Aequat. Math. 90 (2016), 671–681

0001-9054/16/040671-11
published online May 12, 2016
DOI 10.1007/s00010-016-0421-3

Aequationes Mathematicae

A note on solutions of a functional equation arising in a queuing model for a LAN gateway

Janusz Brzdęk, El-sayed El-hady, Wolfgang Förg-Rob and Zbigniew Leśniak

Dedicated to Professor Roman Ger on the occasion of his 70th birthday

Abstract. We discuss some issues concerning solutions of the functional equation

$$\begin{aligned}(M(x,y)-xy)\,P(x,y) &= (1-y)(M(x,0)+\widehat{r}_1\xi_2xy)P(x,0)\\ &\quad+(1-x)(M(0,y)+\widehat{r}_2\xi_1xy)P(0,y)\\ &\quad-(1-x)(1-y)M(0,0)P(0,0)\end{aligned}$$

in the class of analytic functions P mapping $\overline{D}^2$ ($\overline{D}$ stands for the closure of the unit disc D in the complex plane $\mathbb{C}$) into $\mathbb{C}$. Here $r_j, s_j \in (0,1)$ for $j=1,2$ are fixed, $\xi_j = r_j s_j$, $\widehat{q} = 1-q$ for every $q \in \mathbb{R}$ and

$$M(x,y) = (\widehat{r}_1 + r_1\widehat{s}_1 y + \xi_1 xy)(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 xy).$$

The equation arises in a two-dimensional queueing model for a LAN gateway.

Mathematics Subject Classification. 30D05, 30E25, 39B32, 65Q20.

Keywords. Functional equation, queueing model, network gateway, boundary value problem.

1. Introduction

During the past five decades a certain class of functional equations (FEs) arose in connection with numerous issues in queuing theory and communication networks (see, e.g., [1]). Unfortunately, there is no universal technique for finding solutions to these FEs, except for some special cases.

The general form of all equations in that class is

$$\begin{aligned}C_1(x,y)P(x,y) &= C_2(x,y)P(x,0)+C_3(x,y)P(0,y)\\ &\quad+C_4(x,y)P(0,0)+C_5(x,y),\end{aligned}\tag{1.1}$$

where C_j, for $j = 1, \dots, 5$, are given functions in two complex variables x, y. The unknown function P is defined for $x, y \in \overline{D}$, where

$$D := \{z \in \mathbb{C} \colon |z| < 1\}, \quad \overline{D} := \{z \in \mathbb{C} \colon |z| \le 1\}.$$

It is the probability generating function (PGF) of a sequence of nonnegative real numbers $p_{m,n}$ $(m, n = 0, 1, 2, \dots)$ with the normalization condition

$$\sum_{m,n=0}^{\infty} p_{m,n} = 1; \tag{1.2}$$

this means that it is always of the following form

$$P(x, y) = \sum_{m,n=0}^{\infty} p_{m,n} x^m y^n, \quad x, y \in \overline{D}. \tag{1.3}$$

The value $P(0, 0) = p_{0,0}$ is, in general, the probability that the underlying system is empty.

Clearly, P is analytic with respect to either variable separately, that is, for every fixed $x \in \overline{D}$ the functions $P(\cdot, x)$ and $P(x, \cdot)$ are analytic in D and continuous in $\overline{D}$.

It worth stating that Malyshev [2] pioneered the approach of transforming such functional equations to boundary value problems in the early 1970s. The idea to reduce a functional equation for a generating function to a standard Riemann–Hilbert boundary value problem stems from the work of Fayolle and Iasnogorodski [3] (on two parallel M/M/1 queues with coupled processors). Extensive treatments of the boundary value technique for some functional equations can be found in Cohen and Boxma [4] and Fayolle et al. [5].

The boundary value problem technique of solving equations of form (1.1) seems to be the only known one which is somehow universal, that is, has already been applied for numerous particular forms of (1.1). Unfortunately, it is connected with many difficulties (see, e.g., [1]). One of them is that very often it is hard to verify that a description that we obtain in this way really depicts a solution to a particular equation. Therefore it seems to be desirable to investigate equations of form (1.1) further to work out some other general and systematic methods of solving them.

There are many special cases of (1.1) arising in various models of applications (see [6–12]). In this paper we investigate a particular example of Eq. (1.1) of the following form

$$\begin{aligned}(M(x, y) - xy)\, P(x, y) = {} & (1 - y)(M(x, 0) + \widehat{r}_1 \xi_2 xy) P(x, 0) \\ & + (1 - x)(M(0, y) + \widehat{r}_2 \xi_1 xy) P(0, y) \\ & - (1 - x)(1 - y) M(0, 0) P(0, 0), \end{aligned} \tag{1.4}$$

where $r_j, s_j \in (0,1)$ for $j = 1,2$ are fixed, $\xi_j = r_j s_j$, $\widehat{q} = 1 - q$ for every $q \in \mathbb{R}$ and

$$M(x,y) = (\widehat{r}_1 + r_1\widehat{s}_1 y + \xi_1 xy)(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 xy). \tag{1.5}$$

The equation appears in [13] (see also [14]) in investigations of a two-dimensional queueing model for a LAN gateway. To the best of our knowledge, no result on its solutions has been published so far. We present a description of its solutions in a particular situation, under suitable restrictions on parameters r_i and s_i.

Note that we can write (1.4) in form (1.1) with

$$\begin{aligned} C_1(x,y) &= (\widehat{r}_1 + r_1\widehat{s}_1 y + \xi_1 xy)(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 xy) - xy,\\ C_2(x,y) &= (1-y)\widehat{r}_1(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 xy),\\ C_3(x,y) &= (1-x)\widehat{r}_2(\widehat{r}_1 + r_1\widehat{s}_1 y + \xi_1 xy),\\ C_4(x,y) &= -(1-x)(1-y)\widehat{r}_1\widehat{r}_2,\\ C_5(x,y) &= 0, \quad x,y \in \mathbb{C}. \end{aligned} \tag{1.6}$$

In what follows we always assume that functions C_1, C_2, C_3, C_4 have the forms described by (1.6).

2. A general auxiliary result

We present now some general observations on solutions of (1.4). The form of the equation makes it possible to consider it for x, y belonging to, e.g., a subset T of $\mathbb{C}$ with $0 \in T$.

So, let $T \subset \mathbb{C}$ and $0 \in T$. Write

$$\begin{aligned} \mathcal{K} &:= \{(x,y) \in T^2 \colon C_1(x,y) = 0\},\\ \mathcal{K}_0 &:= \{x \in T \colon (x,0) \in \mathcal{K}\}, \quad \mathcal{K}^0 := \{x \in T \colon (0,x) \in \mathcal{K}\}. \end{aligned}$$

The next very simple theorem provides a useful description of all solutions $P \colon T^2 \to \mathbb{C}$ (in particular, also analytic solutions, for $T = \overline{D}$) of (1.4).

Theorem 2.1. *If a function $P \colon T^2 \to \mathbb{C}$ satisfies Eq. (1.4) for every $x, y \in T$, then there exist functions $f, g \colon T \to \mathbb{C}$ such that $f(0) = g(0)$,*

$$C_2(x,y)f(x) + C_3(x,y)g(y) + C_4(x,y)g(0) = 0, \quad (x,y) \in \mathcal{K}, \tag{2.1}$$

and

$$P(x,y) = \frac{C_2(x,y)f(x) + C_3(x,y)g(y) + C_4(x,y)g(0)}{C_1(x,y)}, \quad (x,y) \in T^2 \setminus \mathcal{K}; \tag{2.2}$$

in particular,

$$P(x,0) = f(x), \quad P(0,y) = g(y), \quad x \in T \setminus \mathcal{K}_0,\ y \in T \setminus \mathcal{K}^0. \tag{2.3}$$

Moreover, if $T = \overline{D}$*, then every function* $P: T^2 \to \mathbb{C}$ *fulfilling* (2.2)*, with some continuous functions* $f, g: T \to \mathbb{C}$ *such that* $f(0) = g(0)$ *and* (2.1) *holds, satisfies Eq.* (1.4) *for every* $x, y \in T$*.*

Proof. First assume that $P: T^2 \to \mathbb{C}$ satisfies Eq. (1.4) for every $x, y \in T$. Write

$$f(x) = P(x, 0), \quad g(x) = P(0, x), \quad x \in T.$$

Take $(x, y) \in \mathcal{K}$. Then, in view of the definition of $\mathcal{K}$, $C_1(x, y) = 0$ and consequently

$$\begin{aligned} 0 &= C_1(x,y)P(x,y) = C_2(x,y)P(x,0) + C_3(x,y)P(0,y) + C_4(x,y)P(0,0) \\ &= C_2(x,y)f(x) + C_3(x,y)g(y) + C_4(x,y)g(0). \end{aligned}$$

Thus we have proved (2.1).

Now, take $(x, y) \in T^2 \backslash \mathcal{K}$. Then $C_1(x, y) \neq 0$ and, by the equation,

$$\begin{aligned} 0 &\neq C_1(x,y)P(x,y) \\ &= C_2(x,y)P(x,0) + C_3(x,y)P(0,y) + C_4(x,y)P(0,0) \\ &= C_2(x,y)f(x) + C_3(x,y)g(y) + C_4(x,y)g(0). \end{aligned}$$

So, dividing both sides by $C_1(x, y)$ we obtain

$$P(x,y) = \frac{C_2(x,y)f(x) + C_3(x,y)g(y) + C_4(x,y)g(0)}{C_1(x,y)}.$$

This proves (2.2).

Further, it is easy to check that, by (1.6) and (2.2), for each $x \in T \backslash \mathcal{K}_0$ (i.e., $C_1(x, 0) \neq 0$),

$$P(x,0) = \frac{C_2(x,0)f(x) + C_3(x,0)g(0) + C_4(x,0)g(0)}{C_1(x,0)} = f(x),$$

and, for each $y \in T \backslash \mathcal{K}^0$,

$$P(0,y) = \frac{C_2(0,y)f(0) + C_3(0,y)g(y) + C_4(0,y)g(0)}{C_1(0,y)} = g(y).$$

Consequently, we get (2.3).

Assume now that P has form (2.2), f and g are continuous and (2.1) holds. We show that P is a solution to (1.4).

So, take $x, y \in T$. If $(x, y) \in \mathcal{K}$, then $C_1(x, y) = 0$ and (2.1) implies (1.4). If $(x, y) \notin \mathcal{K}$, then (2.3) implies that $f(x) = P(x, 0)$ and $g(y) = P(0, y)$. Hence (1.4) results from condition (2.2), because each of the sets $\mathcal{K}_0$ and $\mathcal{K}^0$ has at most one element. □

Theorem 2.1 shows that the main issue in solving Eq. (1.1) in the class of analytic (or continuous) functions $P: \overline{D}^2 \to \mathbb{C}$ is to find all pairs of suitable (analytic or continuous) functions $f, g: \overline{D} \to \mathbb{C}$ satisfying condition (2.1) (such functions are uniquely determined for each P in view of (2.3)). So, in the remaining parts of the paper we focus on condition (2.1).

3. Necessary conditions

In this part we present several simple observations which are useful in further investigations of solutions to (2.1).

Notice that

$$\mathcal{K} := \{(x, y) \in \overline{D}^2 \colon M(x, y) = xy\}.$$

The condition $M(x, y) = xy$ can be written in the form

$$(\widehat{r}_1 + r_1\widehat{s}_1 y + \xi_1 xy)(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 xy) = xy, \tag{3.1}$$

which means that, for each fixed x, it is a quadratic equation (with respect to y) of the form

$$a(x)y^2 + b(x)y + c(x) = 0,$$

where

$$\begin{aligned} a(x) &\equiv \xi_1\xi_2 x^2 + r_1\widehat{s}_1\xi_2 x,\\ b(x) &\equiv r_2\widehat{s}_2\xi_1 x^2 + (\widehat{r}_1\xi_2 + r_1 r_2\widehat{s}_1\widehat{s}_2 + \widehat{r}_2\xi_1 - 1)x + r_1\widehat{s}_1\widehat{r}_2,\\ c(x) &\equiv \widehat{r}_1 r_2\widehat{s}_2 x + \widehat{r}_1\widehat{r}_2, \quad \Delta(x) \equiv b(x)^2 - 4a(x)c(x). \end{aligned} \tag{3.2}$$

Clearly $a(x) \neq 0$ for $x \neq 0, -\widehat{s}_1/s_1$. So, without loss of generality, we may assume that there exist functions $y_1, y_2 \colon \mathbb{C}\backslash\{0, -\widehat{s}_1/s_1\} \to \mathbb{C}$ with

$$a(x)y^2 + b(x)y + c(x) = a(x)(y - y_1(x))(y - y_2(x)),$$

which means that

$$(\widehat{r}_1 + r_1\widehat{s}_1 y_j(x) + \xi_1 x y_j(x))(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 x y_j(x)) = x y_j(x) \tag{3.3}$$

for every $x \in \widehat{\mathbb{C}} := \mathbb{C}\backslash\{0, -\widehat{s}_1/s_1\}$ and $j = 1, 2$. Clearly, if $s_1 < 1/2$, then $-\widehat{s}_1/s_1 \notin \overline{D}$ and consequently $\widehat{D} := \overline{D} \cap \widehat{\mathbb{C}} = \overline{D}\backslash\{0\}$.

Note that the set

$$E := \{x \in \widehat{\mathbb{C}} \colon y_1(x) = y_2(x)\} \tag{3.4}$$

is nonempty and has at most four elements, because $E \subset \{x \in \mathbb{C} \colon \Delta(x) = 0\}$. Next, it is easy to check that if $x = 1$ in (3.1), then

$$y \in \left\{1, \frac{\widehat{\xi}_2\widehat{r}_1}{\xi_2 r_1}\right\};$$

if $y = 1$ in (3.1), then

$$x \in \left\{1, \frac{\widehat{\xi}_1\widehat{r}_2}{\xi_1 r_2}\right\}.$$

This means that we can assume that

$$y_1^{-1}(1) = \{1\}, \quad y_2(1) = \frac{\widehat{\xi}_2\widehat{r}_1}{\xi_2 r_1}, \quad y_2^{-1}(1) = \left\{\frac{\widehat{\xi}_1\widehat{r}_2}{\xi_1 r_2}\right\}. \tag{3.5}$$

Analogously we obtain that if $(-\widehat{s}_1/s_1, \widetilde{y}_0) \in \mathcal{K}$, with some $\widetilde{y}_0 \in \mathbb{C}\setminus\{0\}$, then

$$\widetilde{y}_0 = \frac{\widehat{r}_1(\widehat{r}_2 s_1 - r_2\widehat{s}_1\widehat{s}_2)}{\widehat{s}_1(\widehat{r}_1\xi_2 - 1)}. \tag{3.6}$$

Also note that if $(\widetilde{x}, 0), (0, \widetilde{y}) \in \mathcal{K}$, then (3.3) yields

$$\widetilde{x} = -\frac{\widehat{r}_2}{r_2\widehat{s}_2}, \quad \widetilde{y} = -\frac{\widehat{r}_1}{r_1\widehat{s}_1}. \tag{3.7}$$

Write

$$\mathcal{K}_j := \{(x, y_j(x))\colon x \in \widehat{D}\} \cap \overline{D}^2 = \{(x, y_j(x))\colon x \in \widehat{D}, |y_j(x)| \le 1\},$$
$$\overline{D}_j := \{x \in \widehat{D}\colon (x, y_j(x)) \in \mathcal{K}_j\} = \{x \in \widehat{D}\colon |y_j(x)| \le 1\}, \quad j = 1, 2.$$

Then

$$\mathcal{K}_j = \{(x, y_j(x))\colon x \in \overline{D}_j\}, \quad j = 1, 2,$$
$$\mathcal{K} \subset \mathcal{K}_1 \cup \mathcal{K}_2 \cup \{(-\widehat{s}_1/s_1, \widetilde{y}_0), (0, \widetilde{y})\}.$$

Clearly, if for instance $\widehat{s}_1/s_1, \widetilde{y} \notin \overline{D}$, then $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$.

Moreover, condition (2.1) implies that

$$\begin{aligned}&\widehat{r}_1(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 x y_j(x))(1 - y_j(x))f(x)\\ &\quad = -\widehat{r}_2(\widehat{r}_1 + r_1\widehat{s}_1 y_j(x) + \xi_1 x y_j(x))(1 - x)g(y_j(x))\\ &\qquad + \widehat{r}_1\widehat{r}_2(1 - x)(1 - y_j(x))g(0), \quad x \in \overline{D}_j,\ j = 1, 2.\end{aligned} \tag{3.8}$$

Consequently, for each $j \in \{1, 2\}$ we obtain

$$\begin{aligned}f(x) &= \frac{\widehat{r}_2(1 - x)g(0)}{\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 x y_j(x)}\\ &\quad - \frac{\widehat{r}_2(\widehat{r}_1 + r_1\widehat{s}_1 y_j(x) + \xi_1 x y_j(x))(1 - x)g(y_j(x))}{\widehat{r}_1(1 - y_j(x))(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 x y_j(x))}\\ &\qquad x \in \overline{D}_j, \widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 x y_j(x) \neq 0, y_j(x) \neq 1.\end{aligned} \tag{3.9}$$

4. A particular case

Consider now the particular case when

$$s_1 < 1/2 \tag{4.1}$$

and

$$r_2 < \frac{1 - r_1}{2 - s_2}. \tag{4.2}$$

Note that (4.2) holds, in particular, in the situation when $r_1 + 2r_2 < 1$, because $s_2 \in (0,1)$.

Clearly, Eq. (4.2) implies that $1 + r_2 s_2 > 2r_2 + r_1$. Moreover, $2r_1 r_2 > 2r_1 \xi_2$, whence

$$1 + r_2 s_2 + 2r_1 r_2 > 2r_2 + r_1 + 2r_1 \xi_2.$$

Easy calculations show that this implies

$$\widehat{r}_1 \widehat{r}_2 > r_1 \xi_2 + \widehat{r}_1 r_2 \widehat{s}_2$$

and next

$$\widehat{r}_1 \widehat{r}_2 > r_1 \widehat{s}_1 \xi_2 + \xi_1 \xi_2 + \widehat{r}_1 r_2 \widehat{s}_2.$$

Consequently

$$\widehat{r}_1 \widehat{r}_2 - \widehat{r}_1 r_2 \widehat{s}_2 > r_1 \widehat{s}_1 \xi_2 + \xi_1 \xi_2,$$

which yields

$$\widehat{r}_1 \widehat{r}_2 - \widehat{r}_1 r_2 \widehat{s}_2 |x| > |x| (r_1 \widehat{s}_1 \xi_2 + \xi_1 \xi_2 |x|), \quad |x| \leq 1.$$

Hence

$$|\widehat{r}_1 \widehat{r}_2 + \widehat{r}_1 r_2 \widehat{s}_2 x| > |r_1 \widehat{s}_1 \xi_2 x + \xi_1 \xi_2 x^2|, \quad |x| \leq 1.$$

This implies that

$$\left| \frac{\widehat{r}_1 \widehat{r}_2 + \widehat{r}_1 r_2 \widehat{s}_2 x}{r_1 \widehat{s}_1 \xi_2 x + \xi_1 \xi_2 x^2} \right| > 1, \quad |x| \leq 1, x \neq 0,$$

because on account of (4.1) we have

$$-\frac{\widehat{s}_1}{s_1} \notin \overline{D}.$$

Thus, by Vieta's formulas, we get

$$|y_1(x) y_2(x)| = \left| \frac{c(x)}{a(x)} \right| > 1, \quad x \in \widehat{D} = \overline{D} \backslash \{0\}.$$

Now, we can choose the values of the functions y_1 and y_2 appropriately; namely, we take $|y_1(x)| \leq |y_2(x)|$ for $x \in \widehat{D}$. Hence $|y_2(x)| > 1$ for $x \in \widehat{D}$ and therefore $\overline{D}_2 = \emptyset$. Clearly, then (3.5) and (3.9) yield

$$\begin{aligned} f(x) = {} & \frac{\widehat{r}_2 (1-x) g(0)}{\widehat{r}_2 + r_2 \widehat{s}_2 x + \xi_2 x y_1(x)} \\ & - \frac{\widehat{r}_2 (\widehat{r}_1 + r_1 \widehat{s}_1 y_1(x) + \xi_1 x y_1(x))(1-x) g(y_1(x))}{\widehat{r}_1 (1 - y_1(x))(\widehat{r}_2 + r_2 \widehat{s}_2 x + \xi_2 x y_1(x))} \\ & x \in \overline{D}_1, \widehat{r}_2 + r_2 \widehat{s}_2 x + \xi_2 x y_1(x) \neq 0, x \neq 1. \end{aligned} \tag{4.3}$$

Further, let us notice that, in view of (3.3), if

$$\widehat{r}_2 + r_2 \widehat{s}_2 x_0 + \xi_2 x_0 y_1(x_0) = 0$$

for some $x_0 \in \widehat{\mathbb{C}}$, then $y_1(x_0) = 0$ and consequently, again by (3.3),

$$x_0 = -\frac{\widehat{r}_2}{r_2 \widehat{s}_2}.$$

So, if $\widehat{r}_2 > r_2\widehat{s}_2$, then $x_0 \notin \overline{D}$ and therefore (4.3) holds for all $x \in \overline{D}_1$, $x \neq 1$. Since $y_1(1) = 1$ (see (3.5)), it is easily seen that (3.8) is valid for $j = 1$ and $x = 1$ with any value of $f(1)$. Next, in view of (3.2) and (3.7), we can assume that

$$y_1(0) = \widetilde{y} = -\frac{\widehat{r}_1}{r_1 \widehat{s}_1}. \tag{4.4}$$

Hence, we can take

$$\begin{aligned} f(x) = {} & \frac{\widehat{r}_2(1-x)g(0)}{\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 x y_1(x)} \\ & - \frac{\widehat{r}_2(\widehat{r}_1 + r_1\widehat{s}_1 y_1(x) + \xi_1 x y_1(x))(1-x)g(y_1(x))}{\widehat{r}_1(1-y_1(x))(\widehat{r}_2 + r_2\widehat{s}_2 x + \xi_2 x y_1(x))} \end{aligned} \tag{4.5}$$

for $x \in \overline{D}_1$, $x \neq 1$; then in particular $f(0) = g(0)$.

Thus we have shown that, in this case, condition (2.1) holds if and only if functions $f, g \colon \overline{D} \to \mathbb{C}$ satisfy (4.5).

If $x_0 \in \overline{D}$ (i.e., $\widehat{r}_2 \leq r_2\widehat{s}_2$), then (3.7) implies that $y_1(x_0) = 0$ and consequently

$$\widehat{r}_1(\widehat{r}_2 + r_2\widehat{s}_2 x_0 + \xi_2 x_0 y_1(x_0)) f(x_0) = 0 \tag{4.6}$$

which is (3.8) for $x = x_0$ and $j = 1$ and means that also when $x_0 \in \overline{D}$ we do not get any additional restriction on f at x_0 [that is $f(x_0)$ can be quite arbitrary, if there is no other conditions on f concerning for instance its regularity]. So, if (4.5) holds for $x \in \overline{D}_1$, $x \neq 1, x_0$, then so does (2.1) also when $x_0 \in \overline{D}$.

This means that we can use Theorem 2.1 to obtain a description of all continuous or analytic solutions $P \colon \overline{D}^2 \to \mathbb{C}$ of (1.4) (in the case where (4.1) and (4.2) hold). In this way we get the following main result of this paper.

Theorem 4.1. *Assume that (4.1) and (4.2) are valid. A continuous function $P \colon \overline{D}^{\,2} \to \mathbb{C}$ satisfies Eq. (1.4) for every $x, y \in \overline{D}$ if and only if there exists a continuous function $g \colon \overline{D} \to \mathbb{C}$ such that*

$$P(x,y) = \frac{C_2(x,y)f(x) + C_3(x,y)g(y) + C_4(x,y)g(0)}{C_1(x,y)}, \quad (x,y) \in \overline{D}^{\,2}, y \neq y_1(x), \tag{4.7}$$

where f is given by (4.5). In particular, $g(x) = P(0,x)$ and $f(x) = P(x,0)$ for $x \in \overline{D}$.

So, to obtain an analytic solution $P \colon \overline{D}^{\,2} \to \mathbb{C}$ of (1.4) we must find a suitable analytic function $g \colon \overline{D} \to \mathbb{C}$ such that the function $f \colon \overline{D} \to \mathbb{C}$, described by (4.5), is also analytic in $\overline{D}$. This task seems to be nontrivial, so we are not going to study it in this paper, in which we only want to show some general results.

5. Final remarks

Observe that Eq. (1.4) remains the same if we exchange x, s_1, r_1 and y, s_2, r_2, respectively. Therefore, the considerations from the previous part can be repeated (after suitable modifications) also in the case where

$$s_2 < 1/2, \quad r_1 < \frac{1-r_2}{2-s_1}. \tag{5.1}$$

Moreover, it is easy to notice that in the previous part it is enough to replace (4.2) by the weaker assumption that $\overline{D}_2 = \emptyset$. The same concerns (5.1).

The remaining situation when $\overline{D}_i \neq \emptyset$ for $i = 1, 2$ seems to be much more complicated. It occurs for some particular values of r_i and s_i. For instance, according to (3.7),

$$y_j\Big(-\frac{\widehat{r}_2}{r_2\widehat{s}_2}\Big) = 0$$

for some $j \in \{1, 2\}$. Then for $i \in \{1, 2\}$, $i \neq j$, from (3.3) we have

$$-\frac{\widehat{r}_2}{r_2\widehat{s}_2} \in D, \quad y_i\Big(-\frac{\widehat{r}_2}{r_2\widehat{s}_2}\Big) = \frac{(1-\widehat{r}_1\xi_2)\widehat{s}_2}{s_2(r_2\widehat{s}_2 r_1\widehat{s}_1 - \xi_1\widehat{r}_2)} \in D,$$

when for example $1 - \widehat{r}_1\xi_2$ and $\widehat{r}_2$ are small enough, which means that

$$-\frac{\widehat{r}_2}{r_2\widehat{s}_2} \in \overline{D}_1 \cap \overline{D}_2.$$

We hope to provide some useful results in this case in the future.

Compliance with ethical standards

Funds El-sayed El-hady has been a joint-scholarship holder. One scholarship was fully funded by *the Egyptian Ministry of Higher Education and Scientific Research*, Cairo, Egypt during the period January 25, 2013 till January 24, 2015. He also gets a scholarship number 2014/3/MIP-12 funded by Innsbruck University (Austria) starting from March 1, 2015.

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Janusz Brzdęk and Zbigniew Leśniak
Department of Mathematics
Pedagogical University
Kraków
Poland
e-mail: jbrzdek@up.krakow.pl

El-sayed El-hady
Faculty of Computers and Informatics
Suez Canal University
Ismailia
Egypt

Wolfgang Förg-Rob
Institute of Mathematics
University of Innsbruck
Innsbruck
Austria

Received: February 10, 2016
Revised: April 9, 2016