0, \infty)$, then $B_f = H$.

Proof. Assume that $f : (0, \infty) \rightarrow (0, \infty)$ is a differentiable function. Since B_f is a mean in $(0, \infty)$, it satisfies (3.3). Taking the logarithm of both sides, we can write this equation as follows

$$2(\log \circ f)(x) = (\log \circ f)(2x) + \log x, \quad x > 0.$$

Differentiating both sides gives

$$2(\log \circ f)'(x) = 2(\log \circ f)'(2x) + \frac{1}{x}, \quad x > 0,$$

which can be written in the form

$$2(\log \circ f)' \circ \exp(\log x) = 2(\log \circ f)' \circ \exp(\log x + \log 2) + \frac{1}{\exp(\log x)}, \quad x > 0.$$

Setting here

$$h := (\log \circ f)' \circ \exp, \quad u = \log x, \quad (3.11)$$

we conclude that a convex function h satisfies the functional equation

$$h(u + \log 2) = h(u) - \frac{e^{-u}}{2}, \quad u \in \mathbb{R}. \quad (3.12)$$

Since the function $F : \mathbb{R} \rightarrow \mathbb{R}$, $F(u) := -\frac{e^{-u}}{2}$ is concave and

$$\lim_{u \rightarrow \infty} [F(u + \log 2) - F(u)] = 0,$$

Krull's theorem [2, 3] (see also [4], pp. 114–115) gives us the existence of a (unique up to a constant) convex solution h to (3.12). Since, for any real constant k , the function $u \mapsto e^{-u} + k$ satisfies the functional equation (3.12) and is convex, it follows that for some k ,

$$h(u) = e^{-u} + k, \quad u \in \mathbb{R}.$$

Thus, from (3.11), we get

$$(\log \circ f)' \circ \exp(u) = e^{-u} + k, \quad u \in \mathbb{R}.$$

It follows that

$$(\log \circ f)'(x) = \frac{1}{x} + k, \quad x > 0,$$

whence, setting $c = e^k$, we get

$$f(x) = bxc^x, \quad x > 0,$$

for some $b > 0$. □

4. Means in terms of beta-type functions

We use Theorem 1 on the solutions of reflexive beta-type functions to answer the question when a beta-type function is a mean.

Theorem 2. *Let $I = (0, \infty)$ or $I = [a, \infty)$ for some $a > 0$ and $f : I \rightarrow (0, \infty)$ be an arbitrary function.*

The following conditions are equivalent:

- (i) *the beta-type function $B_f : I^2 \rightarrow (0, \infty)$ is a mean;*
- (ii) *there is an additive function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$f(x) = 2xe^{\alpha(x)}, \quad x \in I;$$

- (iii) *B_f is the harmonic mean in I .*

Proof. To prove the implication (i) $\implies$ (ii), assume first that $I = (0, \infty)$ and B_f is a mean in I . Put $f_0 := f|_{[1,2]}$. Since every mean is reflexive, the function $f : (0, \infty) \rightarrow (0, \infty)$ satisfies (3.3). Thus, by part (ii) of Theorem 1 with $a = 1$, we have, for every $n \in \mathbb{Z}$,

$$x \in [2^n, 2^{n+1}) \implies f(x) = 2^{1+(n-1)2^n} x^{1-2^n} \left(f_0 \left(\frac{x}{2^n} \right) \right)^{2^n};$$

whence, using the definition of beta-type functions,

$$\begin{aligned} B_f(x, y) &= \frac{f(x)f(y)}{f(x+y)} \\ &= 2^{1-2^{n+1}} \frac{xy}{x+y} \left[\frac{(x+y)^2}{xy} \frac{f_0\left(\frac{x}{2^n}\right)f_0\left(\frac{y}{2^n}\right)}{\left(f_0\left(\frac{x+y}{2^{n+1}}\right)\right)^2} \right]^{2^n}. \end{aligned}$$

Since B_f is a mean, we have, for all $x, y \in [2^n, 2^{n+1})$,

$$x \leq y \implies x \leq 2^{1-2^{n+1}} \frac{xy}{x+y} \left[\frac{(x+y)^2}{xy} \frac{f_0\left(\frac{x}{2^n}\right)f_0\left(\frac{y}{2^n}\right)}{\left(f_0\left(\frac{x+y}{2^{n+1}}\right)\right)^2} \right]^{2^n} \leq y.$$

Choose s, t arbitrarily from $[1, 2)$. Then, there exists unique $n \in \mathbb{Z}$ such that

$$x, y \in [2^n, 2^{n+1})$$

and

$$s = \frac{x}{2^n} \text{ and } t = \frac{y}{2^n}.$$

Since $x \leq y$ implies $s \leq t$, the latter inequality reads

$$s \leq 2^{1-2^{n+1}} \frac{st}{s+t} \left[\frac{(s+t)^2}{st} \frac{f_0(s)f_0(t)}{\left(f_0\left(\frac{s+t}{2}\right)\right)^2} \right]^{2^n} \leq t, \quad s, t \in [1, 2). \quad (4.1)$$

The first inequality gives us

$$\frac{s+t}{2t} \leq \left[\frac{(s+t)^2}{2st} \frac{f_0(s)f_0(t)}{2\left(f_0\left(\frac{s+t}{2}\right)\right)^2} \right]^{2^n}, \quad s, t \in [1, 2),$$

whence, taking the 2^n -th root of both sides,

$$\left(\frac{s+t}{2t} \right)^{1/2^n} \leq \frac{(s+t)^2}{2st} \frac{f_0(s)f_0(t)}{2\left(f_0\left(\frac{s+t}{2}\right)\right)^2}, \quad s, t \in [1, 2).$$

Letting n tend to ∞ , we obtain

$$1 \leq \frac{(s+t)^2}{2st} \frac{f_0(s)f_0(t)}{2\left(f_0\left(\frac{s+t}{2}\right)\right)^2}, \quad s, t \in [1, 2),$$

and thus

$$\frac{4st}{(s+t)^2} \leq \frac{f_0(s)f_0(t)}{\left(f_0\left(\frac{s+t}{2}\right)\right)^2}, \quad s, t \in [1, 2).$$

Proceeding analogously, the second inequality in (4.1) gives us

$$\frac{4st}{(s+t)^2} \geq \frac{f_0(s)f_0(t)}{\left(f_0\left(\frac{s+t}{2}\right)\right)^2}, \quad s, t \in [1, 2);$$

thus we have

$$\frac{4st}{(s+t)^2} = \frac{f_0(s)f_0(t)}{\left(f_0\left(\frac{s+t}{2}\right)\right)^2}, \quad s, t \in [1, 2),$$

and consequently

$$\left(\frac{f_0\left(\frac{s+t}{2}\right)}{\frac{s+t}{2}} \right)^2 = \frac{f_0(s)f_0(t)}{s} \frac{f_0(t)}{t}, \quad s, t \in [1, 2).$$

Putting $g : [1, 2) \rightarrow (0, \infty)$ defined by

$$g(s) = \frac{f_0(s)}{s},$$

we obtain

$$\left(g\left(\frac{s+t}{2}\right)\right)^2 = g(s)g(t) \quad s, t \in [1, 2).$$

After taking the logarithm of both sides, we get

$$2 \log g\left(\frac{s+t}{2}\right) = \log g(s) + \log g(t), \quad s, t \in [1, 2),$$

thus, dividing both sides by 2, we observe that $h := \log \circ g$ satisfies the Jensen equation

$$h\left(\frac{s+t}{2}\right) = \frac{h(s) + h(t)}{2}, \quad s, t \in [1, 2).$$

By Kuczma [5], p. 351, there exist an additive function $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ and $k \in \mathbb{R}$ such that

$$h(t) = \alpha(t) + k, \quad t \in [1, 2).$$

Substituting back, we thus have

$$f_0(t) = bte^{\alpha(t)}, \quad t \in [1, 2),$$

where $b := e^k$. Hence, by (3.4), we have

$$f(x) = 2^{1-2^n} x b^{2^n} e^{\alpha(x)}$$

and hence the corresponding beta-type function reads

$$\begin{aligned} B_f(x, y) &= \frac{f(x)f(y)}{f(x+y)} \\ &= \frac{2^{1-2^n} x c^{2^n} e^{\alpha(x)} \cdot 2^{1-2^n} y c^{2^n} e^{\alpha(y)}}{2^{1-2^{n+1}} (x+y) c^{2^{n+1}} e^{\alpha(x+y)}} \\ &= 2 \frac{xy}{x+y} \\ &= H(x, y). \end{aligned}$$

This finishes the proof in the case when $I = (0, \infty)$. If $I = [a, \infty)$ for some $a > 0$, applying part (i) of Theorem 1, we can argue similarly. $\square$

In connection with part (ii) of this result, note that very irregular generators may produce quite regular beta-type functions, which is an interesting and characteristic phenomenon in the theory of functional equations. This is a consequence of

Remark 8. (Equality of beta-type functions) We have $B_f = B_g$ if, and only if, there is a function $E: \mathbb{R} \rightarrow (0, \infty)$ such that $\frac{g}{f} = E|_I$ and E is exponential, i.e.

$$E(x+y) = E(x)E(y), \quad x, y \in \mathbb{R}.$$

As an application of Theorem 2 and Proposition 1 we get the following

Theorem 3. *If a function $f : (0, \infty) \rightarrow (0, \infty)$ is logarithmically concave and satisfies the functional equation*

$$[f(x)]^2 = xf(2x), \quad x > 0,$$

then there is $c > 0$ such that

$$f(x) = 2xc^x, \quad x > 0.$$

Hence, taking Remark 4 into account, we obtain

Corollary 1. *If a function $f : (0, \infty) \rightarrow (0, \infty)$ is logarithmically convex and satisfies the functional equation*

$$f(2x) = x[f(x)]^2, \quad x > 0,$$

then, for some $c > 0$,

$$f(x) = \frac{1}{2x}c^x, \quad x > 0.$$

5. Final remarks

Our results, formulated for positive-valued functions defined on intervals of the form (a, ∞) for $a \geq 0$, can be transformed to negative-valued functions defined on the domain $(-\infty, -a)$, which is shown in the following

Remark 9. Let $a \geq 0$. If $\bar{f} : (-\infty, -a) \rightarrow (-\infty, 0)$ satisfies

$$[\bar{f}(x)]^2 = x\bar{f}(2x), \quad x < -a,$$

then $f : (a, \infty) \rightarrow (0, \infty)$ defined by

$$f(x) = -\bar{f}(-x), \quad x > a$$

satisfies

$$[f(x)]^2 = xf(2x), \quad x > a,$$

and thus, by Theorem 2 and part (i) of Theorem 1, the function $B_{\bar{f}} : (-\infty, -a)^2 \rightarrow (-\infty, 0)$ is a mean if, and only if,

$$B_f(x, y) = -H(-x, -y), \quad x, y < -a.$$

From the latter remark it follows that the harmonic mean $H : (0, \infty)^2 \rightarrow (0, \infty)$ and its negative counterpart $\bar{H} : (-\infty, 0)^2 \rightarrow (-\infty, 0)$ defined by

$$\bar{H}(x, y) = -H(-x, -y), \quad x, y < 0,$$

are means of beta-type. These two means can be “glued” together to obtain a unique increasing mean defined in $\mathbb{R}^2$, which can be treated as a *bivariable harmonic mean* in $\mathbb{R}$; namely we have the following

Remark 10. The function $M : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$M(x, y) := \begin{cases} H(x, y) & \text{if } x, y > 0 \\ -H(-x, -y) & \text{if } x, y < 0 \\ 0 & xy \leq 0 \end{cases}$$

is the unique increasing one with respect to each variable function such that $M|_{(0, \infty)^2} = H$ and $M|_{(-\infty, 0)^2} = \bar{H}$. Moreover, M is a continuous bivariable mean in $\mathbb{R}$; its restrictions $M|_{(0, \infty)^2}$ and $M|_{(-\infty, 0)^2}$ are of beta-type but, of course, M is not of beta-type.

Proof. Of course, H and $\bar{H}$ are increasing. Assume that $M : \mathbb{R}^2 \rightarrow \mathbb{R}$ is an increasing function such that $M|_{(0, \infty)^2} = H$ and $M|_{(-\infty, 0)^2} = \bar{H}$. Take $(x, y) \in \mathbb{R}^2$ such that $xy \leq 0$. We consider first the case $x \geq 0, y \leq 0$. Since M is increasing in both variables, we have

$$\begin{aligned} M(x, y) &\geq M(0, y) \geq \lim_{v \rightarrow y-} M(0, v) \leq \lim_{u \rightarrow 0-} \left(\lim_{v \rightarrow y-} M(u, v) \right) \\ &= \lim_{u \rightarrow 0-} \left(\lim_{v \rightarrow y-} \bar{H}(u, v) \right) = 0, \end{aligned}$$

and

$$\begin{aligned} M(x, y) &\leq M(x, 0) \leq \lim_{u \rightarrow x+} M(u, 0) \leq \lim_{v \rightarrow 0+} \left(\lim_{u \rightarrow x+} M(u, v) \right) \\ &= \lim_{v \rightarrow 0+} \left(\lim_{u \rightarrow x+} H(u, v) \right) = 0, \end{aligned}$$

implying

$$M(x, y) = 0, \quad x \geq 0, y \leq 0.$$

The case $x \leq 0, y \geq 0$ is treated analogously yielding

$$M(x, y) = 0, \quad x \leq 0, y \geq 0.$$

Indeed, the function M is a mean in $\mathbb{R}$ since H and $\bar{H}$ are means in $(0, \infty)$ and $(-\infty, 0)$, respectively, and, for $xy \leq 0$ with $x \leq y$, obviously

$$x \leq 0 \leq y,$$

holds. The results of the “moreover” part are obvious. $\square$

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