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# On polynomial congruences

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**Abstract.** We deal with functions which fulfil the condition $\Delta_h^{n+1}\varphi(x) \in \mathbb{Z}$ for all $x, h$ taken from some linear space $V$. We derive necessary and sufficient conditions for such a function to be decent in the following sense: there exist functions $f\colon V \to \mathbb{R}$, $g\colon V \to \mathbb{Z}$ such that $\varphi = f + g$ and $\Delta_h^{n+1} f(x) = 0$ for all $x, h \in V$.

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## 1. Introduction

Let $V$ be a linear space over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $n \in \mathbb{N}$ (we assume that $0 \in \mathbb{N}$). The symbol $\equiv$ stands for a congruence modulo $\mathbb{Z}$ (so $a \equiv b \iff a - b \in \mathbb{Z},\ a, b \in \mathbb{R}$), the symbol $[x]$ denotes the integer part of a real number $x$ and $\tilde{x}$ denotes the fractional part of $x$ (so $x = [x] + \tilde{x},\ \tilde{x} \in [0, 1)$).
Following e.g. [10], we define the difference operator:

**Definition 1.1.** Let $f\colon V \to \mathbb{R}$ be a function. Then

$$\begin{aligned}
\Delta_h^0 f &= f,\\
\Delta_h^1 f(x) &= \Delta_h f(x) = f(x+h) - f(x) \quad (x, h \in V),\\
\Delta_h^{p+1} f &= \Delta_h(\Delta_h^p f) \quad (p \in \mathbb{N}).
\end{aligned}$$

A function $f\colon V \to \mathbb{R}$ which satisfies the condition

$$\Delta_h^{n+1} f(x) = 0 \quad (x, h \in V) \tag{1.1}$$

is called a *polynomial function* of degree $n$.

The aim of this paper is to examine functions $\varphi\colon V \to \mathbb{R}$ fulfilling a less restrictive condition than (1.1), namely

$$\Delta_h^{n+1}\varphi(x) \in \mathbb{Z} \qquad (x, h \in V). \tag{1.2}$$

We call condition (1.2) *polynomial congruence* of degree $n$.
This study is inspired by several works (e.g. [1–4]), in which the so called *Cauchy's congruence* (or *Cauchy equation modulo* $\mathbb{Z}$) i.e.

$$\varphi(x+y) - \varphi(x) - \varphi(y) \in \mathbb{Z} \qquad (x, y \in V,\ \varphi\colon V \to \mathbb{R}) \tag{1.3}$$

is considered. In these works the problem of decency in the sense of Baker of solutions of (1.3) is discussed (see e.g. [1]; the solution $\varphi$ of (1.3) is called *decent* iff there exist an additive function $a\colon V \to \mathbb{R}$ and a function $g\colon V \to \mathbb{Z}$ such that $\varphi = a + g$).

In many cases Cauchy's congruence can be easily transformed to the congruence $\Delta_h^2\varphi(x) \in \mathbb{Z},\ x, h \in V$. To be more precise, if $\varphi$ fulfills (1.3), then

$$\begin{aligned}\Delta_h^2\varphi(x) &= \varphi(x+2h) - 2\varphi(x+h) + \varphi(x)\\ &= (\varphi((x+h)+h) - \varphi(x+h) - \varphi(h)) - (\varphi(x+h) - \varphi(x) - \varphi(h)) \in \mathbb{Z}\ (x, h \in V).\end{aligned}$$

Almost conversely, if $\Delta_h^2\varphi(x) \in \mathbb{Z}$ for $x, h \in V$, then the function $\hat{\varphi} = \varphi - \varphi(0)$ fulfills $\hat{\varphi}(x+y) - \hat{\varphi}(x) - \hat{\varphi}(y) \in \mathbb{Z}$. Indeed, observe first that $\hat{\varphi}(0) = 0$. Moreover, $\Delta_h^2\varphi(0) = \Delta_h^2\hat{\varphi}(0) = \hat{\varphi}(2h) - 2\hat{\varphi}(h) + \hat{\varphi}(0) \in \mathbb{Z}$ for $h \in V$, so $\hat{\varphi}(h) \equiv 2\hat{\varphi}(\frac{h}{2})$ for $h \in V$. Therefore,

$$\begin{aligned}\hat{\varphi}(x+y) - \hat{\varphi}(x) - \hat{\varphi}(y) &\equiv 2\hat{\varphi}\left(\frac{x+y}{2}\right) - \hat{\varphi}(x) - \hat{\varphi}(y)\\ &= -\Delta^2_{\frac{y-x}{2}}\hat{\varphi}(x) \in \mathbb{Z}\ (x, h \in V).\end{aligned}$$

Obviously, if $\varphi = f + g,\ f\colon V \to \mathbb{R}$ is a polynomial function of degree $n$ and $g\colon V \to \mathbb{Z}$, then $\varphi$ solves the congruence (1.2). In analogy to Baker [1], we call such functions $\varphi$ decent solutions of (1.2).

Examples of Á. Száz and G. Száz from [13] and Godini from [8] prove that there exist non-decent solutions of (1.3). Thus the natural question arises: what conditions should be imposed on the solution of the congruence $\Delta_h^{n+1}\varphi(x) \in \mathbb{Z}$ to ensure its decency.

In the present paper we obtain results which correspond to those of Baron et al. from [2] and results of Baron and Volkmann from [3]. Namely, we present analogues of results from [2,3] for polynomial congruences of degree greater than 1. Below we cite one of the characterizations of decent solutions of the Cauchy's congruence from [2], because we use it in Remark 1.3:

**Theorem 1.2.** (Baron et al. [2]) *A solution $\varphi\colon V \to \mathbb{R}$ of Cauchy's congruence is decent if and only if for every vector $v \in V$ there is a real $\alpha$ such that $\varphi(\xi v) \equiv \xi\alpha$ for all $\xi \in \mathbb{Q}$.*

When dealing with polynomial functions the inductional approach may always come in mind. In our situation one could expect that a solution of the congruence $\Delta_h^{n+1}\varphi(x) \in \mathbb{Z}$ is a decent iff for every $h \in V$ the function $V \ni$

$v \longrightarrow \varphi(v+h)-\varphi(v)$ is a decent solution of the polynomial congruence of degree $n-1$. However, this is not the case as it is visible from the following remark:

**Remark 1.3.** There exists a function $\varphi$ such that $\Delta_h^3\varphi(x) \in \mathbb{Z}$ for all $x, h \in \mathbb{R}$, $\Delta_h\varphi$ is a decent solution of the polynomial congruence of degree 1 for every $h \in V$, but $\varphi$ is not a decent solution of the polynomial congruence of degree 2.

*Proof.* Let $\alpha\colon \mathbb{R} \to \mathbb{R}$ be a function fulfilling $\alpha(x+y)-\alpha(x)-\alpha(y) = m(x,y) \in \mathbb{Z}$ for all $x, y \in \mathbb{R}$, which cannot be expressed as a sum of an additive function and an integer-valued function (the existence of such a function is proved in [8], [13]). Then $\alpha$ fulfills the congruence $\Delta_h^2\alpha(x) \in \mathbb{Z},\ x, h \in \mathbb{R}$ (which is proved on the previous page).

Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by the formula $\varphi(x) = \alpha(x) + x^2$. Then of course $\Delta_h^3\varphi(x) = \Delta_h^3\alpha(x) = \Delta_h\left(\Delta_h^2\alpha\right)(x) \in \mathbb{Z}$ for all $x, h \in \mathbb{R}$ and $\Delta_h\varphi(x) = \varphi(x+h)-\varphi(x) = \alpha(x+h)-\alpha(x)+(x+h)^2-x^2 = 2xh+h^2+\alpha(h)+m(x,h)$. The function $\mathbb{R} \ni x \longrightarrow 2xh + h^2 + \alpha(h)$ is a polynomial function of degree 1 and the function $\mathbb{R} \ni x \longrightarrow m(x,h)$ is integer-valued, thus the function $\mathbb{R} \ni x \longrightarrow \Delta_h\varphi(x)$ is a decent solution of the polynomial congruence of degree 1 (for every fixed $h \in V$). Suppose that the function $\varphi$ is a decent solution of the polynomial congruence of degree 2. Then from Theorem 2.2, which is proved in the second part of this paper, it follows that for every $v \in \mathbb{R}$ there exist constants $a_v, b_v, c_v \in \mathbb{R}$ such that for every $\xi \in \mathbb{Q}$ we have $\varphi(\xi v) \equiv a_v\xi^2+b_v\xi+c_v$. Thus $\alpha(\xi v) = (a_v-v^2)\xi^2+b_v\xi+c_v+n_v(\xi)$, where $n_v\colon \mathbb{Q} \to \mathbb{Z}$. The expression $\alpha(x+y)-\alpha(x)-\alpha(y)$ is an integer for $x, y \in \mathbb{R}$, so $2(a_v-v^2)\xi\mu - c_v \in \mathbb{Z}$ for all $\xi, \mu \in \mathbb{Q}$. This condition holds only if $a_v = v^2,\ c_v \in \mathbb{Z}$. Then $\alpha(\xi v) \equiv b_v\xi$ for $\xi \in \mathbb{Q}$ and Theorem 1.2 implies that $\alpha$ is a decent solution of Cauchy's congruence, which is in contradiction to our choice of the function $\alpha$. $\square$

We make use of the following, easy to check, properties of (decent) solutions of the congruence (1.2):

**Remark 1.4.** Let $\varphi\colon V \to \mathbb{R},\ m\colon V \to \mathbb{Z}$ and $v_0 \in V,\ c \in \mathbb{R}$. Then:

(i) $\varphi$ is a (decent) solution of the congruence (1.2) if and only if the function $\psi\colon V \longrightarrow \mathbb{R},\ \psi(v) = \varphi(v+v_0)$ is a (decent) solution of (1.2),

(ii) $\varphi$ is a (decent) solution of the congruence (1.2) if and only if the function $\varphi + c$ is a (decent) solution of (1.2),

(iii) $\varphi$ is a (decent) solution of the congruence (1.2) if and only if the function $\varphi+m$ is a (decent) solution of (1.2). Hence, in particular, $\varphi$ is a (decent) solution of the congruence (1.2) if and only if the function $\tilde{\varphi}$ is a (decent) solution of (1.2).

*Proof.* Ad (i) The first part is a consequence of the equality $\Delta_h^{n+1}\psi(v) = \Delta_h^{n+1}\varphi(v+v_0)$.

Observe that $\varphi$ is of the form $\varphi = f+g$, with $f\colon V \to \mathbb{R}$ being a polynomial function of degree $n$ and $g\colon V \to \mathbb{Z}$ if and only if $\psi = \hat{f} + \hat{g}$ where $\hat{f}(v) =$

$f(v+v_0)$ is a polynomial function of degree $n$ and $\hat{g}(v) = g(v+v_0)$ is an integer-valued function.

Ad (ii) The first part follows from the identity $\Delta_h^{n+1}(\varphi+c)(v) = \Delta_h^{n+1}\varphi(v)$.

The function $\varphi = f+g$, where $f\colon V \to \mathbb{R}$ is a polynomial function of degree $n$ and $g\colon V \to \mathbb{Z}$ if and only if we have $\varphi + c = (f+c)+g$, where $f+c$ is a polynomial function of degree $n$ and $g$ is an integer-valued function.

Ad (iii) Obviously

$$\Delta_h^{n+1}(\varphi+m)(v) = \Delta_h^{n+1}\varphi(v) + \Delta_h^{n+1}m(v) \equiv \Delta_h^{n+1}\varphi(v),$$

which proves the first part.

We have $\varphi = f+g$, where $f\colon V \to \mathbb{R}$ is a polynomial function of degree $n$ and $g\colon V \to \mathbb{Z}$ if and only if $\varphi + m = f + (g+m)$, which means that $\varphi + m$ can also be split into a polynomial and an integer-valued part. □

We can also notice that $\varphi$ fulfills the congruence $\Delta_h^{n+1}\varphi(x) \in \mathbb{Z}$ for all $x, h \in V$ if and only if the function $\Phi = \pi \circ \varphi$ ($\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ denotes the natural projection of $\mathbb{R}$ onto $\mathbb{R}/\mathbb{Z}$) is a solution of the Frechét equation $\Delta_h^{n+1}\Phi(x) = \hat{0}$ for all $x, h \in V$, where $\hat{0}$ means the neutral element of the quotient group $(\mathbb{R}/\mathbb{Z}, +)$. We recall the well-known result (see e.g. [14], Theorem 9.1, p.70) describing solutions of the Frechét equation in a wide class of spaces. It will be useful for us in our further considerations (Theorem 2.2) and, moreover, it will clarify why we cannot use it for the group $\mathbb{R}/\mathbb{Z}$ (the group $\mathbb{R}/\mathbb{Z}$ is not divisible by $n!$ for $n > 1$). For the simplicity of the statement we assume that a 0-additive function is an arbitrary function, whose domain is the linear space $\{0\}$ (see e.g. [7]).

**Theorem 1.5.** (Székelyhidi [14]) *Let $n \in \mathbb{N}$. Let $(S,+)$ be an abelian semigroup with identity and $(H,+)$ be an abelian group uniquely divisible by $n!$. Then a function $f\colon S \to H$ fulfills the equation $\Delta_h^{n+1}f(x) = 0$ for all $x, h \in S$ if and only if $f$ is of the form $f = \sum_{i=0}^{n} a^i$, where $a^i$ is a diagonalisation of the $i$-additive and symmetric function $A^i\colon S^i \to H$.*

## 2. Main result

We start with the result which corresponds to Theorem 2.1 from [2]. In the proof we make use of Theorem 1.5 and the following, very obvious remark:

**Remark 2.1.** If $p$ is a polynomial from $\mathbb{R}[X]$, which takes only integer values for rational arguments, then $p$ is constantly equal to $p(0)$.

Our first theorem reads as follows:

**Theorem 2.2.** *Let $n \in \mathbb{N}$ and let $\varphi\colon V \to \mathbb{R}$ fulfill (1.2). Then $\varphi$ is a decent solution of the polynomial congruence of degree $n$ if and only if for every vector*

*$v \in V$ there exists a polynomial $p_v$ of degree smaller than $n+1$ with real coefficients so that $\varphi(\xi v) \equiv p_v(\xi)$ for all $\xi \in \mathbb{Q}$.*

*Proof.* Firstly, assume that $\varphi$ is a decent solution of the polynomial congruence of degree $n$. Then there exist functions $f\colon V \to \mathbb{R}, g\colon V \to \mathbb{Z}$ such that $\varphi = f + g$ and $\Delta_h^{n+1} f(x) = 0$ for all $x, h \in V$. From Theorem 1.5 it follows that $f = \sum_{i=0}^{n} a_i$, where $a_i(v) = A_i(v, \ldots, v)$ for an $i$-additive and symmetric function $A_i\colon V^i \to \mathbb{R}$. Thus

$$\begin{aligned}\varphi(\xi v) &= f(\xi v) + g(\xi v) = \sum_{i=0}^{n} a_i(\xi v) + g(\xi v) = \sum_{i=0}^{n} A_i(\xi v, \ldots, \xi v) + g(\xi v)\\ &= \sum_{i=0}^{n} A_i(v, \ldots, v)\xi^i + g(\xi v) = \sum_{i=0}^{n} a_i(v)\xi^i + g(\xi v) \equiv \sum_{i=0}^{n} a_i(v)\xi^i =: p_v(\xi).\end{aligned}$$

Now, assume that for every $v \in V$ there exists $p_v \in \mathbb{R}_n[X]$ such that $\varphi(\xi v) \equiv p_v(\xi)$ for all $\xi \in \mathbb{Q}$. At the beginning, let us consider the case $\varphi(0v) = 0$ for $v \in V$. Then we can choose polynomials $p_v \in \mathbb{R}_n[X]$ in such a way that $p_v(v) = 0$ for $v \in V$. For $v, h \in V,\ \xi \in \mathbb{Q}$ we have $\Delta_{\xi h}^{n+1}\varphi(\xi v) \in \mathbb{Z}$, so

$$\begin{aligned}0 \equiv \Delta_{\xi h}^{n+1}\varphi(\xi v) &= \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} \varphi(\xi(v + kh))\\ &\equiv \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} p_{v+kh}(\xi).\end{aligned}$$

From the above congruence it follows that $\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} p_{v+kh}$ is the polynomial with integer values for rational arguments. Moreover, $p_{v+kh}(0) = 0$ for $k = 0, 1, \ldots, n+1$, so $\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} p_{v+kh}$ is the polynomial constantly equal to 0. Define the function $f\colon V \to \mathbb{R}$ by the formula $f(v) = p_v(1)$. Then $\Delta_h^{n+1} f(v) = \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} p_{v+kh}(1) = 0$ and $f(v) = p_v(1) \equiv \varphi(v)$. Thus the function $g = \varphi - f$ is integer-valued.

For an arbitrary function $\varphi$ consider $\hat{\varphi} = \varphi - \varphi(0)$. Using the already proved part of the theorem to the function $\hat{\varphi}$, we obtain that $\hat{\varphi}$ is decent. From Remark 1.4 (ii) it follows that it is equivalent to the decency of the function $\varphi$. □

Considering (i) of Remark 1.4 we can rewrite Theorem 2.2 in the following manner:

**Remark 2.3.** Let $\varphi\colon V \to \mathbb{R}$ fulfill the condition $\Delta_h^{n+1}\varphi(x) \in \mathbb{Z}$ for all $x, h \in V$.

Then $\varphi$ is a decent solution of the polynomial congruence of degree $n$ if and only if for any vectors $v, w \in V$ there exists a polynomial $p_{v,w}$ of degree smaller than $n+1$ with real coefficients so that $\varphi(v + \xi w) \equiv p_{v,w}(\xi)$ for all $\xi \in \mathbb{Q}$.

In our main theorem we apply the following result:

**Theorem 2.4.** (Ger [6]) *Let $X$ and $Y$ be two $\mathbb{Q}$-linear spaces and let $D$ be a nonempty $\mathbb{Q}$-convex subset of $X$. If $\operatorname{algint}_{\mathbb{Q}} D \neq \emptyset$ then for every function $f\colon D \to Y$ fulfilling $\Delta_h^{n+1} f(x) = 0$ for all $x, h \in X$ such that $x, x+(n+1)h \in D$ there exists exactly one function $F\colon X \to Y$ fulfilling $\Delta_h^{n+1} F(x) = 0$ for all $x, h \in X$ and $F|D = f$.*

Now we present our main result, which provides necessary and sufficient conditions for a function $\varphi$ fulfilling $\Delta_h^{n+1}\varphi(x) \in \mathbb{Z}$ for all $x, h \in V$ to be a decent solution of this congruence.

**Theorem 2.5.** *Let $\varphi\colon V \to \mathbb{R}$ fulfill the condition $\Delta_h^{n+1}\varphi(x) \in \mathbb{Z}$ for all $x, h \in V$.*

*Then the following conditions are equivalent:*

(i) *$\varphi$ is a decent solution of the polynomial congruence of degree $n$,*

(ii) *For every vector $v \in V$ there exists a polynomial $p_v$ of degree smaller than $n+1$ with real coefficients so that $\varphi(\xi v) \equiv p_v(\xi)$ for all $\xi \in \mathbb{Q}$,*

(iii) *For every vector $v \in V$ there exist $\varepsilon > 0$ and a polynomial $p_v$ of degree smaller than $n+1$ with real coefficients so that $\varphi(\xi v) \equiv p_v(\xi)$ for all $\xi \in \mathbb{Q} \cap (0, \varepsilon)$,*

(iv) *For every vector $v \in V$ there exist $\varepsilon > 0$ and a polynomial $p_v$ of degree smaller than $n+1$ with real coefficients so that $\tilde{\varphi}(\xi v) \equiv \tilde{p}_v(\xi)$ for all $\xi \in \mathbb{Q} \cap (0, \varepsilon)$,*

(v) *For every vector $v \in V$ there exist $\varepsilon > 0$ and $\alpha \in [0, 1]$ such that for every $\xi \in \mathbb{Q} \cap (0, \varepsilon)$ we have $\tilde{\varphi}(\xi v) \in (\alpha, \alpha + \frac{1}{2^{n+1}})$,*

(vi) *For every vector $v \in V$ there exists $\varepsilon > 0$ such that the function $\xi \ni \mathbb{Q} \to \tilde{\varphi}(\xi v)$ is monotone on $\mathbb{Q} \cap (0, \varepsilon)$.*

*Proof.* The equivalence $(i) \iff (ii)$ has already been proved.
The implication $(ii) \Longrightarrow (iii)$ is obvious.
Now we show that $(iii) \Longrightarrow (ii)$.

For this aim, denote $\Omega = \{\xi \in \mathbb{Q}\colon \varphi(\xi v) \equiv p_v(\xi)\}$. From our assumption it follows that $\mathbb{Q} \cap (0, \varepsilon) \subseteq \Omega$.

First, we prove that if $\mathbb{Q} \cap (0, \alpha) \subseteq \Omega$, then $\mathbb{Q} \cap (0, (1 + \frac{1}{n})\alpha) \subseteq \Omega$. Indeed, for fixed $\xi \in \mathbb{Q} \cap [\alpha, (1 + \frac{1}{n})\alpha)$ and arbitrarily taken $h \in \mathbb{Q} \cap (\xi - \alpha, \frac{1}{n+1}\xi)$ put $x = \xi - (n+1)h$. We have $x, x+h, \ldots, x+nh \in \mathbb{Q} \cap (0, \alpha)$ and $x + (n+1)h = \xi$. Therefore

$$
\begin{aligned}
0 \equiv \Delta_{hv}^{n+1}\varphi(xv) &= \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} \varphi((x+kh)v) \\
&\equiv \varphi((x+(n+1)h)v) + \sum_{k=0}^{n} \binom{n+1}{k} (-1)^{n+1-k} p_v(x+kh) \\
&= \varphi(\xi v) - p_v(\xi) + \Delta_h^{n+1} p_v(x) = \varphi(\xi v) - p_v(\xi),
\end{aligned}
$$

which means that $\xi \in \Omega$.

From the above we obtain $\bigcup_{k=0}^{+\infty}(0,(1+\frac{1}{n+1})^k\varepsilon)\cap\mathbb{Q}=(0,+\infty)\cap\mathbb{Q}\subseteq\Omega$. For $\xi\in\mathbb{Q}\cap(-\infty,0]$ put $h=-2\xi+1$. Then $\xi+h,\xi+2h,\ldots,\xi+(n+1)h\in\Omega$ and by similar considerations as above for $\Delta_{hv}^{n+1}\varphi(\xi v)$, we obtain that $\xi\in\Omega$.
$(iii)\iff(iv)$
Since $\tilde{\varphi}(\xi v),\tilde{p}_v(\xi)\in[0,1)$, we have

$$(\varphi(\xi v)\equiv p_v(\xi))\iff(\tilde{\varphi}(\xi v)\equiv\tilde{p_v}(\xi))\iff(\tilde{\varphi}(\xi v)=\tilde{p}_v(\xi))$$

for $\xi\in(0,\varepsilon)\cap\mathbb{Q}$.
$(v)\implies(iv)$
Fix $v\in V$ and take $\xi\in(0,\varepsilon)\cap\mathbb{Q},\ \eta\in\mathbb{Q}$ such that $\xi+(n+1)\eta\in(0,\varepsilon)\cap\mathbb{Q}$. We have

$$\begin{aligned}
&\Delta_{\eta v}^{n+1}\tilde{\varphi}(\xi v)\\
&\quad=\sum_{n+1-k\in E_{n+1}}\binom{n+1}{k}\tilde{\varphi}((\xi+\eta k)v)-\sum_{n+1-k\in O_{n+1}}\binom{n+1}{k}\tilde{\varphi}((\xi+\eta k)v)\\
&\quad\le\sum_{n+1-k\in E_{n+1}}\binom{n+1}{k}\left(\alpha+\frac{1}{2^{n+1}}\right)-\sum_{n+1-k\in O_{n+1}}\binom{n+1}{k}\alpha\\
&\quad=\left(\alpha+\frac{1}{2^{n+1}}\right)2^n-\alpha 2^n=\frac{1}{2},
\end{aligned}$$

where $E_{n+1}$ denotes the set of all natural even numbers smaller than or equal to $n+1$ and $O_{n+1}$ denotes the set of all natural odd numbers smaller than or equal to $n+1$.

Arguing similarly as above one can obtain that $\Delta_{\eta v}^{n+1}\tilde{\varphi}(\xi v)\ge-\frac{1}{2}$ for $\xi\in(0,\varepsilon)\cap\mathbb{Q},\ \eta\in\mathbb{Q}$ such that $\xi+(n+1)\eta\in(0,\varepsilon)\cap\mathbb{Q}$. Thus $\Delta_{\eta v}^{n+1}\tilde{\varphi}(\xi v)\in[-\frac{1}{2},\frac{1}{2}]\cap\mathbb{Z}=\{0\}$ for $\xi\in(0,\varepsilon)\cap\mathbb{Q},\ \eta\in\mathbb{Q}$ such that $\xi+(n+1)\eta\in(0,\varepsilon)\cap\mathbb{Q}$. From Theorem 2.4 it follows that there exists a function $F\colon\mathbb{Q}\to Y$ such that $F(\xi)=\tilde{\varphi}(\xi v)$ for $\xi\in\mathbb{Q}\cap(0,\varepsilon)$ and $\Delta_\eta^{n+1}F(\xi)=0$ for all $\xi,\eta\in\mathbb{Q}$. In particular $F(\xi)=\sum_{i=0}^n a_i\xi^i$ for $\xi\in\mathbb{Q}$. Thus we get $\tilde{\varphi}(\xi v)=\sum_{i=0}^n a_i\xi^i$ for $\xi\in\mathbb{Q}\cap(0,\varepsilon)$. This completes the proof.
$(iv)\implies(vi)$
There exists $\varepsilon'\in(0,\varepsilon)$ such that $p_v|_{(0,\varepsilon')}$ is monotone and the function $(0,\varepsilon')\cap\mathbb{Q}\ni\xi\to[p_v(\xi)]$ is constant. Therefore, the function $(0,\varepsilon')\cap\mathbb{Q}\ni\xi\to\tilde{p_v}(\xi)=p_v(\xi)-[p_v(\xi)]$ is monotone, too.
To finish the proof it is enough to demonstrate that $(vi)\implies(v)$.

Let $\xi\ni\mathbb{Q}\to\tilde{\varphi}(\xi v)$ be increasing on $\mathbb{Q}\cap(0,\varepsilon)$. Take an arbitrary sequence $(\xi_n)_{n\in\mathbb{N}}\in((0,\varepsilon)\cap\mathbb{Q})^{\mathbb{N}}$, which is decreasing and with limit 0. Then $(\tilde{\varphi}(\xi_n v))_{n\in\mathbb{N}}$ is monotone, with elements in $[0,1]$ and say converges to some limit, call it $g\in[0,1]$. Thus, we have $N\in\mathbb{N}$ such that $0<g-\tilde{\varphi}(\xi_m v)\le\frac{1}{2^{n+1}}$ for $m\ge N\ m\in\mathbb{N}$. From the monotonicity of the function $\xi\ni\mathbb{Q}\to\tilde{\varphi}(\xi v)$ it follows that $g-\frac{1}{2^{n+1}}\le\tilde{\varphi}(\xi v)\le g$ for sufficiently small $\xi$. In case of a decreasing function $\xi\ni\mathbb{Q}\to\tilde{\varphi}(\xi v)$ on $\mathbb{Q}\cap(0,\varepsilon)$ the proof is analogical. $\square$

**Remark 2.6.** Considering part (i) of Remark 1.4 we can replace conditions $(ii) - (vi)$ from Theorem 2.5 with slightly more general ones. For example, the condition (v) may be replaced by the following one:

(v') there exists a point $v_0 \in V$ such that for every vector $v \in V$ there exist $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ such that for every $\xi \in \mathbb{Q} \cap (0, \varepsilon)$ we have $\tilde{\varphi}(v_0 + \xi v) \in (\alpha, \alpha + \frac{1}{2^{n+1}})$.

*Proof.* Indeed, for fixed $v_0 \in V$ define a function $\psi\colon V \to \mathbb{R}$ by the formula $\psi(v) = \varphi(v_0 + v)$. From (v') it follows that for every vector $v \in V$ there exist $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ such that for every $\xi \in (0, \varepsilon) \cap \mathbb{Q}$ we have $\tilde{\psi}(\xi v) \in (\alpha, \alpha + \frac{1}{2^{n+1}})$. Moreover, from part (i) of Remark 1.4 it follows that $\psi$ fulfills a polynomial congruence of degree $n$. Therefore, the already proved part (v) of Theorem 2.5 implies that $\psi$ is a decent solution of the polynomial congruence of degree $n$. Then also $\varphi$ is a decent solution of the polynomial congruence of degree $n$ (see part (i) of Remark 1.4). $\square$

## 3. Regular solutions of polynomial congruences

Now we are going to make use of Theorem 2.5 [equivalence (i) and (v)] to obtain that regular (continuous with respect to a suitable topology or measurable with respect to a suitable $\sigma$-field) solutions of polynomial congruences are decent.

At first we recall the notions of core topology and $\mathbb{Q}$-radial continuity of a function:

**Definition 3.1.** Let $X$ be a linear space over $\mathbb{Q}$ and let $A \subseteq X$. A point $v_0 \in A$ is said to be algebraically interior to $A$ iff for every vector $v \in V$ there exists $\varepsilon > 0$ such that for every $\lambda \in \mathbb{Q},\ |\lambda| < \varepsilon$ we have $v_0 + \lambda v \in A$.

The set $A$ is called algebraically open iff each of its points is algebraically interior to $A$.

The family of all algebraically open sets in a linear space $X$ is a topology in $X$, which is called the core topology.

**Definition 3.2.** Let $h\colon V \to \mathbb{R}$ be a function and $v_0 \in V$. Then we say that $h$ is $\mathbb{Q}$-radial continuous at the point $v_0$ provided that for every vector $v \in V$ the function $\mathbb{Q} \ni \xi \to h(v_0 + \xi v)$ is continuous at 0.

**Corollary 3.3.** *Let $\varphi\colon V \to \mathbb{R}$ be a solution of (1.2), which is $\mathbb{Q}$-radial continuous at 0. Then $\varphi$ is decent.*

*Proof.* We can assume that $\varphi(0) = \frac{1}{2}$ (take $c = \frac{1}{2} - \varphi(0)$ in Remark 1.4 (ii)). Now fix $v \in V$ and choose $\varepsilon > 0$ such that $|\varphi(\xi v) - \varphi(0)| < \frac{1}{2^{n+2}}$ for $\xi \in (-\varepsilon, \varepsilon) \cap \mathbb{Q}$. Then

$$\varphi(\xi v) \in \left(\varphi(0) - \frac{1}{2^{n+2}}, \varphi(0) + \frac{1}{2^{n+2}}\right) = \left(\frac{1}{2} - \frac{1}{2^{n+2}}, \frac{1}{2} + \frac{1}{2^{n+2}}\right) \subseteq \left(\frac{1}{4}, \frac{3}{4}\right),$$

so $[\varphi(\xi v)] = 0$ for $\xi \in (-\varepsilon, \varepsilon) \cap \mathbb{Q}$. Thus $\tilde{\varphi}(\xi v) \in (\tilde{\varphi}(0) - \frac{1}{2^{n+2}}, \tilde{\varphi}(0) + \frac{1}{2^{n+2}})$ for $\xi \in (-\varepsilon, \varepsilon) \cap \mathbb{Q}$, so condition (v) from Theorem 2.5 is fulfilled. □

From the Remark 2.6 it follows that it is enough to assume that the solution of the congruence $\Delta_h^{n+1}\varphi(x) \in \mathbb{Z}$ is $\mathbb{Q}$-radial continuous at some point to get its decency.

Obviously, every function continuous with respect to the core topology in $V$ is $\mathbb{Q}$-radial continuous at this point, thus it is decent.

Now we focus our attention on Lebesgue measurable and on Baire measurable solutions of (1.2).

**Definition 3.4.** (see e.g. [5]) Let $X$ be a linear space over $\mathbb{R}$ and let $n \in \mathbb{N}$. For arbitrary $E \subseteq X$ we define the set $H(E)$ as follows

$$H(E) = \{x \in X \colon \exists_{h \in X} x + kh, x - kh \in E \text{ for } k = 1, 2, \ldots, n + 1\}.$$

Moreover,

$$\begin{aligned} H^0(E) &= E, \\ H^1(E) &= H(E), \\ H^{k+1}(E) &= H(H^k(E)), \quad k \in \mathbb{N}. \end{aligned}$$

The following remarks (see [5,9]) show important properties of the operation $H$.

**Remark 3.5.** (Ger [5]) Suppose $K$ is a field containing the set of rationals and $X$ is a linear space over $K$. If $E \subseteq X$ is of the second category and with the Baire property, then $intH(E) \neq \emptyset$.

**Remark 3.6.** (Kemperman [9]) If $E \subseteq \mathbb{R}^m$, $m \in \mathbb{N}$, has got a positive inner Lebesgue measure, then $intH(E) \neq \emptyset$.

In the proof of Theorem 3.8 we will make use of the following results:

**Theorem 3.7.** (Ger [5]) *Let $X$ be a real Hausdorff linear topological space, $\emptyset \neq D \subseteq X$ is a convex and open set and let $Y$ be a real normed space. Suppose that an $n$-convex function $f \colon D \to Y$ is bounded on a set $E \subseteq D$. If there exists a nonnegative integer $k$ such that $intH^k(E) \neq \emptyset$, then $f$ is continuous in $D$.*

**Theorem 3.8.** *Let $X$ be a real Hausdorff locally convex linear topological space and let $E \subseteq X$ be such a set that $intH(E) \neq \emptyset$. If $\varphi \colon V \to \mathbb{R}$ is a solution of the congruence $\Delta_h^{n+1}\varphi(x) \equiv 0$, $x, h \in X$ such that $\varphi(x) \in \mathbb{Z} + (-\alpha, \alpha)$ for $x \in E$ and some $0 < \alpha < \frac{1}{2^{n+1}(2^{n+1}-1)}$, then $\varphi$ is a decent solution of the polynomial congruence of degree $n$. Moreover, $\varphi = f + g$ with $f$ being a continuous polynomial function of degree $n$ and $g$ being an integer-valued function.*

*Proof.* First we prove that $\varphi(x) \in \mathbb{Z} + (-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})$ for arguments $x$ taken from some nonempty open subset of $X$.

From our assumptions it follows that there exist functions $m\colon E \to \mathbb{Z}$ and $q\colon E \to (-\alpha, \alpha)$ such that $\varphi|_E = m + q$. Since $X$ is a locally convex linear topological space and $intH(E) \neq \emptyset$, there exists an open and convex set $U$ such that $\emptyset \neq U \subseteq H(E)$. Fix $x \in U$ and choose $h \in X$ such that $x+kh, x-kh \in E$ for $k = 1, 2, \ldots, n+1$. Then

$$\begin{aligned}\mathbb{Z} \ni \Delta_h^{n+1}\varphi(x) &= (-1)^{n+1}\varphi(x) + \sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}m(x+kh)\\ &\quad + \sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}q(x+kh) < (-1)^{n+1}\varphi(x) + M\\ &\quad + \sum_{k>0}\binom{n+1}{k}\alpha\\ &= (-1)^{n+1}\varphi(x) + M + (2^{n+1}-1)\alpha,\end{aligned}$$

where $M = \sum_{k=1}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}m(x+kh)$.

Similarly, one can show that

$$\mathbb{Z} \ni \Delta_h^{n+1}\varphi(x) > (-1)^{n+1}\varphi(x) + M - (2^{n+1}-1)\alpha.$$

Putting $N = \Delta_h^{n+1}\varphi(x) - M \in \mathbb{Z}$, we have

$$(-1)^{n+1}\varphi(x) \in N + (-(2^{n+1}-1)\alpha, (2^{n+1}-1)\alpha),$$

so $\varphi(x) \in \mathbb{Z} + (-(2^{n+1}-1)\alpha, (2^{n+1}-1)\alpha) \subseteq \mathbb{Z} + (-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})$ for $x \in U$.

Thus there exist functions $\hat{m}\colon U \to \mathbb{Z}$, $\hat{q}\colon U \to (-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}})$ such that $\varphi|_U = \hat{m} + \hat{q}$.

Now we fix $x \in U$ and choose $h \in X$ such that $x+h, \ldots, x+(n+1)h \in U$. Then we have

$$\begin{aligned}\Delta_h^{n+1}\varphi(x) &= \Delta_h^{n+1}\hat{m}(x) + \sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}\hat{q}(x+kh)\\ &< \Delta_h^{n+1}\hat{m}(x) + \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{2^{n+1}} = \Delta_h^{n+1}\hat{m}(x) + 1.\end{aligned}$$

Similarly, $\Delta_h^{n+1}\varphi(x) > \Delta_h^{n+1}\hat{m}(x) - 1$. Therefore $\Delta_h^{n+1}\varphi(x) = \Delta_h^{n+1}\hat{m}(x)$ and $\Delta_h^{n+1}\hat{q}(x) = 0$ for $x \in U$ and $h \in X$ such that $x+h, \ldots, x+(n+1)h \in U$.

Theorem 2.4 applied for the function $\hat{q}$, the space $X$ and the set $U$ implies that there exists a polynomial function $F\colon X \to \mathbb{R}$ of degree $n$ such that $F|_U = \hat{q}$. Therefore $F$ is bounded from both sides on $U$, so it is continuous (Theorem 3.7).

We get $\varphi(x) = \hat{m}(x) + \hat{q}(x) \equiv F(x)$ for $x \in U$.

Now take $c \in U$ and define $\psi\colon X \to \mathbb{R}$ by the formula $\psi(x) = \varphi(x+c)$. Take $x \in U - c$. Then $x + c \in U$, so

$$\psi(x) = \varphi(x+c) \equiv F(x+c) =: G(x).$$

Obviously, $G$ is a continuous polynomial function of degree $n$.

Denote $\Omega = \{x \in X : \psi(x) \equiv G(x)\}$. We know that $U - c \subseteq \Omega$ and $U - c$ is a convex neighbourhood of 0. We show that if $W$ is a convex neighbourhood of 0, then $W \subseteq \Omega$ implies that $(1 + \frac{1}{n})W \subseteq \Omega$. Choose arbitrary $x \in W$. From the convexity of $W$ and $0 \in W$ it follows that $\frac{1}{n}x, \dots, \frac{n-1}{n}x \in W$. Thus

$$\begin{aligned}
\Delta^{n+1}_{\frac{1}{n}x}\psi(0) &= \psi\left(\frac{n+1}{n}x\right) + \sum_{k=0}^{n}(-1)^{n+1-k}\binom{n+1}{k}\psi\left(\frac{k}{n}x\right) \\
&\equiv \psi\left(\frac{n+1}{n}x\right) + \sum_{k=0}^{n}(-1)^{n+1-k}\binom{n+1}{k}G\left(\frac{k}{n}x\right) \\
&= \psi\left(\frac{n+1}{n}x\right) + \Delta^{n+1}_{\frac{1}{n}x}G(0) - G\left(\frac{n+1}{n}x\right) \\
&= \psi\left(\frac{n+1}{n}x\right) - G\left(\frac{n+1}{n}x\right),
\end{aligned}$$

which means that $\frac{n+1}{n}x \in \Omega$.

Since $\lim_{k\to+\infty}(\frac{n+1}{n})^k = +\infty$, we get $X = \bigcup_{k\in\mathbb{N}}(\frac{n+1}{n})^k U \subseteq \Omega$.

Thus $\varphi(x) = \psi(x - c) \equiv G(x - c) = F(x)$ for $x \in X$. $\square$

**Theorem 3.9.** *Let $X$ be a linear space and let $\varphi\colon X \to \mathbb{R}$ be a solution of the polynomial congruence of degree $n$. Assume that one of the following two hypotheses is valid*

1. *$X = \mathbb{R}^m$, with some positive $m$ and $\varphi$, is Lebesgue measurable.*
2. *$X$ is a real Fréchet space and $\varphi$ is a Baire measurable function.*

*Then $\varphi$ is a decent solution of the polynomial congruence of degree $n$. Moreover, $\varphi = f + g$ with $f$ being a continuous polynomial function of degree $n$ and $g$ being an integer-valued and Lebesgue (resp. Baire) measurable function.*

*Proof.* Let $\alpha = \frac{1}{2^{n+2}(2^{n+1}-1)}$. Put $A_0 = \varphi^{-1}(\mathbb{Z} + [-\alpha, \alpha])$ and for $k = 1, \dots, 2^{n+2}(2^{n+1}-1) - 2$

$$A_k = \varphi^{-1}(\mathbb{Z} + [k\alpha, (k+1)\alpha]).$$

The function $\varphi$ is Lebesgue measurable in case (1) and Baire measurable in case (2) measurable, therefore each of the sets $A_k,\ k = 0, 1, \dots, 2^{n+2}(2^{n+1}-1)-2$ is Lebesgue measurable in case (1) and has got a Baire property in case (2). Moreover, $\bigcup_{k=0}^{2^{n+2}(2^{n+1}-1)-2} A_k = X$, so some of the sets $A_k,\ k = 0, 1, \dots, 2^{n+2}(2^{n+1}-1) - 2$, say $A_{k_0}$, is of positive Lebesgue measure in case (1) and is of the second category in case (2).

If $k_0 = 0$, then the previous theorem and Remark 3.5 in case (1) and Remark 3.6 in case (2) implies the decency of $\varphi$ and the continuity of its polynomial part in a decomposition of $\varphi$ on a polynomial function and an integer-valued function.

If $k_0 \in \{1, \ldots, 2^{n+2}(2^{n+1} - 1) - 2\}$, then consider the function

$$\hat{\varphi} = \varphi - \left(k_0 + \frac{1}{2}\right)\alpha.$$

Of course, the function $\hat{\varphi}$ is a solution of the polynomial congruence of degree $n$ and

$$\begin{aligned}\hat{\varphi}^{-1}\left(\mathbb{Z} + \left[-\frac{1}{2}\alpha, \frac{1}{2}\alpha\right]\right) &= \left(\varphi - \left(k_0 + \frac{1}{2}\right)\alpha\right)^{-1}\left(\mathbb{Z} + \left[-\frac{1}{2}\alpha, \frac{1}{2}\alpha\right]\right)\\ &= \varphi^{-1}\left(\mathbb{Z} + [k_0\alpha, (k_0+1)\alpha]\right) = A_{k_0}.\end{aligned}$$

Therefore, from Remark 3.5 in case (1) and Remark 3.6 in case (2) and the previous theorem it follows that $\hat{\varphi}$ is a decent solution of the polynomial congruence and a polynomial part of its decomposition is continuous, but then also $\varphi$ is a decent solution of the polynomial congruence of degree $n$ with continuous polynomial part in the decomposition.

We proved that $\varphi = f + g$, where $f$ is a continuous polynomial function and $g$ is an integer-valued function. Since $f$ is continuous, it is Lebesgue measurable in case (1) and Baire measurable in case (2). Therefore, $g = \varphi - f$ is Lebesgue measurable in case (1) and Baire measurable in case (2), too. □



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