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Aequationes Mathematicae

A note on the orthogonality equation with two functions

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Abstract. The aim of this paper is to describe the solution (f, g) of the equation

$$\langle f(x)|g(y)\rangle = \langle x|y\rangle, \quad x, y \in D,$$

where $f, g\colon D \to Y$, X, Y are Hilbert spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, D is a dense subspace of X.

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1. Introduction

Throughout this paper X, Y are Hilbert spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\langle\cdot|\cdot\rangle$ denotes the inner product and $\|\cdot\|$ the norm associated with it. We shall not distinguish between the symbols used for X and Y, $D(T)$ denotes the domain of the operator T.

It is known that $h\colon X \to Y$ is a solution of the orthogonality equation:

$$\langle h(x)|h(y)\rangle = \langle x|y\rangle, \quad x, y \in X$$

if and only if h is a linear isometry.

Chmieliński [2] studied the generalized orthogonality equation

$$\langle f(x)|g(y)\rangle = \langle x|y\rangle, \quad x, y \in X,$$

with two unknown functions $f, g\colon X \to Y$. The form of solutions of the above equation was presented by Łukasik and Wójcik [4]. We would like to present solutions in the case of inner product spaces.

We use some facts from the theory of adjoint operators.

Definition 1. Let $D(f)$ be a linear subspace of X, $f\colon D(f) \to Y$ be a linear operator. Further, let

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$$D(f^*) = \left\{ y \in Y \colon \bigvee_{z \in X} \bigwedge_{x \in D(f)} \langle f(x)|y\rangle = \langle x|z\rangle \right\}.$$

A function $f^* \colon Y \to X$ is called an adjoint operator iff

$$\langle f(x)|y\rangle = \langle x|f^*(y)\rangle, \quad x \in D(f), \quad y \in D(f^*).$$

Remark 2 (*See* [1, § 39]). Let $D(f)$ be a dense linear subspace of X, $f \colon D(f) \to Y$ be a linear operator. Then the adjoint operator exists and it is unique. Moreover $D(f^*)$ is a linear subspace of Y, f^* is a linear operator and

$$\ker f^* = (\operatorname{im} f)^\perp, \quad \operatorname{cl} \operatorname{im} f = (\ker f^*)^\perp.$$

Definition 3. Let $D(f)$ be a linear subspace of X. A linear operator $f \colon D(f) \to Y$ is called bounded iff there exist $M > 0$ such that

$$\|f(x)\| \leq M \|x\|, \quad x \in D(f).$$

Remark 4 (*See* [1, § 19]). Let $D(f)$ be a dense linear subspace of X, $f \colon D(f) \to Y$ be a linear and bounded operator. Then f can be uniquely extended to a linear and bounded operator on X.

Definition 5. Let $D(f)$ be a dense linear subspace of X, $f \colon D(f) \to Y$ be a linear and injective operator. A function $f^{-1} \colon \operatorname{im} f \to X$ given by the formula

$$f^{-1}(y) = x, \quad y \in \operatorname{im} f, \quad \text{where } x \in D(f) \text{ satisfies } y = f(x)$$

is called an inversion of f.

Remark 6. The inversion of the linear and injective operator is linear and injective.

We also use some lemmas.

Lemma 7. (See [3]) *Let $D \neq \emptyset$ be a set, $f, g \colon D \to Y$ be arbitrary mappings. Then there exist a subspace Y_0 of $\operatorname{cl} \operatorname{Lin} \operatorname{im} g$ and mappings $f_1, g_1 \colon D \to Y_0$, $f_2 \colon D \to (\operatorname{im} g)^\perp$, $g_2 \colon D \to Y_0^\perp \cap \operatorname{cl} \operatorname{Lin} \operatorname{im} g$ such that*

$$\langle f_1(x)|g_1(y)\rangle = \langle f(x)|g(y)\rangle, \quad x, y \in D,$$
$$f = f_1 + f_2, \quad g = g_1 + g_2,$$
$$\operatorname{cl} \operatorname{Lin} \operatorname{im} f_1 = \operatorname{cl} \operatorname{Lin} \operatorname{im} g_1 = Y_0.$$

Lemma 8. (See [2]) *Let D be a dense linear subspace of X, $f, g \colon D \to Y$ satisfy the equation*

$$\langle f(x)|g(y)\rangle = \langle x|y\rangle, \quad x, y \in D. \tag{1}$$

Then f and g are injective. Moreover, if $\operatorname{cl} \operatorname{im} g = Y$ then f is linear.

From the above lemmas we almost directly obtain

Theorem 9. *Let D be a dense linear subspace of X, $f, g\colon D \to Y$ satisfy Eq.* (1)*. Then there exist a subspace Y_0 of* $\operatorname{cl}\operatorname{Lin}\operatorname{im} g$*, linear, injective mappings $f_1, g_1\colon D \to Y_0$ and maps $f_2\colon D \to (\operatorname{im} g)^{\perp}$, $g_2\colon D \to Y_0^{\perp} \cap \operatorname{cl}\operatorname{Lin}\operatorname{im} g$ such that*

$$\begin{aligned} \langle f_1(x)|g_1(y)\rangle &= \langle x|y\rangle, \quad x, y \in D, \\ f = f_1 + f_2, &\quad g = g_1 + g_2, \\ \operatorname{cl}\operatorname{im} f_1 &= \operatorname{cl}\operatorname{im} g_1 = Y_0. \end{aligned}$$

In the main result we also use some lemma.

Lemma 10. *Let D be a dense linear subspace of X, $T\colon X \to Y$ be a linear and injective operator such that* $\operatorname{cl}\operatorname{im} T = Y$*. Then*

$$D \subset \operatorname{im} T^* \iff D \subset D((T^{-1})^*).$$

Proof. ($\Longrightarrow$) Assume that $D \subset \operatorname{im} T^*$. Let $y \in D$, then $y = T^*(v)$ for some $v \in D(T^*)$. We have

$$\left\langle T^{-1}(T(x))\middle|y\right\rangle = \langle x|y\rangle = \langle x|T^*(v)\rangle = \langle T(x)|v\rangle, \quad x \in D.$$

Hence $y \in D((T^{-1})^*)$.
($\Longleftarrow$) Assume that $D \subset D((T^{-1})^*)$. Since $T, T^{-1}, T^{-1} \circ T = Id_D$ are densely defined, $(Id_D)^* = (T^{-1} \circ T)^*$ is an extension of $T^* \circ (T^{-1})^*$. Hence $D = T^*((T^{-1})^*(D)) \subset \operatorname{im} T^*$. □

2. Main result

Theorem 11. *Let D be a dense linear subspace of X. Then $f, g\colon D \to Y$ satisfy Eq.* (1) *iff there exist subspaces Y_0, Y_1, Y_2 of Y orthogonal each to other, a linear, injective operator $T\colon D \to Y_0$ and maps $A\colon D \to Y_1$, $B\colon D \to Y_2$ such that*

$$\begin{aligned} D \subset \operatorname{im} T^*, &\quad \operatorname{cl}\operatorname{im} T = Y_0, \\ f = T + A, &\quad g = (T^{-1})^* + B. \end{aligned}$$

Proof. ($\Longrightarrow$) Assume that $f, g\colon D \to Y$ satisfy Eq. (1). In view of Theorem 9 we have the existence of the subspace Y_0 of $\operatorname{cl}\operatorname{Lin}\operatorname{im} g$, linear, injective mappings $f_1, g_1\colon D \to Y_0$ and maps $f_2\colon D \to (\operatorname{im} g)^{\perp}$, $g_2\colon D \to Y_0^{\perp} \cap \operatorname{cl}\operatorname{Lin}\operatorname{im} g$ such that

$$\begin{aligned} \langle f_1(x)|g_1(y)\rangle &= \langle x|y\rangle, \quad x, y \in D, \\ f = f_1 + f_2, &\quad g = g_1 + g_2, \\ \operatorname{cl}\operatorname{im} f_1 &= \operatorname{cl}\operatorname{im} g_1 = Y_0. \end{aligned}$$

Let $Y_1 = (\operatorname{im} g)^\perp$, $Y_2 = Y_0^\perp \cap \operatorname{cl}\operatorname{Lin}\operatorname{im} g$, $A = f_2$, $B = g_2$, $T = f_1$. Hence T^{-1} exists and its domain $\operatorname{im} T = \operatorname{im} f_1$ is dense in Y_0 so there exists the adjoint operator $(T^{-1})^*$. We observe that

$$\left\langle T^{-1}(T(x))\middle|y\right\rangle = \langle x|y\rangle = \langle f_1(x)|g_1(y)\rangle = \langle T(x)|g_1(y)\rangle, \quad x, y \in D,$$

so $D \subset D((T^{-1})^*)$ and $(T^{-1})^*(y) = g_1(y)$ for $y \in D$.

Hence, in view of Lemma 10 we get $D \subset \operatorname{im} T^*$ and also we have

$$\begin{aligned} f &= f_1 + f_2 = T + A, \\ g &= g_1 + g_2 = (T^{-1})^* + B, \\ \operatorname{cl}\operatorname{im} T &= \operatorname{cl}\operatorname{im} f_1 = Y_0. \end{aligned}$$

($\Longleftarrow$) Assume that there exist subspaces Y_0, Y_1, Y_2 of Y orthogonal to each other, a linear, injective operator $T\colon D \to Y_0$ and maps $A\colon D \to Y_1$, $B\colon D \to Y_2$ such that

$$\begin{aligned} &D \subset \operatorname{im} T^*, \quad \operatorname{cl}\operatorname{im} T = Y_0, \\ &f = T + A, \quad g = (T^{-1})^* + B. \end{aligned}$$

In view of Lemma 10 we get $D \subset D((T^{-1})^*)$ and we obtain

$$\begin{aligned} \langle f(x)|g(y)\rangle &= \left\langle T(x) + A(x)\middle|(T^{-1})^*(y) + B(y)\right\rangle = \left\langle T(x)\middle|(T^{-1})^*(y)\right\rangle \\ &= \left\langle T^{-1}(T(x))\middle|y\right\rangle = \langle x|y\rangle, \quad x, y \in D. \end{aligned}$$

□

Now we show an example that the assumption $D \subset \operatorname{im} T^*$ is independent from other assumptions and cannot be omitted in the previous theorem.

Example 12. Let $\{e_n\colon n \in \mathbb{N}\}$ be a standard orthonormal base of ℓ^2, $D = \{(x_n)_{n\in\mathbb{N}} \in \ell^2\colon \sum_{n=1}^\infty n^2 x_{2n}^2 < \infty\}$, $T\colon D \to \ell^2$ be given by the formula

$$T((x_n)_{n\in\mathbb{N}}) = \sum_{n=1}^\infty \frac{1}{n} x_{2n-1} e_{2n-1} + \sum_{n=1}^\infty n x_{2n} e_{2n}, \quad (x_n)_{n\in\mathbb{N}} \in D.$$

It is easy to see that D is a dense linear subspace of ℓ^2, T is linear, injective and $\operatorname{im} T$ is dense in ℓ^2 (it contains the standard base).

Let $y \in D(T^*)$ then

$$\begin{aligned} &\sum_{n=1}^\infty \frac{1}{n} x_{2n-1} y_{2n-1} + \sum_{n=1}^\infty n x_{2n} y_{2n} \\ &\quad = \langle T(x)|y\rangle = \langle x|T^*(y)\rangle = \sum_{n=1}^\infty x_n T^*(y)_n, \quad x \in D. \end{aligned}$$

Putting $x = e_k$ for $k \in \mathbb{N}$ we obtain

$$T^*(y) = \sum_{n=1}^{\infty} \frac{1}{n} y_{2n-1} e_{2n-1} + \sum_{n=1}^{\infty} n y_{2n} e_{2n}.$$

Let $y = \sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1}$. Then $y \in D$. Suppose that $y = T^*(x)$ for some $x \in D(T^*)$. Hence

$$\sum_{n=1}^{\infty} \frac{1}{n} e_{2n-1} = y = T^*(x) = \sum_{n=1}^{\infty} \frac{1}{n} x_{2n-1} e_{2n-1} + \sum_{n=1}^{\infty} n x_{2n} e_{2n}.$$

Then we have $x_{2n-1} = 1$, $x_{2n} = 0$ for $n \in \mathbb{N}$ which gives a contradiction. So we obtain that $D \not\subset \operatorname{im} T^*$.

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