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**Aequationes Mathematicae**

# On the structure and solutions of functional equations arising from queueing models

El-sayed El-hady, Janusz Brzdęk, and Hamed Nassar

**Abstract.** It is a survey on functional equations of a certain type, for functions in two complex variables, which often arise in queueing models. They share a common pattern despite their apparently different forms. In particular, they invariably characterize the probability generating function of the bivariate distribution characterizing a two-queue system and their forms depend on the composition of the underlying system. Unfortunately, there is no general methodology of solving them, but rather various ad-hoc techniques depending on the nature of a particular equation; most of the techniques involve advanced complex analysis tools. Also, the known solutions to particular cases of this type of equations are in general of quite involved forms and therefore it is very difficult to apply them practically. So, it is clear that the issues connected with finding useful descriptions of solutions to these equations create a huge area of research with numerous open problems. The aim of this article is to stimulate a methodical study of this area. To this end we provide a survey of the queueing literature with such two-place functional equations. We also present several observations obtained while preparing it. We hope that in this way we will make it easier to take some steps forward on the road towards a (more or less) general solving theory for this interesting class of equations.

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**Keywords.** Two-place functional equation, Boundary value problem, Two-queue system, Probability generating function, PGF, Canonical form.

## 1. Introduction

Queueing theory [8,10–13,18,39,41,43,52,60,64–66,69] plays an important role in computer sciences and operations research. This kind of applied probability has been extensively used in modeling numerous queueing phenomena, which can be observed on the roads, when waiting in line in post offices, emergency rooms in hospitals, when being on a waiting list for surgery [62]. In general a queueing process can be defined as the process of waiting before getting some kind of service. In telecommunication networks, buffers are used to store information that cannot be sent instantly to a next destination. The cause of this is that an instantaneous overload of arriving information may occur, when in a certain time period more information arrives than can be simultaneously transmitted.

In recent years, computer scientists have ended up in their analysis of different queueing systems with challenging two-place functional equations. The whole story can be summarized as follows: Some computer scientists, who are interested in the performance evaluation of communication systems, start with any system (like a switch or a multiplexer) and model it as a queueing system. Since they are interested in studying some processes (like occupancy of the system or delays), they describe the systems mathematically and such a model, in nearly all the cases consists of a set of difference equations that are called balance equations.

Next, the difference equations can be transformed into a functional equation in two complex variables that fits in the general pattern considered in this survey. The transformation can be done by a function called the generating function see [25,31, 68], which in fact is a probability generating function (PGF). In this way some complex analysis tools can be applied in solving these equations. Solutions of such equations immediately provide the various performance measures, which are of interest.

Let us mention that there are three more analysis techniques that can be applied in investigations of these processes; namely, numerical, experimental, and simulation and each of them has its own advantages and disadvantages. The advantage of the analytical approach, described above, is the obvious parameter-dependance of the results obtained.

This article is concerned with the class of FEs arising in models of two-queue systems. The two queues can be either physical or logical; an example of the latter case being a queue with two types of customers: high priority and low priority. In such a case, the two-place FE appears nearly automatically for the two-dimensional probability generating function (PGF) of the double probability sequence, which characterizes the underlying system [22–24]. Namely, if the double sequence $p_{m,n}, m, n = 0, 1, 2, \dots$, characterizes a system, then its PGF $f(x, y)$ is defined as

$$f(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n} x^m y^n, \qquad |x| \leq 1, |y| \leq 1. \tag{1}$$

Applying this definition to a definite two-queue system, one does not get an explicit expression for $f(x, y)$ in terms of $x, y$, but rather a FE defining $f(x, y)$ (see, e.g., [68]), where $f(x, y)$ depends on $f(x, 0)$, $f(0, y)$, and $f(0, 0)$, with

$$f(x,0) = \sum_{m=0}^{\infty} p_{m,0} x^m, \qquad |x| \leq 1,$$

$$f(0,y) = \sum_{n=0}^{\infty} p_{0,n} y^n, \qquad |y| \leq 1,$$

and $f(0, 0) = p_{0,0}$.

As will be seen in this article, two-place FEs arising from queueing models take different forms, depending on the details of the underlying models. The original aim of the present article is to merely survey the literature for these FEs and the various techniques used to solve them. As the survey went ahead, we observed that these FEs, if rearranged, fit in the following common pattern:

$$C_1(x,y)f(x,y) = C_2(x,y)f(x,0) + C_3(x,y)f(0,y) + C_4(x,y)f(0,0) + C_5(x,y) \tag{2}$$

where $C_i, i = 1, \ldots, 5$, are given functions (actually mainly polynomials, not necessarily of the same degree) in two complex variables $x, y$. This form in effect says that each such FE has in general three unknowns and is characterized by five polynomials.

The equations will be presented in a nearly chronological order, the oldest first. A brief description of the solution technique and the underlying model will accompany it. A comparison table is provided at the end of the article showing the differences and similarities between the FEs.

## 2. Auxiliary observations

There is actually mainly one approach used so far in solving equations of type (2). It can be depicted in the following very easy observations, stated below in the form of two theorems.

In what follows $D := \{x \in \mathbb{C} : |x| < 1\}$ and $\overline{D}$ denotes the closure of $D$, i.e.,

$$\overline{D} := \{x \in \mathbb{C} : |x| \leq 1\}.$$

Write

$$\mathcal{K} := \{(x,y) \in \overline{D}^{\,2} : C_1(x,y) = 0\},$$

$$\mathcal{K}_0 := \{x \in \overline{D} : (x,0) \in \mathcal{K}\}, \qquad \mathcal{K}^0 := \{x \in \overline{D} : (0,x) \in \mathcal{K}\}.$$

We need the following two hypotheses.

($\mathcal{A}$) $C_1(x,0) = C_2(x,0), C_3(x,0) = -C_4(x,0), C_5(x,0) = 0$ for $x \in \overline{D} \backslash \mathcal{K}_0$,
($\mathcal{B}$) $C_1(0,y) = C_3(0,y), C_2(0,y) = -C_4(0,y), C_5(0,y) = 0$ for $y \in \overline{D} \backslash \mathcal{K}^0$.

**Theorem 1.** *If a function $P : \overline{D}^{\,2} \to \mathbb{C}$ satisfies Eq. (2) for every $x, y \in \overline{D}$, then there exist functions $g, h : \overline{D} \to \mathbb{C}$ such that $h(0) = g(0)$,*

$$C_2(x,y)h(x) + C_3(x,y)g(y) + C_4(x,y)g(0) + C_5(x,y) = 0, \qquad (x,y) \in \mathcal{K}, \tag{3}$$

*and*

$$P(x,y) = \frac{C_2(x,y)h(x) + C_3(x,y)g(y) + C_4(x,y)g(0) + C_5(x,y)}{C_1(x,y)}, \quad (x,y) \in \overline{D}^{\,2} \backslash \mathcal{K}. \tag{4}$$

*Moreover, if hypotheses ($\mathcal{A}$) and ($\mathcal{B}$) hold and the sets $\mathcal{K}_0$ and $\mathcal{K}^0$ are at most countable, then each continuous function $P : \overline{D}^{\,2} \to \mathbb{C}$, given by (4), with some* *continuous functions* $g, h : \overline{D} \to \mathbb{C}$ *such that* $h(0) = g(0)$ *and* (3) *is valid, satisfies Eq.* (2) *for every* $x, y \in \overline{D}$*; in particular,*

$$P(x,0) = h(x), \qquad P(0,x) = g(x), \qquad x \in \overline{D}. \tag{5}$$

*Proof.* First assume that $P : \overline{D}^{\,2} \to \mathbb{C}$ satisfies Eq. (2) for every $x, y \in \overline{D}$. Write

$$h(x) = P(x,0), \qquad g(x) = P(0,x), \qquad x \in \overline{D}.$$

Take $(x, y) \in \mathcal{K}$. Then, by the definition of $\mathcal{K}$, $C_1(x, y) = 0$

$$\begin{aligned} 0 &= C_1(x,y)P(x,y) \\ &= C_2(x,y)P(x,0) + C_3(x,y)P(0,y) + C_4(x,y)P(0,0) + C_5(x,y) \\ &= C_2(x,y)h(x) + C_3(x,y)g(y) + C_4(x,y)g(0) + C_5(x,y). \end{aligned}$$

Thus we have proved (3).

Now, take $(x, y) \in \overline{D}^{\,2} \setminus \mathcal{K}$. Then $C_1(x, y) \neq 0$ and, by the equation,

$$\begin{aligned} 0 &\neq C_1(x,y)P(x,y) \\ &= C_2(x,y)P(x,0) + C_3(x,y)P(0,y) + C_4(x,y)P(0,0) + C_5(x,y) \\ &= C_2(x,y)h(x) + C_3(x,y)g(y) + C_4(x,y)g(0) + C_5(x,y). \end{aligned}$$

So, dividing both sides by $C_1(x, y)$ we obtain

$$P(x,y) = \frac{C_2(x,y)h(x) + C_3(x,y)g(y) + C_4(x,y)g(0) + C_5(x,y)}{C_1(x,y)}.$$

This proves (4).

Assume now that $(\mathcal{A})$, $(\mathcal{B})$ are fulfilled, $P : \overline{D}^{\,2} \to \mathbb{C}$ is continuous, (4) is valid with some continuous $g, h : \overline{D} \to \mathbb{C}$ such that $h(0) = g(0)$ and (3) holds, and the sets $\mathcal{K}_0$ and $\mathcal{K}^0$ are at most countable. We show that $P$ is a solution to (2).

To this end observe that, by $(\mathcal{A})$, $(\mathcal{B})$ and (4), we have

$$P(x,0) = \frac{C_2(x,0)h(x) + C_3(x,0)g(0) + C_4(x,0)g(0) + C_5(x,0)}{C_1(x,0)} = h(x),$$

$$P(0,y) = \frac{C_2(0,y)h(0) + C_3(0,y)g(y) + C_4(0,y)g(0) + C_5(0,y)}{C_1(0,y)} = g(y)$$

for every $x, y \in \overline{D}$ with $x \notin \mathcal{K}_0$ and $y \notin \mathcal{K}^0$. Consequently, we get (5), because each of the sets $\mathcal{K}_0$ and $\mathcal{K}^0$ is at most countable.

Take $x, y \in \overline{D}$. If $(x, y) \in \mathcal{K}$, then $C_1(x, y) = 0$ and (3) implies (2). If $(x, y) \notin \mathcal{K}$, then (2) results from (5) and (4). □

Theorem 1 shows that, under hypotheses $(\mathcal{A})$ and $(\mathcal{B})$, the main issue in solving equation (2) in some class of functions $P : \overline{D}^{\,2} \to \mathbb{C}$ (e.g., analytic or continuous) is to find all pairs of suitable (analytic or continuous, respectively) functions $g, h : \overline{D} \to \mathbb{C}$ satisfying condition (3) (such functions are uniquely determined for each $P$ in view of (5), provided the sets $\mathcal{K}_0$ and $\mathcal{K}^0$ are 'small' enough).

The next theorem describes a somewhat different case, where

($\mathcal{C}$) $C_2(x,y)=0$ for $x,y\in\overline{D}$.

In this situation we also need the following hypothesis.

($\mathcal{D}$) $C_4(0,y)=C_5(0,y)=0$, $C_3(0,y)=C_1(0,y)$ for $y\in\overline{D}\backslash\mathcal{K}^0$.

**Theorem 2.** *Let hypothesis* ($\mathcal{C}$) *be valid. Assume that* $P:\overline{D}^2\to\mathbb{C}$ *satisfies Eq.* (2) *for every* $(x,y)\in\overline{D}$. *Then there exists a function* $g:\overline{D}\to\mathbb{C}$ *such that*

$$C_3(x,y)g(y)+C_4(x,y)g(0)+C_5(x,y)=0,\qquad (x,y)\in\mathcal{K},\tag{6}$$

*and*

$$P(x,y)=\frac{C_3(x,y)g(y)+C_4(x,y)g(0)+C_5(x,y)}{C_1(x,y)},\qquad (x,y)\in\overline{D}^2\backslash\mathcal{K}.\tag{7}$$

*Moreover, if hypothesis* ($\mathcal{D}$) *holds and the set* $\mathcal{K}^0$ *is at most countable, then each continuous function* $P:\overline{D}^2\to\mathbb{C}$, *given by* (7) *with some continuous function* $g:\overline{D}\to\mathbb{C}$ *fulfilling* (6), *satisfies Eq.* (2) *for every* $x,y\in\overline{D}$; *in particular,*

$$g(y)=P(0,y),\qquad y\in\overline{D}.\tag{8}$$

The proof of Theorem 2 is analogous to that of Theorem 1. Therefore we omit it.

Theorems 1 and 2 show that the main issue in solving equation (2), under hypotheses ($\mathcal{A}$) and ($\mathcal{B}$) (($\mathcal{C}$) and ($\mathcal{D}$), respectively) is finding functions $g,h:\overline{D}\to\mathbb{C}$ ($g:\overline{D}\to\mathbb{C}$, resp.) satisfying condition (3) ((6), resp.). We will see in the further parts of this survey that, in numerous particular cases of Eq. (2), hypotheses ($\mathcal{A}$), ($\mathcal{B}$) or ($\mathcal{C}$), ($\mathcal{D}$) are fulfilled, which seems to be a bit surprising. So, conditions (3) and (6) are very important in the process of solving numerous cases of (2). Actually, condition (6) is a particular case (under ($\mathcal{C}$)) of (3), which has been somehow used in many papers (also in somewhat different forms), only sometimes it is not clearly stated. Unfortunately, we cannot say that there are known successful methods to study it, but several tools have already been applied effectively in investigations of (3) and in the next part we describe some of them.

## 3. Tools used in the investigation of condition (3)

In this section we briefly describe the different tools used throughout the literature to solve numerous particular cases of Eq. (2). Actually they have been mainly applied in investigations of various forms of condition (3) or conditions derived from it.

### 3.1. Rouché Theorem (RT)

Named after the French mathematician Eugene Rouché $(1832-1910)$.

**Statement:** *Let* $L$ *be a closed contour in the complex domain,* $L^+$ *is the interior of the contour* $L$. *Furthermore, let*

$$h_1,h_2:L^+\cup L\to\mathbb{C},$$

*be continuous functions, analytic in* $L^+$, *such that* $|h_2(z)| < |h_1(z)|$ *for all* $z$ *on* $L$. *Then the functions* $h_1(z)$ *and* $h_1(z) + h_2(z)$ *have the same number of zeros inside* $L$.

Rouché's Theorem (see e.g. [5,6,38,58]) is used to simplify the problem of locating zeros: Given an analytic function, we write it as the sum of two parts, one of which is simpler and grows faster than (thus dominates) the other part. We can then locate the zeros by looking at the dominating part only. Examples of applications of Rouché's Theorem can be found in, e.g., [53,54].

Let $a \in \mathbb{C}$ be surrounded by $L$ and $h : L^+ \cup L \to \mathbb{C}$ be a continuous function, which is analytic in $L^+$. Then taking $h_1(z) = z - a$ and $h_2(z) = wh(z)$ for a suitable $w \in \mathbb{C}$, we obtain the following particular case of the Rouché theorem, sometimes called the Lagrange Theorem (see [62, Ch. 1]) and used in, e.g., [53,54].

**Statement:** *Let* $w \in \mathbb{C}$ *be such that the inequality* $|wh_2(z)| < |z - a|$ *holds at all points* $z \in L$. *Then the equation*

$$z = a + wh(z)$$

*in* $z$ *has exactly one root inside* $L$.

### 3.2. Riemann–Hilbert boundary value problem (R–H BVP)

Named after the German mathematicians Bernhard Riemann (1826–1866) and David Hilbert (1862–1943).

**Statement:** *Let* $L$ *be a smooth contour in* $\mathbb{C}$ *and* $a, b, c : L \to \mathbb{R}$ *be Hölder continuous functions such that* $a^2(t) + b^2(t) \neq 0$ *for every* $t \in L$. *Find* (*see e.g.* [7,15,17,47]) *a continuous function* $h : L \cup L^+ \to \mathbb{C}$ *such that*

**R-H1** $h$ *is analytic in* $L^+$;
**R-H2** $\Re[(a(t) - ib(t))h(t)] = c(t)$ *for* $t \in L$.

Let us recall that, as usual, $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of a complex number $z$.

Condition **R-H2** can be written in the following equivalent form

$$a(t)u(t) + b(t)v(t) = c(t),$$

with $u(t) = \Re(h(t))$ and $v(t) = \Im(h(t))$.

The problem can sometimes be solved by a reduction to the Dirichlet Boundary Value Problem (D BVP), which is a particular case of R–H BVP with $L$ being the unit circle $S$ and $a = 1, b = 0$. The reduction is usually made by a conformal mapping used to map the problem onto the closure $\overline{D}$ of the open unit disk $D$ (see, e.g., [21, 59]). Unfortunately the description of such mapping is usually quite complicated, which causes significant difficulties.

### 3.3. Dirichlet boundary value problem (D BVP)

Named after the German mathematician Peter Gustav Lejeune Dirichlet (1805 – 1859). It is well-known that D BVP is a special case of R–H BVP (see Sect. 3.2), where the contour is the unit circle, and $a = 1, b = 0$.

**Statement:** *Let* $v : S \to \mathbb{R}$ *be continuous. Find a continuous function* $h : \overline{D} \to \mathbb{C}$, *which is analytic in* $D$ *and such that* $h(t) = v(t)$ *for* $t \in S$.

The solution of the D BVP is given by $h(z) = \Re u(z)$, where

$$u(z) := \frac{1}{2\pi} \int_S v(t) \frac{t+z}{t-z} \frac{dt}{t} + i\alpha, \quad z \in D \quad \text{(Poisson formula)}$$

and $\alpha$ is a real constant (see, e.g., [17]).

There are many modifications of the original R–H BVP depending on the contour $L$ and the properties of the unknown function. For instance, $L$ could be the boundary of a disjoint union of simply connected domains and then we obtain the Riemann Boundary Value Problem.

### 3.4. Riemann BVP (R BVP)

Named after Bernhard Riemann.

**Statement:** *Let* $L$ *be a smooth contour,* $h_1, h_2 : L \to \mathbb{C}$ *be Hölder continuous functions, and* $h_1(t) \neq 0$ *for every* $t \in L$. *Find a sectionally analytic function* $h : L^+ \cup L^- \to \mathbb{C}$ *such that*

**R1** *the restriction of* $h$ *to* $L^+$ *has a continuous extension* $h^+ : L \cup L^+ \to \mathbb{C}$;
**R2** *the restriction of* $h$ *to* $L^-$ *has a continuous extension* $h^- : L \cup L^- \to \mathbb{C}$;
**R3** $h$ *is bounded at infinity;*
**R4** $h^+(t) = h_1(t)h^-(t) + h_2(t)$ *for* $t \in L$.

If $h_2(t) \equiv 0$, then the problem is called the homogenous R BVP; otherwise it is called non-homogenous. For details concerning solutions of R BVP we refer to, e.g., [15,17,30,47].

The main difficulties in applications of R BVP in solving equations of type (2) arise while selecting a proper smooth contour on the Riemann surface $\mathcal{K} := \{(x, y) \in \mathbb{C}^2 : C_1(x, y) = 0\}$.

### 3.5. Analytic continuation (AC) principle

Let $h_i$, $i = 1, 2$, be an analytic functions defined on open subsets $A_i \subset \mathbb{C}$. Assume that $A_1 \cap A_2 \neq \emptyset$ and $h_1(z) = h_2(z)$ for all $z \in A_1 \cap A_2$. Then we say that $h_2$ is an AC of $h_1$ (see e.g. [5,6,38,58]); clearly, $h_2$ must be unique (if it exists). A possibility of existence of such AC is described in the following theorem (see e.g. [6,38,58]).

**Monodromy Theorem:** *Let* $A_2 \subset \mathbb{C}$ *be a simply connected domain and* $h_1$ *be analytic in a domain* $A_1 \subset A_2$. *If the function* $h_1$ *can be analytically continued along*

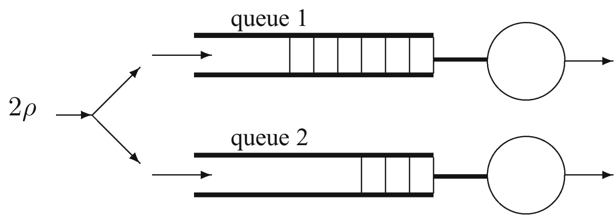


FIGURE 1. Two parallel queues with the customer joining the shorter

*any curve in* $A_2$*, then there is a single-valued analytic function* $h_2 : A_2 \to \mathbb{C}$ *with* $h_1(z) = h_2(z)$ *for* $z \in A_1$.

## 4. Particular cases of Eq. (2)

### 4.1. FE 1: Two parallel queues with the customer joining the shorter

This equation arises [40] from a queueing model consisting of two unbounded single server queues, illustrated in Fig. 1. The arrivals are assumed to be a Poisson process with mean $2\rho$, and the service time is assumed to be exponential with mean 1. If one queue is shorter, the customer joins it. Else, if the two queues are equal, the customer joins either with probability $1/2$. A state of statistical equilibrium is reached whenever $\rho < 1$.

The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$\begin{aligned}(x(2\rho x + 1) - 2(1+\rho)xy + y^2)f(x,y) &= y(y-x)f(0,y)\\ &+ (x(2\rho x+1) - (1+\rho)xy - \rho xy^2)f(x,0). \end{aligned} \tag{9}$$

A description of solutions has been provided in [40].

This FE can be rewritten in the form (2) with $C_4(x,y) = 0$, $C_5(x,y) = 0$,

$$\begin{aligned}&C_1(x,y) = x(1+2\rho x) - 2(1+\rho)xy + y^2,\\ &C_2(x,y) = x(1+2\rho x) - (1+\rho)xy - \rho xy^2, \qquad C_3(x,y) = y\,(y-x)\,,\end{aligned}$$

whence hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ hold.

The same FE has been considered in [1,3,4] with the compensation approach [9], which yielded explicit relations for the coefficients in infinite linear combination of product forms and thus an explicit characterization of the equilibrium probabilities.

### 4.2. FE 2: Symmetric two-node aloha network

This equation arises [61] from a packet radio model having two symmetric, interfering queues, as illustrated in Fig. 2. It is assumed that the arrivals into the queues are

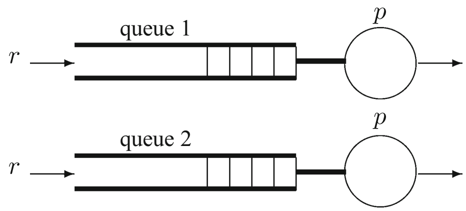


FIGURE 2. Symmetric two-node aloha network

independent Bernoulli processes with an equal rate $< r \in (0, 1/4)$, served with an equal rate $p \in (0, 1)$. The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system fulfils a FE written in the form

$$\begin{aligned} f(x,y) = g(x,y)p&\frac{(x(y-1) - p(2xy-x-y))f(x,0)}{xy - g(x,y)((x+y)p\widetilde{p} + xy(p^2+\widetilde{p}^2))} \\ +g(x,y)p&\frac{(y(x-1) - p(2xy-x-y))f(0,y)}{xy - g(x,y)((x+y)p\widetilde{p} + xy(p^2+\widetilde{p}^2))} \\ +g(x,y)p&\frac{p(2xy-x-y)f(0,0)}{xy - g(x,y)((x+y)p\widetilde{p} + xy(p^2+\widetilde{p}^2))}, \end{aligned} \tag{10}$$

where

$$\widetilde{p} = 1 - p, \qquad g(x,y) = (xr + \widetilde{r})(yr + \widetilde{r}), \qquad \widetilde{r} = 1 - r.$$

Clearly, with some manipulation this FE can be rewritten in the form (2) with

$$\begin{aligned} C_1(x,y) &= xy - (xr+\widetilde{r})(yr+\widetilde{r})((x+y)p\widetilde{p} + xy(p^2+\widetilde{p}^2)), \\ C_2(x,y) &= p(xr+\widetilde{r})(yr+\widetilde{r})(x(y-1) - p(2xy-x-y)), \\ C_3(x,y) &= p(xr+\widetilde{r})(yr+\widetilde{r})(y(x-1) - p(2xy-x-y)), \\ C_4(x,y) &= p^2(xr+\widetilde{r})(yr+\widetilde{r})(2xy-x-y), \qquad C_5(x,y) = 0, \end{aligned}$$

and then hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ are valid.

Up to now there is no description of solutions available to Eq. (10).

### 4.3. FE 3: Inventory control of database systems

Now, we present the equation, which arises [27] from a double queue model, illustrated in Fig. 3, where the arriving customers simultaneously place two demands handled independently by two servers. The arrivals are assumed to be a Poisson process with mean 1, and the two servers have exponential service times with rates $\alpha, \beta$, with the stability condition $1 < \alpha \leq \beta$. The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$Q(x,y)f(x,y) = \beta x(y-1)f(x,0) + \alpha y(x-1)f(0,y), \tag{11}$$

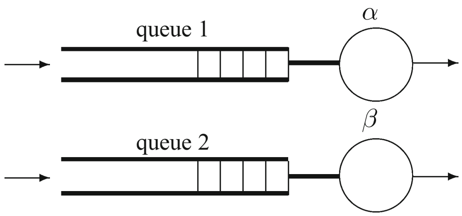


FIGURE 3. Inventory control of database systems

where

$$Q(x,y) = (1+\alpha+\beta)xy - \alpha y - \beta x - x^2y^2.$$

By a parametrization of the kernel $\mathcal{K} := \{(x,y) : Q(x,y) = 0\}$ with a pair of elliptic functions $x = x(t), y = y(t)$, the functional equation has been converted into a set of conditions on $f(x(t),0)$ and $f(0,y(t))$, which in turn lead to the formulas for $f(x,0)$ and $f(0,y)$. Clearly the FE has the form (2) with hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ fulfilled.

Let us also mention that Flatto [26] analyzed the asymptotic behavior of $p_{m,n}$ (see (1)) as $m, n \to \infty$.

### 4.4. FE 4: Two parallel queues with batch server

The next equation of the form (2) arises [20,21] from a model consisting of two parallel $M/M/1$ queues with infinite capacities, illustrated in Fig. 4. It is assumed that the arrivals form two independent Poisson processes with parameters $\lambda_1, \lambda_2$, and that the service times are distributed exponentially with instantaneous service rates $S_1$ and $S_2$ depending on the system state in the following manner:

- $S_i = \mu_i$ for $i = 1, 2$, if both queues are nonempty;
- $S_i = \mu_i^*$ for $i = 1, 2$, if queue $i$ is empty.

The PGF $f(x,y)$ of the two-dimensional distribution characterizing the system yields the FE

$$T(x,y)f(x,y) = a(x,y)f(x,0) + b(x,y)f(0,y) + c(x,y)f(0,0), \tag{12}$$

where

$$a(x,y) = \mu_1\Big(1-\frac{1}{x}\Big) + q\Big(1-\frac{1}{y}\Big),$$
$$b(x,y) = \mu_2\Big(1-\frac{1}{y}\Big) + p\Big(1-\frac{1}{x}\Big),$$
$$c(x,y) = p\Big(\frac{1}{x}-1\Big) + q\Big(\frac{1}{y}-1\Big),$$

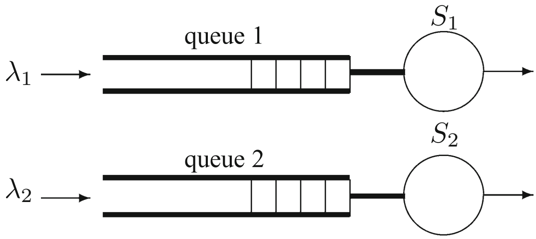


FIGURE 4. Two parallel queues with batch server

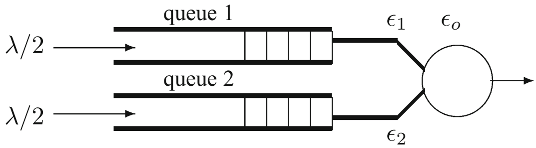


FIGURE 5. Two types of customers with one server

$$T(x,y) = \lambda_1(1-x) + \mu_1\Big(1-\frac{1}{x}\Big) + \lambda_2(1-y) + \mu_2\Big(1-\frac{1}{y}\Big),$$

and $p = \mu_1 - \mu_1^*$, $q = \mu_2 - \mu_2^*$.

Having done a suitable analysis of the set $\mathcal{T} := \{(x,y) : xyT(x,y) = 0\}$, the authors obtained descriptions of solutions by solving a corresponding D BVP problem for a circle (when $pq = \mu_1\mu_2$) and a homogenous R–H BVP problem for a circle (when $pq \neq \mu_1\mu_2$).

Note that multiplying both sides of (12) by $xy$, we obtain (2) with

$$C_1(x,y) = \lambda_1(xy - x^2y) + \mu_1(xy - y) + \lambda_2(xy - xy^2) + \mu_2(xy - x),$$
$$C_2(x,y) = \mu_1(xy-y) + q(xy-x), \qquad C_3(x,y) = \mu_2(xy-x) + p(xy-y),$$

and

$$C_4(x,y) = p(y - xy) + q(x - xy), \qquad C_5(x,y) = 0.$$

Certainly, in this way we may additionally admit the values $x = 0$ and $y = 0$ in the domain of $f$. Then it is easily seen that hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ are fulfilled only for some special values of $\mu_1$, $\mu_2$, $q$ and $p$.

### 4.5. FE 5: Two types of customers with one server

Now we present an equation, which arises [15] from a one-server-two-queue model with two types of customers, as illustrated in Fig. 5.

The arrival streams of both types of customers are assumed to be independent Poisson processes each with arrival rate $\frac{1}{2}\lambda$, $\epsilon_i$ being the probability that a customer of type $i$ is served for $i = 1, 2$, and $\epsilon_0$ the probability that a single customer of type 1 or of type 2 is served. The positive constants $\epsilon_0, \epsilon_1$ are such that

$$2\epsilon_0 + \epsilon_1 = 1, \qquad \epsilon_0 < \frac{1}{2}.$$

The PGF $f$ of the two-dimensional distribution characterizing the system yields a FE, which is written in [15] in the form

$$\begin{aligned}
&\Big(1 - \Big(\frac{1}{x} + \frac{1}{y}\Big)\epsilon_0\alpha(\chi) - \frac{\epsilon_1}{xy}\alpha^2(\chi)\Big)xyf(x,y) \\
&\quad = \Big(\frac{1}{x}\beta(\chi) - \Big(\frac{1}{x} + \frac{1}{y}\Big)\epsilon_0\alpha(\chi) - \frac{\epsilon_1}{xy}\alpha^2(\chi)\Big)xyf(x,0) \\
&\qquad + \Big(\frac{1}{y}\beta(\chi) - \Big(\frac{1}{x} + \frac{1}{y}\Big)\epsilon_0\alpha(\chi) - \frac{\epsilon_1}{xy}\alpha^2(\chi)\Big)xyf(0,y) \\
&\qquad + \Big(\gamma(\chi) - \Big(\frac{1}{x} + \frac{1}{y}\Big)\beta(\chi) + \Big(\frac{1}{x} + \frac{1}{y}\Big)\epsilon_0\alpha(\chi) + \frac{\epsilon_1}{xy}\alpha^2(\chi)\Big)xyf(0,0),
\end{aligned}$$

with $\chi = \lambda(1 - \frac{x+y}{2})$ and

$$\alpha(\rho) = \int_0^\infty e^{-\rho t} dA(t), \qquad \beta(\rho) = \int_0^\infty e^{-\rho t} dB(t), \qquad \gamma(\rho) = \int_0^\infty e^{-\rho t} dC(t)$$

for $\Re(\rho) \geq 0$, where $A$, $B$, and $C$ are probability distributions with

$$\int_0^\infty t dA(t) > 0, \qquad \int_0^\infty t dB(t) > 0, \qquad \int_0^\infty t dC(t) > 0.$$

A description of solutions to the equation has been obtained by a reduction to a quite involved R BVP.

The FE can be rewritten in the form (2) with

$$\begin{aligned}
C_1(x,y) &= xy - (y+x)\epsilon_0\alpha(\chi) - \epsilon_1\alpha^2(\chi), \\
C_2(x,y) &= y\beta(\chi) - (y+x)\epsilon_0\alpha(\chi) - \epsilon_1\alpha^2(\chi), \\
C_3(x,y) &= x\beta(\chi) - (y+x)\epsilon_0\alpha(\chi) - \epsilon_1\alpha^2(\chi), \\
C_4(x,y) &= \gamma(\chi)xy - (y+x)\beta(\chi) + (y+x)\epsilon_0\alpha(\chi) + \epsilon_1\alpha^2(\chi),
\end{aligned}$$

and $C_5(x,y) = 0$. It is easily seen that then hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ are fulfilled.

### 4.6. FE 6: Two parallel processors with coupled inputs

The model considered in [70] concerns a system consisting of two processors serving three job streams, illustrated in Fig. 6, generated by independent Poisson sources.

The central job stream of rate $\nu$ consists of jobs which place resource demands on both processors, which are handled separately by each processor once the request is made. In addition, the first processor receives background work at a rate $\lambda$ while

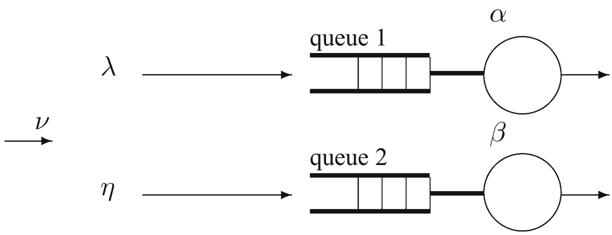


FIGURE 6. Two parallel processors with coupled inputs

the second receives similar tasks at a rate $\eta$. Each processor has exponentially distributed service times with rates $\alpha$ and $\beta$, respectively. The system considered here is a generalization of a two-server system considered by Flatto and Hahn [27] and Flatto [26].

The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields FE (2) with

$$C_1(x, y) = \gamma xy - \lambda x^2 y - \eta x y^2 - \alpha y - \beta x - \nu x^2 y^2,$$
$$C_2(x, y) = \beta x(y - 1), \quad C_3(x, y) = \alpha(x - 1)y, \quad C_4(x, y) = C_5(x, y) = 0,$$

where $\gamma = \alpha + \beta + \nu + \lambda + \eta$. Clearly, hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ are valid.

A suitable parametrization of the curve $C_1(x, y) = 0$ by a pair of elliptic functions $x(\xi)$ and $y(\xi)$ allowed in [70] to obtain the relation between $f(x(\xi), 0)$ and $f(0, y(\xi))$ (persisting also for their analytic continuations as elliptic functions), which yielded a description of a solution.

The same model was used in [42] with the same FE to show that the kernel method can be applied to a two-dimensional queueing system for exact tail asymptotics in stationary joint distribution and also with two marginal distributions.

### 4.7. FE 7: Asymmetric clocked buffered switch

A model of the asymmetric $2 \times 2$ clocked buffered switch, as illustrated in Fig. 7, has been considered in [16] with the message handling process of this switch modeled as a two-server, time slotted, queueing process with a state space consisting of pairs of the numbers of messages $(x_n, y_n)$ present at the servers at the end of a time slot. The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$(xy - \phi(x, y))f(x, y) = (x - 1)(y - 1)\phi(x, y)\Big[\frac{f(x, 0)}{(x - 1)} + \frac{f(0, y)}{(y - 1)} + f(0, 0)\Big], \tag{13}$$

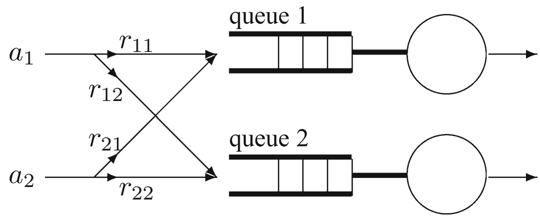


FIGURE 7. Asymmetric 2 × 2 switch

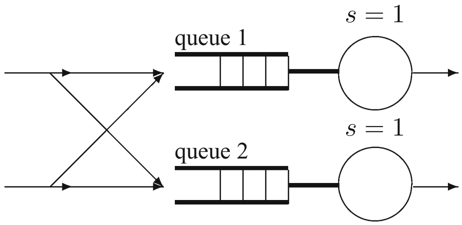


FIGURE 8. Symmetric 2 × 2 switch

where

$$\phi(x, y) = [1 - a_1 + a_1(r_{11}x + r_{12}y)][1 - a_2 + a_2(r_{21}x + r_{22}y)],$$

$a_1, a_2$ are the probabilities that the arrival stream generated at the start of a slot is of stream 1 or 2 respectively, and $r_{i,j}$ is the probability that an $i-$arrival joins the queue of the $j-$th service facility, for $i, j = 1, 2$. Solutions to the equation have been investigated by analysis of the zeros and poles of the functions $f(x, 0)$ and $f(0, y)$ with a support of AC.

Since Eq. (13) has the form (2) with

$$C_1(x, y) = xy - \phi(x, y), \qquad C_2(x, y) = (y - 1)\phi(x, y),$$
$$C_3(x, y) = (x - 1)\phi(x, y), \qquad C_4(x, y) = (x - 1)(y - 1)\phi(x, y),$$

and $C_5(x, y) = 0$, it is easily seen that hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ are valid.

### 4.8. FE 8: 2 × 2 symmetric switch

A model of a clocked buffered switch with two input ports and two output ports has been investigated in [2]; it is illustrated in Fig. 8.

The server serves exactly one job per time unit, provided the buffer is nonempty. The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields

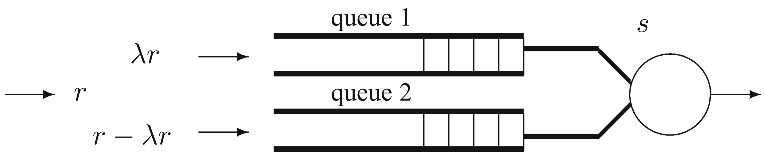


FIGURE 9. Multimedia multiplexers

the FE

$$(xy - r(x,y))f(x,y) = (y-1)r(x,0)f(x,0) + (x-1)r(0,y)f(0,y) + (x-1)(y-1)r(0,0)f(0,0), \tag{14}$$

where

$$r(x,y) = (1 - p + \frac{p}{2}(x+y))^2,$$

with $p = r_1 = r_2$, where the number of arriving jobs of type $i$ in a time unit (i.e., clock cycle) is one with probability $r_i$ and zero with probability $1 - r_i$, $i = 1, 2$. A reduction to a R–H BVP yielded a description of solutions to (14).

Note that (14) can be written in the form (2) and hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ are valid.

### 4.9. FE 9: Multimedia multiplexers

A model of a multimedia multiplexer, handling traffic of two classes, has been studied in [54]. It is illustrated in Fig. 9.

One class represents real-time traffic, for instance packets of live audio or video transmissions, and the other nonreal-time traffic packets of file transfer transmissions. The system thus can be looked at as having two logical queues, one for each class. The packets arrive into the server (multiplexer) in batches, with $r$ being the packet arrival rate regardless of class, and is given by

$$r = r_1 + r_2,$$

where $r_1 = \lambda r$ is the class 1 arrival rate, $r_2 = \widetilde{\lambda} r$ is the class 2 arrival rate, and $\lambda, \widetilde{\lambda}$ are the probabilities that a cell of class 1 or class 2 arrive respectively. The service time is assumed to be geometric with rate $s$. The PGF $f(x,y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$f(x,y) = \frac{A_1(x)A_2(y)s[(x-y)f(0,y) + x(y-1)f(0,0)]}{y[x - A_1(x)A_2(y)(s + \widetilde{s}x)]}, \tag{15}$$

with

$$A_1(x) = \sum_{n=0}^{\infty} \Pr[A_1^k = n]\, x^n, \qquad A_2(y) = \sum_{n=0}^{\infty} \Pr[A_2^k = n]\, y^n,$$

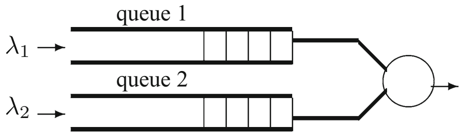


FIGURE 10. ATM queue with a priority scheduling

and

$$f(0,0) = \frac{s-r}{s},$$

where $A_i^k$ stands for the number of packets of class $i$ in the slot $k$.

The unknown $f(0,y)$ in Eq. (15) has been found in the form

$$f(0,y) = \frac{\xi(y)(y-1)(s-r)}{y-\xi(y)}, \quad y \neq \xi(y),$$

where (for a fixed $y$) $\xi(y) \in \overline{D}$ is the zero of the denominator of (15) (which must be unique in view of Rouché's theorem).

Equation (15) can be rewritten in the form (2) with

$$C_1(x,y) = y(x - A_1(x)A_2(y)(s+\widetilde{s}x)), \qquad C_2(x,y) = 0,$$

$$C_3(x,y) = s(x-y)A_1(x)A_2(y), \qquad C_4(x,y) = x(y-1)sA_1(x)A_2(y),$$

and $C_5(x,y) = 0$. So, clearly hypotheses $(\mathcal{C})$ and $(\mathcal{D})$ are valid.

### 4.10. FE 10: ATM queue with a priority scheduling

Another equation of type of (2) arises [67] from a discrete-time queueing system with head-of-line priority. It is assumed that there are two types of traffic arriving in the system, namely cells of class 1 (delay-sensitive traffic) and cells of class 2 (delay-insensitive traffic), which is illustrated in Fig. 10.

The number of arrivals of class $j$ during slot $k$ is denoted by $a_{j,k}$ $(j = 1; 2)$. Both types of cell arrivals are assumed to be independent and identically distributed (i.i.d.) from slot to slot and are characterized by the joint probability mass function (PMF)

$$a(m,n) = \Pr[a_{1,k} = m, a_{2,k} = n]$$

and joint PGF $\check{A}(x,y)$,

$$\check{A}(x,y) = \mathbb{E}[x^{a_{1,k}}y^{a_{2,k}}].$$

The PGF $f(x,y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$f(x,y) = \frac{\check{A}(x,y)(x-y)f(0,y) + \check{A}(x,y)x(y-1)f(0,0)}{y(x-\check{A}(x,y))}; \tag{16}$$

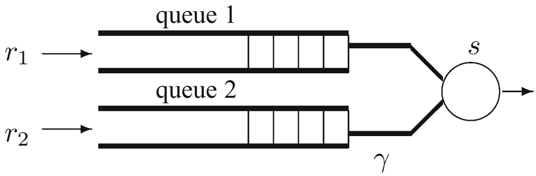


FIGURE 11. One port of the switch transmitting traffic over unreliable channels

in some application to an ATM output-queueing switch considered in [67] the joint PGF is given by

$$\check{A}(x,y) = \Big(1 + \frac{\lambda_1}{N}(x-1) + \frac{\lambda_2}{N}(y-1)\Big)^N,$$

where $N$ denotes the number of inlets and outlets of the switch and the mean arrival rate of class $j$ units is $\lambda j$. The authors solve (16) by showing that, for a given value of $y \in D$, the equation $x = \check{A}(x,y)$ has one solution in the unit disc for $x$.

Equation (16) can be rewritten in the form (2) with

$$C_1(x,y) = y(x - \check{A}(x,y)), \qquad C_2(x,y) = 0,$$
$$C_3(x,y) = \check{A}(x,y)(x-y), \qquad C_4(x,y) = \check{A}(x,y)x(y-1), \qquad C_5(x,y) = 0,$$

so hypotheses $(\mathcal{C})$ and $(\mathcal{D})$ are valid.

### 4.11. FE 11: ATM switch transmitting two-class traffic over unreliable channels

The next equation arises [51] from a model of a space division output buffered switch operating in an ATM multimedia environment. The port is modeled as two logical queues with one server offering two service rates, as illustrated in Fig. 11, with $r_1, r_2$ being the arrival rates of class 1 (representing real time communications), and class 2 (representing nonreal time communications) respectively, $\gamma$ the service rate of class 2 packets, and $s$ the overall packet service rate, regardless of class. The PGF $f(x,y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$f(x,y) = \frac{\widehat{A}(x,y)(\widetilde{\gamma}xy + \gamma x - y)f(0,y) + \widehat{A}(x,y)(\gamma\widetilde{r}_1 - r_2)x(y-1)}{y(x - \widehat{A}(x,y))}, \quad (17)$$

where $r = r_1 + r_2$,

$$\widehat{A}(x,y) = \Big(1 - \frac{r}{N} + \frac{1}{N}(r_1 x + r_2 y)\Big)^N,$$

$N$ is the number of input/output ports, and $s$ is related to $\gamma$ through the relation

$$s = \frac{r\gamma}{r_1\gamma + r_2}.$$

A description of $f(0,y)$ is found to be

$$f(0,y)=\frac{(\gamma\widetilde{r}_1-r_2)\xi_p(y)(1-y)}{\widetilde{\gamma}\xi_p(y)y+\gamma\xi_p(y)-y},$$

where (for a given $y$) $\xi_p(y)\in\overline{D}$ is the zero of the denominator of (17) (the uniqueness of the zero has been shown by Rouché's theorem).

This FE can be rewritten in the form (2) with

$$\begin{aligned}&C_1(x,y)=y(x-\widehat{A}(x,y)), \qquad C_2(x,y)=0,\\ &C_3(x,y)=\widehat{A}(x,y)(\widetilde{\gamma}xy+\gamma x-y), \qquad C_4(x,y)=0,\\ &C_5(x,y)=\widehat{A}(x,y)(\gamma\widetilde{r}_1-r_2)x(y-1).\end{aligned}$$

It is easily seen that hypotheses $(\mathcal{C})$ and $(\mathcal{D})$ are fulfilled.

### 4.12. FE 12: Two tandem queues with coupled servers

Now we present the equation that arises [59] (see also [63]) from a two-stage tandem queue model, illustrated in Fig. 12, where customers (jobs) arrive at queue-1 according to a Poisson process with rate $\lambda$. Each job demands service at both queues before leaving the system, with the time spent in each server $j$ being exponential with parameter $\nu_j$ at station $j; j=1,2$. It is assumed that the total service capacity equals one unit of work per time unit. Whenever both stations are nonempty, a proportion $0\le p\le 1$ of the capacity is allocated to station 1, and the remaining part $(1-p)$ is allocated to station 2, with $\rho_j=\lambda/\nu_j$ being the average amount of work per time unit required at station $j, j=1,2$. The PGF $f(x,y)$ of the two-dimensional distribution characterizing the system yields the FE

$$\begin{aligned}&((\lambda+p\nu_1+(1-p)\nu_2)xy-\lambda x^2y-p\nu_1y^2-(1-p)\nu_2x)f(x,y)\\ &\quad=(1-p)(\nu_1y(y-x)+\nu_2x(y-1))f(x,0)+p(\nu_2x(1-y)\\ &\qquad+\nu_1y(x-y))f(0,y)+(p\nu_2x(y-1)+(1-p)\nu_1y(x-y))f(0,0). \quad (18)\end{aligned}$$

Different cases of $p$ have been considered; for example, for $0<p<1$, a description of $f(0,y)$ has been obtained by solving R–H BVP, in a classical way, by transforming it via some conformal mapping to the unit disk.

Clearly, (18) can be written in the form (2) and hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ hold.

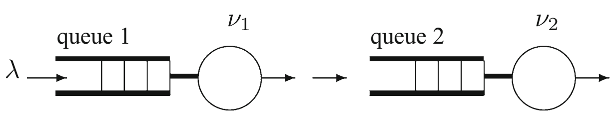


FIGURE 12. Two tandem queues with coupled servers

### 4.13. FE 13: Two class buffer system with batch arrivals

The next equation arises [57] from a queueing model of a buffer receiving customers (packets) of two classes, illustrated in Fig. 13, with class 1 receiving higher service priority than class 2.

The service time is geometric, and the packets arrive in batches of general size at the rate of one batch per slot. The PGF $f(x,y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$f(x,y) = \frac{s(x-y)(\lambda X(A_1(x)) + \widetilde{\lambda} X(A_2(y)))f(0,y)}{sy(x-\lambda X(A_1(x)) - \widetilde{\lambda} X(A_2(y)))} + \frac{(\lambda X(A_1(x)) + \widetilde{\lambda} X(A_2(y)))(s-r)x(y-1)}{sy(x-\lambda X(A_1(x)) - \widetilde{\lambda} X(A_2(y)))}, \tag{19}$$

where $X$ is the common PGF of $X^k$, given by

$$X(z) = \frac{sz}{1-\bar{s}z},$$

$A_i$ is the common PGF of $A_i^k$ for $i=1,2$, $\lambda$ is the probability that a batch of $A_1^k = 0,1,\ldots,$ class 1 packets arrive, $\widetilde{\lambda} := 1-\lambda$ is the probability that a batch of $A_2^k = 0,1,\ldots,$ class 2 packets arrive, and $r$, $s$ are the arrival and service rates, respectively.

A description of $f(0,y)$ has been found in the form

$$f(0,y) = -\frac{(s-r)(y-1)\xi}{s(\xi-y)}, \qquad \xi \neq y,$$

where $\xi$ is the zero of the denominator of (19) for a fixed $y$, which is unique in view of Rouché's theorem.

This FE can be rewritten in the form (2) with

$$C_1(x,y) = sy(x-\lambda X(A_1(x)) - \widetilde{\lambda} X(A_2(y))), \qquad C_2(x,y) = 0,$$
$$C_3(x,y) = s(x-y)(\lambda X(A_1(x)) + \widetilde{\lambda} X(A_2(y))), \qquad C_4(x,y) = 0,$$
$$C_5(x,y) = (s-r)x(y-1)(\lambda X(A_1(x)) + \widetilde{\lambda} X(A_2(y))),$$

with hypotheses $(\mathcal{C})$ and $(\mathcal{D})$ fulfilled.

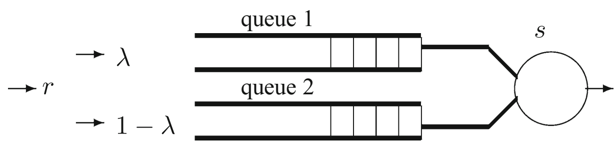


FIGURE 13. Two class system with batch arrivals

### 4.14. FE 14: Multimedia ATM multiplexer with homogenous arrivals

Another equation arises [53] from a queueing model of an ATM multichannel multiplexer (server) handling multimedia traffic (customers) of two classes: real time class and nonreal time class, illustrated in Fig. 14.

The arrivals are assumed to be homogenous batches in the sense that either the arriving batch is all class 1 packets or all class 2 packets. The multiplexer is modeled as a priority discrete time queueing system with batch arrivals and geometric service. The PGF $f(x,y)$ of the two-dimensional distribution characterizing the system yields the FE

$$y(x-\Psi(x,y))sf(x,y)=[(x-y)s\,f(0,y)+x(y-1)(s-r)]\Psi(x,y), \quad (20)$$

where

$$\Psi(x,y)=\lambda X(A_1(x))+\widetilde{\lambda}X(A_2(y)),$$

$\lambda$ is the probability that a batch of $A_1^k=0,1,\dots,$ packets of class 1 arrive, $\widetilde{\lambda}=1-\lambda$ is the probability that a batch of $A_2^k=0,1,\dots,$ packets of class 2 arrive, and

$$X(x)=\frac{sx}{1-\widetilde{s}x},$$

with $r,s$ being the arrival and service rates, respectively. $A_1^k$ and $A_2^k$ are random variables (RVs) with arbitrary distributions, independent and identically distributed (iid).

The authors mainly applied Rouché's theorem to solve the equation, which can be rewritten in the form (2) with

$$C_1(x,y)=y(x-\Psi(x,y))s,\quad C_2(x,y)=0,\quad C_3(x,y)=(x-y)s\Psi(x,y),$$
$$C_4(x,y)=0,\qquad C_5(x,y)=x(y-1)(s-r)\Psi(x,y).$$

As before, hypotheses $(\mathcal{C})$ and $(\mathcal{D})$ are valid.

### 4.15. FE 15: ATM multichannel switch routing multimedia traffic

One more equation arises [55] from a queueing model of a space division output buffered multichannel switch (server) operating in an ATM multimedia environment, illustrated in Fig. 15.

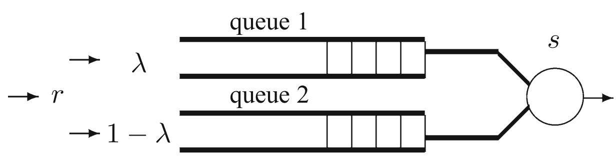


FIGURE 14. ATM multiplexer with homogenous arrivals

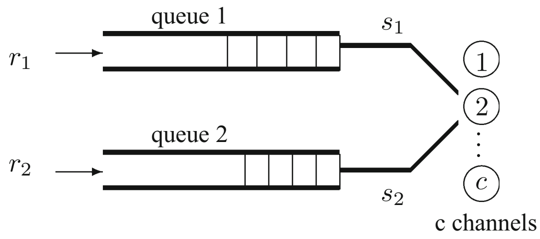


FIGURE 15. ATM multichannel switch

The switch operates at two service rates, one for each class. The switch is modeled as a priority, discrete time, batch arrival, multiserver queueing system, with infinite buffer and two geometric service times. It is assumed that $r_1$and $r_2$ are the arrival rates, and $s_1$ and $s_2$ are the service rates of class 1 and class 2, respectively. In the special case where $c = 1$ (one server), the PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$f(x,y) = \frac{\widehat{A}(x,y)(s_1((s_1-s_2)xy+s_2x-s_1y))f(0,y)}{ys_1(x-(s_1+\widetilde{s}_1x)\widehat{A}(x,y))} + \frac{\widehat{A}(x,y)\,x(s_1(s_2-r_2)-r_1s_2)(y-1)}{ys_1(x-(s_1+\widetilde{s}_1x)\widehat{A}(x,y))}, \tag{21}$$

where $r = r_1 + r_2$,

$$\widehat{A}(x,y) = \left(1 - \frac{r}{N} + \frac{1}{N}(r_1x + r_2y)\right)^N,$$

and $N$ represents the number of ports in the switch. The form of $f(0, y)$ has been found to be

$$f(0,y) = -\frac{(s_1s_2 - r_1s_2 - r_2s_1)\xi(y)(y-1)}{s_1((s_1-s_2)\xi(y)y + s_2\xi(y) - s_2y)},$$

where $\xi(y)$ is defined as the unique (in view of Rouché's theorem) $x \in \overline{D}$ such that the denominator of (21) is equal to zero, with that given $y$.

Note that this FE can be rewritten in the form (2) with

$$C_1(x,y) = ys_1(x-(s_1+\widetilde{s}_1x)\widehat{A}(x,y)), \qquad C_2(x,y) = 0,$$
$$C_3(x,y) = \widehat{A}(x,y)(s_1((s_1-s_2)xy+s_2x-s_1y)), \qquad C_4(x,y) = 0,$$
$$C_5(x,y) = \widehat{A}(x,y)x(s_1(s_2-r_2)-r_1s_2)(y-1),$$

and then hypotheses $(\mathcal{C})$ and $(\mathcal{D})$ are valid.

### 4.16. FE 16: LAN gateway

Now, we present the equation that arises [49] (see also [50]) from a LAN gateway model, illustrated in Fig. 16. The gateway has two buffers, one (Queue I) to queue packets going from LAN I to LAN II, and the other (Queue II) to queue packets going in the other direction.

The queue I arrival rate is $0 < \xi_1 := r_1 s_1 < 1$, where $r_1$ is the traffic load on LAN I, and $s_1$ is the fraction of it destined to LAN II. For queue II, $0 < \xi_2 := r_2 s_2 < 1$, with analogous definitions. The service rates of the two queues are $\widetilde{r}_1$ and $\widetilde{r}_2$, respectively. The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields the FE

$$\begin{aligned}(xy - M(x,y))\, f(x,y) = {} & (y-1)(M(x,0) + \widetilde{r}_1 \xi_2 xy) f(x,0) \\ & + (x-1)(M(0,y) + \widetilde{r}_2 \xi_1 xy) f(0,y) \\ & + (x-1)(y-1) M(0,0) f(0,0), \end{aligned} \tag{22}$$

where $\widetilde{u} := 1 - u$ for each real number $u$ and

$$M(x,y) = (\widetilde{r}_1 + r_1 \widetilde{s}_1 y + \xi_1 xy)(\widetilde{r}_2 + r_2 \widetilde{s}_2 x + \xi_2 xy).$$

The article leaves the equation unsolved, mentioning the traditional difficulty to solve such equations. A quite precise description of solutions, but only in the case $s_1 < 1/2$ and $r_2 < (1 - r_1)/(2 - s_2)$, has been provided in [14]. In the general case, an explicit closed-form description of solution has recently been proposed in [56]. But, actually it has not been proved that it really depicts a solution to (22); only some validation processes have been done.

Clearly, this FE is of the form (2) with hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ fulfilled.

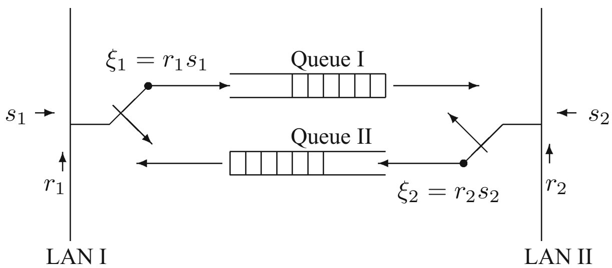


FIGURE 16. Gateway modeled as two back-to-back interfering queues

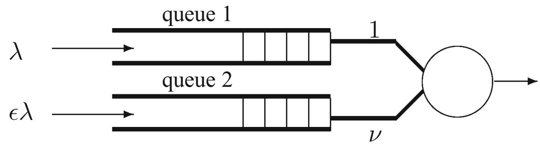


FIGURE 17. Processor sharing for two queues with vastly different rates

### 4.17. FE 17: Processor sharing for two queues with vastly different rates

The following equation arises [45] from a 2−class queueing system, operating under a generalized processor-sharing discipline, depicted in Fig. 17 in an asymptotic regime where the arrival and service rates of the two classes are vastly different.

The jobs arrive as Poisson processes of rates $\lambda$ (the fast class) and $\epsilon\lambda$ (the slow class), where $0 < \epsilon \ll 1$. The fast and slow jobs have exponentially distributed service requirements with parameters 1 and $\nu$, respectively. The server works at unit rate, and if neither queue is empty it devotes fractions $1-\gamma$ and $\gamma$ of its effort to the fast and slow jobs, respectively, where $0 \le \gamma \le 1$. The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields a FE written in the form

$$\begin{aligned}
&\Big[\lambda(1-x) + (1-\gamma)\Big(1-\frac{1}{x}\Big) + \epsilon\lambda(1-y) + \epsilon\gamma\nu\Big(1-\frac{1}{y}\Big)\Big] f(x,y) \\
&\quad = \Big[\epsilon\gamma\nu\Big(1-\frac{1}{y}\Big) - \gamma\Big(1-\frac{1}{x}\Big)\Big] f(x,0) \\
&\qquad + \Big[(1-\gamma)\Big(1-\frac{1}{x}\Big) - \epsilon\nu(1-\gamma)\Big(1-\frac{1}{y}\Big)\Big] f(0,y) \\
&\qquad + \Big[\gamma\Big(1-\frac{1}{x}\Big) + \epsilon\nu(1-\gamma)\Big(1-\frac{1}{y}\Big)\Big] f(0,0) \qquad (23)
\end{aligned}$$

with $f(0, 0)$ given by

$$f(0,0) = 1 - \lambda\Big(1+\frac{1}{\nu}\Big) = 1-\rho,$$

where $\rho$ is the load. No explicit solution is described in [45] to Eq. (23).

Clearly, (23) can be written in the form (2) and hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ hold.

### 4.18. FE 18: Discrete-time queueing system with priority jumps

Now we consider the equation that arises [44] from a discrete-time queueing system with two queues of infinite capacity and one transmission channel, as illustrated in Fig. 18. Two types of packets arrive at the system: packets of type 1, which enter the first queue, and packets of type 2, which enter the second queue. The numbers of both types of packets arriving in slot $k$ are denoted by $a_{1,k}$ and $a_{2,k}$, respectively. The

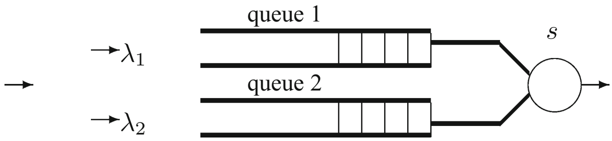


FIGURE 18. Discrete-time queueing system with priority jumps

$a_{1,k}$'s and $a_{2,k}$'s are assumed to be independent and identically distributed (i.i.d.) from slot-to-slot. Within one slot, however, $a_{1,k}$ and $a_{2,k}$ can be correlated. This possible correlation is described by the joint PGF

$$\widehat{A}(x,y) = \lim_{k\to\infty} \mathbb{E}[x^{a_{1,k}} y^{a_{2,k}}],$$

where $\mathbb{E}$ denotes the expected value operator. Next, $A_1(x) = \widehat{A}(x,1)$, $A_2(y) = \widehat{A}(1,y)$, $A_T(x) = \widehat{A}(x,x)$, and $\lambda_i = A_i'(1)$ for $i = 1,2,T$. In particular, $\lambda_T = \lambda_1 + \lambda_2$.

The PGF $f(x,y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$\begin{aligned} f(x,y) = {} & \frac{1}{x(y-\widehat{A}(x,y)+\widehat{A}(x,0)) - y\widehat{A}(x,0)} \\ & \times \big[[y(x-1)\widehat{A}(x,y) + (y-x)\widehat{A}(x,0)]f(0,0) \\ & + (y-x)(\widehat{A}(x,y)-\widehat{A}(x,0))f(x,0) + (x-y)\widehat{A}(x,0)f(0,y)\big]. \end{aligned} \quad (24)$$

Descriptions of $f(0,0)$, $f(x,0)$ and $f(0,y)$ have been been obtained through the application of the Rouché Theorem.

Equation (24) can be rewritten in the form (2) with

$$\begin{aligned} C_1(x,y) &= x(y-\widehat{A}(x,y)+\widehat{A}(x,0)) - y\widehat{A}(x,0), \\ C_2(x,y) &= (y-x)(\widehat{A}(x,y)-\widehat{A}(x,0)), \\ C_3(x,y) &= (x-y)\widehat{A}(x,0), \quad C_4(x,y) = y(x-1)\widehat{A}(x,y) + (y-x)\widehat{A}(x,0), \end{aligned}$$

and $C_5(x,y) = 0$. Note that hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ hold.

### 4.19. FE 19: Two queues with vastly different arrival rates

The next equation arises [46] from a model with two parallel queues of customers of two classes, illustrated in Fig. 19. The jobs arrive as Poisson processes with rates $\lambda$ for the primary class, and $\epsilon\sigma$ for the secondary class, where $0 < \epsilon \ll 1$. The secondary class jobs arrive much less frequently than the primary. The primary and secondary jobs have exponentially distributed service requirements with comparable parameters $\mu$ and 1, respectively. The server works at unit rate, and if none of the queues is empty it devotes fractions $1-k$ and $k$ of its effort to the primary and secondary

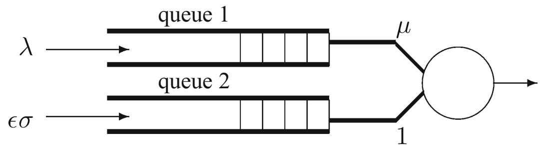


FIGURE 19. Two queues with vastly different arrival rates

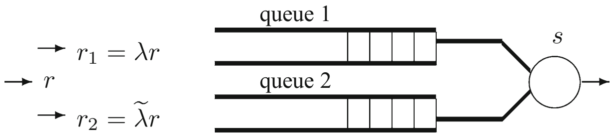


FIGURE 20. Multiserver ATM buffers—special case $c = 1$

queues, respectively, where $k = O(1)$. The corresponding service rates are $(1-k)\mu$ and $k$.

The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields the two-place FE

$$\begin{aligned}\Big[\lambda(1-x) + (1-\epsilon k)\mu\Big(1-\frac{1}{x}\Big) + \epsilon\sigma(1-y) + \epsilon k\Big(1-\frac{1}{y}\Big)\Big] f(x,y) \\ = \epsilon k\Big[\Big(1-\frac{1}{y}\Big) - \mu\Big(1-\frac{1}{x}\Big)\Big] f(x,0) \\ + (1-\epsilon k)\Big[\mu\Big(1-\frac{1}{x}\Big) - \Big(1-\frac{1}{y}\Big)\Big] f(0,y) \\ + \Big[\epsilon k\mu\Big(1-\frac{1}{x}\Big) + (1-\epsilon k)\Big(1-\frac{1}{y}\Big)\Big] f(0,0). \qquad (25)\end{aligned}$$

The article leaves (25) unsolved. The authors use singular perturbation techniques, with a suitable scaling of variables, to derive some asymptotic behaviour.

Note that (25) can be written in the form (2) and hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ hold.

### 4.20. FE 20: Multiserver ATM buffers routing multimedia traffics

The following equation arises [48] from a two dimensional $Geo^{A_1+A_2}/Geo/c$ traffic model, which means batch arrival of two classes of packets with size $A_1$ of class 1 packets and size $A_2$ of class 2 packets, geometric distribution to the inter-arrival time, geometric service time, and $c$ servers. A special case has been considered, illustrated in Fig. 20, where a system has one server (i.e. $c = 1$).

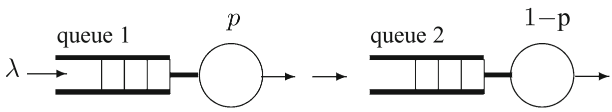


FIGURE 21. Two-stage tandem queues

The PGF in this special case satisfies the FE:

$$\begin{aligned} y(x-\widehat{A}(x,y)(x\widetilde{s}+s))f(x,y) \\ = \widehat{A}(x,y)s(x-y)f(0,y)+\widehat{A}(x,y)(y-1)xsf(0,0), \end{aligned} \tag{26}$$

where $\widetilde{s}=1-s$ and $\widehat{A}(x,y)$ is defined as the joint PGF of the total size of the batch that arrives into the system in slot $k$.

A solution to (26) has been described in [48], but with a small mistake (without some brackets in the denominator). Note that hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ are valid.

### 4.21. FE 21: Rare event asymptotics for a random walk

Now we describe the equation that arises [37] from two-stage tandem queues, illustrated in Fig. 21, where jobs arrive at queue 1 according to a Poisson process with rate $\lambda$, demanding service at both queues before leaving the system. Each job requires an exponential amount of work with parameter $\nu_j$ at queue $j$. The global service rate is set to 1. The service rate for one queue is only a fraction ($p$ for queue 1 and $1-p$ for queue 2) of the global service rate when the other queue is non-empty; when one queue is empty, the other queue has full service rate. Therefore, when both queues are non-empty, the departure rates at queues 1 and 2 are $\nu_1 p$ and $\nu_2(1-p)$, respectively.

In this case the PGF fulfils Eq. (2) with

$$\begin{aligned} C_1(x,y) &= (\lambda+p\nu_1+(1-p)\nu_2)xy-\lambda x^2y-p\nu_1y^2-(1-p)\nu_2x, \\ C_2(x,y) &= (1-p)(\nu_1y(y-x)+\nu_2x(y-1)), \\ C_3(x,y) &= -\frac{p}{1-p}C_2(x,y), \\ C_4(x,y) &= \nu_2x(y-1)-C_2(x,y), \qquad C_5(x,y)=0. \end{aligned}$$

Note that hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ are valid. In particular,

$$f(0,0)=1-\rho, \qquad \rho=\frac{\lambda}{\nu_1}+\frac{\lambda}{\nu_2}.$$

The equation has been solved by a reduction to R–H BVP, giving the formulas for $f(x,0)$ and $f(0,y)$ in some quite involved integral forms. Similar equations to the current equation arise in, e.g., [33–36].

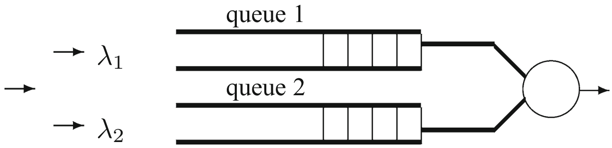


FIGURE 22. Non-work conserving generalized processor sharing queue

### 4.22. FE 22: Non-work conserving generalized processor sharing queue

The next equation arises [32] from a GPS system, composed of two queues in parallel with a server with capacity $r$ (see Fig. 22). The authors assumed that $\phi_1 + \phi_2 > 1$, where $\phi_j \in (0, 1), j = 1, 2$, are some weights. When one queue is empty, the service rate for the other queue, if not empty, is $r/(\phi_1 + \phi_2)$, which is less than $r$, and when both queues are not empty, the service rate of queue $j$ is $\phi_j r/(\phi_1 + \phi_2)$. The system is thus non work-conserving.

The jobs arrive at queue $i$ according to a Poisson process with rate $\lambda_i > 0$ and require exponentially distributed service times with mean $1/\nu_i$. It is also assumed that $\mu_i = \nu_i r/(\phi_1 + \phi_2)$, and $\rho_i = \lambda_i/\mu_i, i = 1, 2$.

The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields a functional equation of the form (2) with

$$\begin{aligned}
C_1(x,y) &= -\lambda_1 x^2 y - \lambda_2 x y^2 + (\lambda_1 + \lambda_2 + \phi_1\mu_1 + \phi_2\mu_2)xy - \phi_1\mu_1 y - \phi_2\mu_2 x,\\
C_2(x,y) &= \phi_2\mu_2 x(y-1) - (1-\phi_1)\mu_1 y(x-1),\\
C_3(x,y) &= \phi_1\mu_1 y(x-1) - (1-\phi_2)\mu_2 x(y-1),\\
C_4(x,y) &= (1-\phi_1)\mu_1 y(x-1) + (1-\phi_2)\mu_2 x(y-1), \qquad C_5(x,y) = 0.
\end{aligned}$$

Clearly, hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ hold.

A description of solutions to the equation has been obtained [28] by a reduction to a R–H BVP, using an intensive analysis of the Riemann surface defined by $\{(x, y) : C_1(x, y) = 0\}$.

### 4.23. FE 23: Data centers in fog computing

Finally, we present the equation that arises [29] from a system consisting of two data centers in parallel (see Fig. 23). It is assumed that if a request arrives at an overloaded data center, it is forwarded to the other data center with a given probability. Both data centers are assumed to have a large number of servers and that external traffic to one of them is causing saturation so that the other data center may help alleviate this saturation regime. The aim is to qualitatively estimate the gain achieved by the collaboration of the two data centers. More specifically, it is assumed that, for $i \in \{1, 2\}$, the arrival process of external requests to data center $i$ is Poisson with

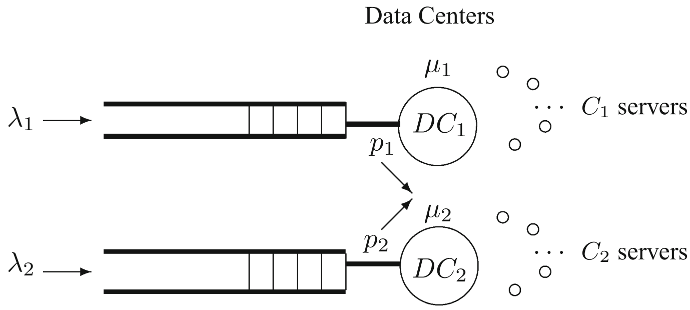


FIGURE 23. Offloading scheme for data centers in fog computing

rates $\lambda_i, i = 1, 2$ and that if one of the $C_i$ servers is idle upon arrival, then the request is processed by this data center. If the data center is saturated, i.e., all the $C_i$ servers are busy, then with probability $p_i$ the request is forwarded to the other data center if it is not saturated too, otherwise with probability $1 - p_i$ the request is rejected. A request allocated at data center $i$ is processed at rate $\mu_i$.

The PGF $f(x, y)$ of the two-dimensional distribution characterizing the system yields FE (2) with

$$\begin{aligned}
C_1(x, y) =& -\mu_1 c_1 x^2 y - \mu_2 c_2 x y^2 + (\lambda_1 + \lambda_2 + \mu_1 c_1 \\
&+ \mu_2 c_2) x y - \lambda_1 y - \lambda_2 x, \\
C_2(x, y) =& \lambda_2((1 - p_2) x y - x + p_2 y), \\
C_3(x, y) =& \lambda_1((1 - p_1) x y - y + p_1 x), \\
C_4(x, y) =& (\lambda_1 p_1 + \lambda_2 p_2) x y - p_2 \lambda_2 y - p_1 \lambda_1 x, \\
C_5(x, y) =& 0,
\end{aligned}$$

where $c_1$ and $c_2$ are some positive constants. Note that hypotheses $(\mathcal{A})$ and $(\mathcal{B})$ are valid.

The equation has been solved by a reduction to a R–H BVP (using the results in [19]).

## 5. Conclusion and comparison table

The survey that we present spans almost the past six decades and describes 23 functional equations in two complex variables, fitting into one particular pattern (2). Each of these FE is characterized by five polynomials also in two complex variables. These polynomials have determined to a certain extent the tools that have been used to obtain the known descriptions of their solutions (if any).

The comparison table that we provide below shortly describes the case of each of the 23 FE of type (2). We hope that it can be useful for the readers.

| FE | Orders of polynomials $C_i$ | | | | | Method of solution |
|---|---|---|---|---|---|---|
| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | |
| 1 | 2 | 3 | 2 | 0 | 0 | AC |
| 2 | 4 | 4 | 4 | 4 | 0 | Not yet solved |
| 3 | 4 | 2 | 2 | 0 | 0 | Parametrization and AC |
| 4 | 3 | 2 | 2 | 2 | 0 | D BVP |
| 5 | 2 | 1 | 1 | 2 | 0 | R BVP |
| 6 | 4 | 2 | 2 | 0 | 0 | Parametrization and AC |
| 7 | 2 | 3 | 3 | 4 | 0 | AC |
| 8 | 2 | 3 | 3 | 2 | 0 | R–H BVP |
| 9 | 2 | 0 | 1 | 0 | 2 | RT |
| 10 | $N+1$ | 0 | $N+1$ | $N+2$ | 0 | RT |
| 11 | $N+1$ | 0 | $N+2$ | 0 | $N+2$ | RT |
| 12 | 2 | 3 | 2 | 2 | 0 | R–H BVP |
| 13 | 2 | 0 | 1 | 0 | 2 | RT |
| 14 | 2 | 0 | 1 | 0 | 2 | RT |
| 15 | $N+2$ | 0 | $N+2$ | 0 | $N+2$ | RT |
| 16 | 4 | 3 | 3 | 2 | 0 | Trial and error |
| 17 | 3 | 2 | 2 | 2 | 0 | Not yet solved |
| 18 | 2 | 1 | 1 | 2 | 0 | RT |
| 19 | 3 | 2 | 2 | 2 | 0 | Not yet solved |
| 20 | 2 | 0 | 1 | 2 | 0 | RT |
| 21 | 3 | 2 | 2 | 2 | 0 | R–H BVP |
| 22 | 3 | 2 | 2 | 2 | 0 | R–H BVP |
| 23 | 3 | 2 | 2 | 2 | 0 | R–H BVP |

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El-sayed El-hady
Faculty of Computers and Informatics
Suez Canal University
Ismailia
Egypt
e-mail: elsayed_elhady@ci.suez.edu.eg

Janusz Brzdęk
Department of Mathematics
Pedagogical University
Kraków
Poland
e-mail: jbrzdek@up.krakow.pl

Hamed Nassar
Computer Science Department, Faculty of Computers and Informatics
Suez Canal University
Ismailia
Egypt
e-mail: nassar@ci.suez.edu.eg

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