Aequat. Math. 90 (2016), 527–540

0001-9054/16/030527-14
published online March 26, 2016
DOI 10.1007/s00010-016-0418-y

Aequationes Mathematicae

Hyperstability of general linear functional equation

Anna Bahyrycz and Jolanta Olko

Abstract. Our purpose is to investigate criteria for hyperstability of linear type functional equations. We prove that a function satisfying the equation approximately in some sense, must be a solution of it. We give some conditions on coefficients of the functional equation and a control function which guarantee hyperstability. Moreover, we show how our outcomes may be used to check whether the particular functional equation is hyperstable. Some relevant examples of applications are presented.

Mathematics Subject Classification. Primary 39B82, 47H14, 47J20; Secondary 39B62, 47H10.

Keywords. Hyers–Ulam stability, hyperstability, fixed point theorem.

1. Introduction

Let X, Y be linear spaces over the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. The functional equation

$$\sum_{i=1}^{m} A_i g\left(\sum_{j=1}^{n} a_{ij} x_j\right) + A = 0, \tag{1.1}$$

where $g\colon X \to Y$, $A, a_{ij} \in \mathbb{F}$, $A_i \in \mathbb{F}\backslash\{0\}$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$, generalizes simultaneously a lot of quite known equations, for example:

linear equation $f(ax + by) = Af(x) + Bf(y)$;

quadratic equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$;

equation of the p-Wright affine function $f(px + (1 - p)y) + f((1 - p)x + py) = f(x) + f(y)$;

Fréchet equation $f(x + y + z) + f(x) + f(y) + f(z) = f(x + y) + f(x + z) + f(y + z)$,

where $a, b, A, B, p \in \mathbb{F}\backslash\{0, 1\}$.

The stability and hyperstability of the particular cases of the functional equation (1.1), among others those mentioned above were studied by many

Birkhäuser

authors (cf., e.g., [1,2,6–10,12,13,17–19]). Stability of the general Eq. (1.1) was considered in [3].

The first well known hyperstability result appeared probably in [4] and concerned some ring homomorphisms. However, the term *hyperstability* was introduced much later (in the meaning applied here probably in [16]; see also [14,15] or [10]).

We say that the equation (1.1) is θ-hyperstable in the class of functions $g\colon X \to Y$ (with a control function $\theta : (X\setminus\{0\})^n \to Y$), if $g : X \to Y$ satisfying the inequality

$$\left\| \sum_{i=1}^{m} A_i g\left(\sum_{j=1}^{n} a_{ij}x_j\right) + A \right\| \le \theta(x_1,\ldots,x_n), \qquad x_1,\ldots,x_n \in X\setminus\{0\},$$

fulfills Eq. (1.1) for all $x_1,\ldots,x_n \in X\setminus\{0\}$.

In the paper we prove, applying the fixed point approach, criteria for the θ-hyperstability of (1.1) under some natural assumptions on θ. In this way we obtain sufficient conditions for the θ-hyperstability of a wide class of functional equations and control functions θ. Moreover, we show how our outcomes may be used to check whether the particular functional equation is θ-hyperstable.

Our investigations have been motivated by a problem of optimality of some estimations arising in stability studies.

From now on, we assume that X, Y are normed spaces over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and the coefficients in Eq. (1.1) are such that

$$A = 0 \quad \text{or} \quad \left(A \neq 0 \text{ and } \sum_{i=1}^{m} A_i \neq 0\right).$$

Denote

$$\begin{aligned} X_0 &:= X\setminus\{0\}, \quad \mathbb{F}_0 := \mathbb{F}\setminus\{0\}, \quad \mathbb{R}_+ := [0,\infty), \\ \mathbb{N}_{\le k} &:= \{1,\ldots,k\}, \quad k \in \mathbb{N}, \\ \mathbb{N}_k &:= \{l \in \mathbb{N}\cup\{0\} : l \ge k\}, \quad k \in \mathbb{N}\cup\{0\}. \end{aligned}$$

Notation Y^D stands for the set of functions $f : D \to Y$.
The equation of the p-Wright affine function will be called shortly the p-Wright equation.

2. The main result

We start with the result concerning the hyperstability of Eq. (1.1). We show, under some suitable assumptions, that a function satisfying Eq. (1.1) approximately (in some sense) must be actually a solution to it.

Theorem 2.1. *Let the functions* $g : X \to Y$, $\omega : \mathbb{F}_0 \to \mathbb{R}_+$, $\theta\colon X_0^n \to \mathbb{R}_+$ *satisfy the inequality*

$$\theta(\beta x_1, \ldots, \beta x_n) \le \omega(\beta)\theta(x_1, \ldots, x_n), \qquad \beta \in \mathbb{F}_0,\ x_1, \ldots, x_n \in X_0, \tag{2.1}$$

and the estimation

$$\left\| \sum_{i=1}^{m} A_i g\left(\sum_{j=1}^{n} a_{ij} x_j\right) + A \right\| \le \theta(x_1, \ldots, x_n), \qquad x_1, \ldots, x_n \in X_0. \tag{2.2}$$

If there exist $\emptyset \neq I \subset \mathbb{N}_{\le m}$ *and a sequence* $\{(c_1^k, \ldots, c_n^k)\}_{k\in\mathbb{N}}$ *of elements of* $\mathbb{F}_0^n$ *such that*

$$\beta_i^k := \sum_{j=1}^{n} a_{ij} c_j^k \in F_0, \qquad i \in \mathbb{N}_{\le m}, \quad k \in \mathbb{N}, \tag{2.3}$$

$$\beta_i^k = 1, \quad i \in I, \quad A_I := \sum_{i\in I} A_i \neq 0, \quad \lim_{k\to\infty} \sum_{i\notin I} \left|\frac{A_i}{A_I}\right| \omega(\beta_i^k) < 1, \tag{2.4}$$

$$\lim_{k\to\infty} \theta(c_1^k x, \ldots, c_n^k x) = 0, \tag{2.5}$$

then g *satisfies*

$$\sum_{i=1}^{m} A_i g\left(\sum_{j=1}^{n} a_{ij} x_j\right) + A = 0, \qquad x_1, \ldots, x_n \in X_0. \tag{2.6}$$

Proof. Note that without loss of generality we can assume that Y is a Banach space, because otherwise we can replace it by its completion. The proof will be divided into two steps. First assume that $A = 0$.

Assume that $\emptyset \neq I \subset \mathbb{N}_{\le m}$ and the sequence $\{(c_1^k, \ldots, c_n^k)\}_{k\in\mathbb{N}}$ of the elements of $\mathbb{F}_0^n$ are such that the conditions (2.3), (2.4) and (2.5) hold. From (2.4) we get that there exists $k_0 \in \mathbb{N}$ such that

$$\gamma_k := \sum_{i\notin I} \left|\frac{A_i}{A_I}\right| \omega(\beta_i^k) < 1, \quad k \in \mathbb{N}_{k_0}. \tag{2.7}$$

For each $k \in \mathbb{N}_{k_0}$ we define

$$\mathcal{T}_k\xi(x) := \sum_{i\notin I} \frac{-A_i}{A_I} \xi(\beta_i^k x), \qquad \xi \in Y^{X_0}, \quad x \in X_0,$$

$$\Lambda_k\delta(x) := \sum_{i\notin I} \left|\frac{-A_i}{A_I}\right| \delta(\beta_i^k x), \quad \delta \in {\mathbb{R}_+}^{X_0}, \quad x \in X_0,$$

$$\varepsilon_k(x) := \frac{\theta(c_1^k x, \ldots, c_n^k x)}{|A_I|}, \qquad x \in X_0.$$

Taking $x \in X_0$ and substituting $x_j = c_j^k x$, $j \in \mathbb{N}_{\le n}$ in (2.2) we have

$$\left\| g(x) - \sum_{i \notin I} \frac{-A_i}{A_I} g(\beta_i^k x) \right\| \le \frac{\theta(c_1^k x, \ldots, c_n^k x)}{|A_I|}, \qquad x \in X_0. \tag{2.8}$$

Thus (2.8) takes the form

$$\|g(x) - \mathcal{T}_k g(x)\| \le \varepsilon_k(x), \qquad x \in X_0.$$

It is easy to prove by induction that for every $x \in X_0$ and $l \in \mathbb{N}_0$

$$\Lambda_k^l \varepsilon(x) \le \varepsilon_k(x) \gamma_k^l.$$

Therefore, using the fact that $\gamma_k < 1$, we have

$$\varepsilon_k^*(x) := \sum_{n=0}^{\infty} (\Lambda_k^n \varepsilon_k)(x) \le \varepsilon_k(x) \sum_{n=0}^{\infty} \gamma_k^n = \frac{\varepsilon_k(x)}{1-\gamma_k}, \qquad x \in X_0.$$

Note that the operators $\mathcal{T}_k$ and Λ_k satisfy the assumptions of Theorem 1 in [5]. Applying this version of the fixed point theorem we obtain that there exists a unique fixed point $G_k : X_0 \to Y$ of $\mathcal{T}_k$ such that

$$\|g(x) - G_k(x)\| \le \frac{\theta(c_1^k x, \ldots, c_n^k x)}{|A_I|(1-\gamma_k)}, \qquad x \in X_0 \tag{2.9}$$

holds and $G_k(x) = \lim_{n\to\infty} (\mathcal{T}_k^n g)(x)$ for $x \in X_0$.
Now, we show that G_k is a solution of Eq. (2.6) (with $A = 0$).
First we prove that for every $l \in \mathbb{N}_0$ and every $x_1, \ldots, x_n \in X_0$

$$\left\| \sum_{i=1}^{m} A_i (\mathcal{T}_k^l g) \left(\sum_{j=1}^{n} a_{ij} x_j \right) \right\| \le \gamma_k^l \theta(x_1, \ldots, x_n). \tag{2.10}$$

Clearly, the case $l = 0$ is just (2.2). Next, fix $l \in \mathbb{N}_0$ and assume that (2.10) holds for every $x_1, \ldots, x_n \in X_0$. Then for every $x_1, \ldots, x_n \in X_0$ we get

$$\begin{aligned} \sum_{i=1}^{m} A_i (\mathcal{T}_k^{l+1} g) \left(\sum_{j=1}^{n} a_{ij} x_j \right) &= \sum_{i=1}^{m} A_i \mathcal{T}_k (\mathcal{T}_k^l g) \left(\sum_{j=1}^{n} a_{ij} x_j \right) \\ &= \sum_{i=1}^{m} A_i \sum_{p \notin I} \frac{-A_p}{A_I} (\mathcal{T}_k^l g) \left(\beta_p^k \sum_{j=1}^{n} a_{ij} x_j \right) \\ &= \sum_{p \notin I} \frac{-A_p}{A_I} \left[\sum_{i=1}^{m} A_i (\mathcal{T}_k^l g) \left(\sum_{j=1}^{n} a_{ij} (\beta_p^k x_j) \right) \right]. \end{aligned}$$

Consequently, applying the inductive assumption and (2.1) we have

$$\begin{aligned}\left\|\sum_{i=1}^{m} A_i(\mathcal{T}_k^{l+1}g)\left(\sum_{j=1}^{n} a_{ij}x_j\right)\right\| &\le \sum_{p\notin I}\left|\frac{-A_p}{A_I}\right|\gamma_k^l\theta(\beta_p^k x_1,\ldots,\beta_p^k x_n)\\ &\le \sum_{p\notin I}\left|\frac{-A_p}{A_I}\right|\gamma_k^l\omega(\beta_p^k)\theta(x_1,\ldots,x_n)\\ &= \gamma_k^{l+1}\theta(x_1,\ldots,x_n).\end{aligned}$$

Thus, by induction we have shown that (2.10) holds for $l\in\mathbb{N}_0$ and $x_1,\ldots,x_n \in X_0$. Letting $l\to\infty$ in (2.10), we obtain that G_k satisfies Eq. (2.6) (with $A=0$).

Consequently, we get the sequence $\{G_k\}_{k\in\mathbb{N}_{k_0}}$ of functions satisfying (1.1) and (2.9) for $k\in\mathbb{N}_{k_0}$. Therefore g is a solution of (2.6), since it is a pointwise limit of the sequence $\{G_k\}_{k\in\mathbb{N}_{k_0}}$.

If $A\neq 0$ and $\sum_{i=1}^m A_i\neq 0$ we define a function $h: X_0\to Y$ in the following way $h(x) := g(x)+\frac{A}{\sum_{i=1}^m A_i}$. From (2.2)

$$\left\|\sum_{i=1}^{m} A_i h\left(\sum_{j=1}^{n} a_{ij}x_j\right)\right\| \le \theta(x_1,\ldots,x_n),\qquad x_1,\ldots,x_n\in X_0,$$

and consequently, according to our previous considerations, the function h satisfies (2.6) with $A=0$, and hence g is a solution of (2.6), which finishes the proof. □

3. Criteria for θ-hyperstability and applications

For the purpose of checking the θ-hyperstability we use the above Theorem 2.1. Namely, we give sufficient conditions for the θ-hyperstability of a wide class of functional equations and control functions θ. In the following two theorems (Theorems 3.2 and 3.8), criteria for determining whether a functional equation of the form (1.1) is θ-hyperstable are stated.

To present the first one we need the following natural assumptions on the control function θ.

(a) Let $\theta\colon X_0^n\to\mathbb{R}_+$, $\omega:\mathbb{F}_0\to\mathbb{R}_+$ satisfy (2.1).
(b) $lim_{k\to\infty}\theta(\beta_1^k x,\ldots,\beta_n^k x)=0$ provided that $lim_{k\to\infty}|\beta_j^k|=+\infty$, $j\in\mathbb{N}_{\le n}$.
(c) $lim_{k\to\infty}\omega(\beta_k)=0$ provided that $lim_{k\to\infty}|\beta_k|=+\infty$.

Remark 3.1. It is easily seen that functions

(i) $\theta_1(x_1,\ldots,x_n)=C\sum_{j=1}^n\|c_jx_j\|^{k_j}$;
(ii) $\theta_2(x_1,\ldots,x_n)=C\max\{\|c_jx_j\|^{k_j}: j\in\{1,\ldots,n\}\}$;
(iii) $\theta_3(x_1,\ldots,x_n)=C\sqcap_{j=1}^n\|x_j\|^{k_j}$;

(iv) $\theta_4(x_1,\ldots,x_n) = C\sqcap_{j=1}^n \|x_j\|^{t_j} + D\sum_{j=1}^n \|c_jx_j\|^{k_j}$;
with some $C, D \in (0,+\infty)$, $c_j \in \mathbb{F}_0$ and with all negative k_j, t_j fulfill (a)–(c) with a suitable function ω. For example in case (iii) we can take $\omega_3(\beta) = |\beta|^{\sum_{j=1}^n k_j}$, for $\beta \in \mathbb{F}_0$, in case (iv) we may take

$$\omega_4(\beta) := \begin{cases} |\beta|^{\max\{k_1,\ldots,k_n,t\}} & \text{for } |\beta| \geq 1 \\ |\beta|^{\min\{k_1,\ldots,k_n,t\}} & \text{for } |\beta| < 1 \end{cases},$$

where $t = \sum_{j=1}^n t_j$.

Now we are in a position to present the above mentioned theorem.

Theorem 3.2. *Let the hypotheses* (a)–(c) *hold and let* $g : X \to Y$ *fulfill* (2.2). *If there exist* $i_0 \in \mathbb{N}_{\leq m}$ *and* $j_0 \in \mathbb{N}_{\leq n}$ *such that*

$$a_{i_0j_0} \neq 0, \qquad \sum_{j\neq j_0} a_{i_0j} \neq 0, \tag{3.1}$$

and

$$\sum_{j\neq j_0} \begin{vmatrix} a_{i_0j_0} & a_{i_0j} \\ a_{ij_0} & a_{ij} \end{vmatrix} \neq 0, \quad \textit{for} \quad i \neq i_0, \tag{3.2}$$

then g *satisfies* (2.6).

Proof. Assume that i_0 and j_0 satisfy the conditions (3.1), (3.2) and put $\beta_i^k := \sum_{j=1}^n a_{ij}c_j^k$. Take any $l \in \mathbb{N}$ such that

$$l \geq \frac{1}{\sum_{j\neq j_0} a_{i_0j}} \quad \text{and} \quad l \geq \max_{i\neq i_0}\left\{-\frac{a_{ij_0}}{a_{i_0j_0}\sum_{j\neq j_0} a_{ij} - a_{ij_0}\sum_{j\neq j_0} a_{i_0j}}\right\}. \tag{3.3}$$

We define the sequence $\{(c_1^k,\ldots,c_n^k)\}_{k\in\mathbb{N}}$ as follows

$$c_j^k := \begin{cases} k+l & \text{for } j \neq j_0 \\ \frac{1-(k+l)\sum_{j\neq j_0} a_{i_0j}}{a_{i_0j_0}} & \text{for } j = j_0 \end{cases}.$$

Observe that $c_j^k \in \mathbb{F}_0$ for $j \in \mathbb{N}_{\leq n}$, $k \in \mathbb{N}$. It is easy to check that

$$\beta_i^k = \begin{cases} 1 & \text{for } i = i_0 \\ \frac{a_{ij_0}+(k+l)\left[a_{i_0j_0}\sum_{j\neq j_0} a_{ij} - a_{ij_0}\sum_{j\neq j_0} a_{i_0j}\right]}{a_{i_0j_0}} & \text{for } i \neq i_0 \end{cases}, \qquad k \in \mathbb{N}.$$

Since the conditions (3.2) and (3.3) are satisfied and

$$a_{i_0j_0}\sum_{j\neq j_0} a_{ij} - a_{ij_0}\sum_{j\neq j_0} a_{i_0j} = \sum_{j\neq j_0} \begin{vmatrix} a_{i_0j_0} & a_{i_0j} \\ a_{ij_0} & a_{ij} \end{vmatrix},$$

we have $\beta_i^k \in \mathbb{F}_0$ for $i \in \mathbb{N}_{\leq m}$ and $\lim_{k\to\infty} |\beta_i^k| = +\infty$ for $i \neq i_0$.

According to our considerations and the conditions (a)–(c), the assumptions of Theorem 2.1 are fulfilled with $I = \{i_0\}$, which completes the proof. □

Remark 3.3. Note that in the case of a functional equation of two variables ($n = 2$), the condition (3.1) means that $a_{i_0 1} \neq 0$, $a_{i_0 2} \neq 0$. The condition (3.2) says that rows $(a_{i_0 1}, a_{i_0 2}), (a_{i1}, a_{i2})$ of the matrix $[a_{ij}]$ of coefficients are linearly independent for $i \neq i_0$.

The coefficients of the equations mentioned in the introduction satisfy (3.1) and (3.2), and from the above theorem we obtain that the linear equation (especially Cauchy and Jensen equations), quadratic equation, p-Wright equation, Fréchet equation are θ-hyperstable in the class of functions $g\colon X \to Y$ with each function $\theta\colon X_0^n \to \mathbb{R}_+$ (for $n = 2$ or $n = 3$, respectively) satisfying the conditions (a)–(c).

For example we present the results for the linear equation, p-Wright equation and Fréchet equation. For these equations we have even stronger results, g satisfying the equation approximately on X_0 is a solution of it on X.

Corollary 3.4. *Suppose that $a, b, A, B \in \mathbb{F}_0$. Let the function θ fulfill conditions* (a)–(c) *with $n = 2$. If $g : X \to Y$ satisfies the estimation*

$$\|g(ax + by) - Ag(x) - B(y)\| \leq \theta(x, y), \qquad x, y \in X_0,$$

then g is a solution of the linear equation on X.

Proof. Take the matrix

$$[a_{ij}] = \begin{bmatrix} a & b \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

of the coefficients of the linear equation. Then the vectors $(a, b), (a_{i1}, a_{i2})$ are linearly independent for every $i \in \{2, 3\}$. From Theorem 3.2 and Remark 3.3 it follows that g satisfies the linear equation for all $x, y \in X_0$. It is easy to check, that g fulfills the linear equation on the whole space X. □

Corollary 3.5. *Assume that $p \in \mathbb{F}\backslash\{0, 1, \frac{1}{2}\}$. Let the function θ fulfill conditions* (a)–(c) *with $n = 2$. If $g : X \to Y$ satisfies the estimation*

$$\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \leq \theta(x, y), \qquad x, y \in X_0,$$

then g is a solution of the p-Wright equation on X.

Proof. Take the matrix

$$[a_{ij}] = \begin{bmatrix} p & 1-p \\ 1-p & p \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

of the coefficients of the p-Wright equation. Then the vectors $(p, 1-p), (a_{i1}, a_{i2})$ are linearly independent for every $i \in \{2, 3, 4\}$. By Theorem 3.2 and Remark 3.3, g satisfies the p-Wright equation for all $x, y \in X_0$. It is easy to check that g is a solution of the p-Wright equation on X. □

For equations with a greater number n of variables and a greater m, this method requires more calculations, but is still not complicated. As an example, consider the Fréchet equation. For the convenience of the reader, we first prove that a function satisfying the Fréchet equation on X_0, fulfills it on the whole space X, since it is not as obvious as in the cases of linear and p-Wright equations.

Lemma 3.6. *If $g : X \to Y$ satisfies*

$$g(x+y+z)+g(x)+g(y)+g(z) = g(x+y)+g(x+z)+g(y+z), \qquad x, y, z \in X_0, \tag{3.4}$$

then it satisfies the Fréchet equation on the whole space X.

Proof. Note that it is enough to show that $g(0) = 0$.
Putting in (3.4) $y = z = x$, and then $y = 2x$, $z = -x$ we get

$$\begin{aligned} g(3x) + 3g(x) &= 3g(2x), && x \in X_0, \\ 2g(2x) + g(-x) &= g(3x) + g(0), && x \in X_0, \end{aligned}$$

respectively, and hence

$$g(2x) = 3g(x) + g(-x) - g(0), \qquad x \in X_0.$$

Replacing y by x and z by $-x$ in (3.4) yields

$$3g(x) + g(-x) = g(2x) + 2g(0), \qquad x \in X_0.$$

Adding the above equalities we conclude that $g(0) = 0$, which completes the proof. □

Corollary 3.7. *Let the function θ fulfill conditions* (a)–(c) *with $n = 3$. If $g : X \to Y$ satisfies*

$$\begin{aligned} &\|g(x + y + z) + g(x) + g(y) + g(z) - g(x + y) - g(x + z) - g(y + z)\| \\ &\quad \le \theta(x, y, z), \quad x, y, z \in X_0, \end{aligned}$$

then g is a solution of the Fréchet equation on X.

Proof. Take a matrix

$$[a_{ij}]^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

the transpose of the matrix of coefficients of this equation and let $i_0 = j_0 = 1$. An easy computation shows that for every $i \neq 1$ the following sum of minors

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{i1} & a_{i2} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{i1} & a_{i3} \end{vmatrix}$$

of $[a_{ij}]$ is nonzero. Theorem 3.2 and Lemma 3.6 complete the proof. □

Theorem 3.2 gives sufficient but not necessary conditions for the θ-hyperstability of functional equations of the form (1.1) with control function θ satisfying the conditions (a)–(c). If the coefficients a_{ij} are such that the conditions (3.1), (3.2) are not satisfied with any $i_0 \in \mathbb{N}_{\leq m}$, $j_0 \in \mathbb{N}_{\leq n}$, then we know nothing about the θ-hyperstability of this equation, which is demonstrated in the following examples.

Example. The coefficients of the p-Wright equation for $p = \frac{1}{2}$ do not satisfy (3.2) for any $i_0 \in \{1, \dots, 4\}$. On the other hand, this equation can be written as a Jensen equation

$$f\Big(\frac{x+y}{2}\Big) = \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

and from Corollary 3.4, we get that the Jensen equation is θ-hyperstable with each function $\theta\colon X_0^2 \to \mathbb{R}_+$ satisfying the conditions (a)–(c).

Example. The coefficients of a special case of the equation of the (p, q)-Wright function generalization of the p-Wright equation

$$f(px - py) + f(py - px) = f(x) + f(y) \tag{3.5}$$

do not satisfy (3.2) for any $i_0 \in \{1, \dots, 4\}$. Moreover, this equation is not θ-hyperstable with the function $\theta(x, y) = |x|^{-2l+1} + |y|^{-2l+1}$ for $l \in \mathbb{N}$. Indeed, the function

$$f(x) = \begin{cases} x^{-2l+1} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases},$$

satisfies the inequality

$$\begin{aligned} |f(px - py) + f(py - px) - f(x) - f(y)| &= |x^{-2l+1} + y^{-2l+1}| \\ &\leq |x|^{-2l+1} + |y|^{-2l+1} = \theta(x, y), \end{aligned}$$

and is not a solution of (3.5).

From Theorem 2.1, we derive another criterion for the θ-hyperstability of Eq. (1.1) with a particular form of the control function, namely for $\theta = \theta_3$ with $C > 0$, $k_j \in \mathbb{R}$ such that $\sum_{j=1}^{n} k_j < 0$. Note that in this case we do not need to assume that all k_j are negative real numbers (in contrast to Theorem 3.2, where the condition (b) must be satisfied).

Theorem 3.8. *Assume that $C > 0$, $k_j \in \mathbb{R}$ are such that $\sum_{j=1}^{n} k_j < 0$. Let $g : X \to Y$ fulfill the estimation*

$$\left\| \sum_{i=1}^{m} A_i g\left(\sum_{j=1}^{n} a_{ij} x_j\right) + A \right\| \leq C \sqcap_{j=1}^{n} \|x_j\|^{k_j}, \qquad x_1, \dots, x_n \in X_0.$$

If there exist $i_0 \in \mathbb{N}_{\le m}$ *and* $j_0 \in \mathbb{N}_{\le n}$ *such that*

$$\sum_{j\neq j_0} k_j < 0, \quad a_{i_0 j_0} \neq 0, \quad \sum_{j\neq j_0} a_{i_0 j} = 0 \quad \text{and} \quad \sum_{j\neq j_0} a_{ij} \neq 0 \quad \text{for} \quad i \neq i_0, \tag{3.6}$$

then g *satisfies* (2.6).

Proof. Assume that $i_0 \in \mathbb{N}_{\le m}$ and $j_0 \in \mathbb{N}_{\le n}$ satisfy the condition (3.6) and define

$$c_j^k := \begin{cases} k & \text{for } j \neq j_0 \\ \frac{1}{a_{i_0 j_0}} & \text{for } j = j_0 \end{cases}, \qquad k \in \mathbb{N}.$$

Then there exists $k_0 \in \mathbb{N}$ such that $\beta_i^k := \sum_{j=1}^n a_{ij} c_j^k \in \mathbb{F}_0$ for $k \in \mathbb{N}_{k_0}$, $i \in \mathbb{N}_{\le m}$. It is easy to check that the assumptions of Theorem 2.1 are fulfilled with $I = \{i_0\}$ and the sequence $\{(c_1^k, \ldots, c_n^k)\}_{k \in \mathbb{N}_{k_0}}$, which completes the proof. □

Applying Theorem 3.8, we obtain that the linear equation (especially Cauchy, Jensen equation), quadratic equation, p-Wright equation are θ_3-hyperstable (with $C \ge 0$ and $k_i < 0$) in the class of functions $g\colon X \to Y$. For example we present the following results for the p-Wright equation and Fréchet equation.

Corollary 3.9. *Let* $C \ge 0$, $k_1 + k_2 < 0$. *If* $g : X \to Y$ *satisfies*

$$\begin{aligned} &\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \\ &\quad \le C\|x\|^{k_1}\|y\|^{k_2}, \qquad x, y \in X_0, \end{aligned}$$

then it is a solution of the p*-Wright equation on* X.

Proof. Since $k_1 + k_2 < 0$, one of k_1, k_2 must be negative. Assume that $k_1 < 0$. Applying Theorem 3.8 with $i_0 = 4$ and $j_0 = 2$ we obtain that g satisfies the p-Wright equation for all $x, y \in X_0$. It may be shown that then g must fulfill the p-Wright equation. □

Corollary 3.10. *Let* $C \ge 0$, $k_1 + k_2 + k_3 < 0$. *If* $g : X \to Y$ *satisfies*

$$\begin{aligned} &\|g(x+y+z) + g(x) + g(y) + g(z) - g(x+y) - g(x+z) - g(y+z)\| \\ &\quad \le C\|x\|^{k_1}\|y\|^{k_2}\|z\|^{k_3}, \qquad x, y, z \in X_0, \end{aligned}$$

then it is a solution of the Fréchet equation on X.

Proof. Since $k_1+k_2+k_3 < 0$, there exists $j_0 \in \{1, 2, 3\}$ such that $\sum_{j\neq j_0} k_j < 0$. Putting $i_0 = j_0 + 1$ and applying Theorem 3.8 and Lemma 3.6, we obtain our claim. □

Finally we use the above outcomes, in order to get the θ-hyperstability of the linear equation with $\theta(x, y) = C\|x\|^p\|y\|^q$, for all $p, q \in \mathbb{R}$ such that $p + q \neq 0$. For this purpose, we prove an analogon of Lemma 3.6 for the linear equation (see also [11, Lemma 4.7]).

Lemma 3.11. *If $g : X \to Y$ satisfies*

$$f(ax + by) = Af(x) + Bf(y) + A_0, \qquad x, y \in X_0, \tag{3.7}$$

then it satisfies the linear equation on X.

Proof. Replacing in (3.7) x by bx, y by $-ax$ and then x by bx, y by ax we get

$$f(0) = Af(bx) + Bf(-ax) - A_0, \qquad x \in X_0, \tag{3.8}$$

and

$$f(2abx) = Af(bx) + Bf(ax) - A_0, \qquad x \in X_0. \tag{3.9}$$

From (3.8) and (3.9) for the even f_e and odd f_o parts of f we get $f_e(2abx) = f_e(0)$, and $Af_o(bx) = Bf_o(ax)$, $f_o(2abx) = Af_o(bx) + Bf_o(ax)$, for $x \in X_0$, respectively. Thus

$$f_o(x) = 2Bf_o\Big(\frac{x}{2b}\Big) = 2Af_o\Big(\frac{x}{2a}\Big), \quad x \in X_0.$$

Hence, using (3.7) for the odd part of f we obtain

$$f_o(x) + f_o(y) = 2Af_o\Big(\frac{x}{2a}\Big) + 2Bf_o\Big(\frac{y}{2b}\Big) = 2f_o\Big(a\frac{x}{2a} + b\frac{y}{2b}\Big) = 2f_o\Big(\frac{x+y}{2}\Big),$$

for $x, y \in X_0$. Therefore, for $x \in X_0$

$$2f_o(x) = f_o(3x) + f_o(-x) \quad \text{and} \quad 2f_o(2x) = f_o(3x) + f_o(x),$$

consequently $f_o(2x) = 2f_o(x)$. In this way we obtain that f_o is an additive function such that

$$f_o(bx) = Bf_o(x) \text{ and } f_o(ax) = Af_o(x), \qquad x \in X,$$

and $f_e \equiv \alpha$ is constant so that

$$\alpha = (A + B)\alpha + A_0,$$

which completes the proof. □

Corollary 3.12. *Let $p, q \in \mathbb{R}$, $a, b, A, B \in \mathbb{F}_0$, $A_0 \in F$, $C \geq 0$ and $A_0 = 0$ or ($A_0 \neq 0$ and $A + B \neq 1$) and $f : X \to Y$ fulfill the estimation*

$$\|f(ax + by) - Af(x) - Bf(y) - A_0\| \leq C\|x\|^p\|y\|^q, \qquad x, y \in X_0.$$

If one of the following conditions is satisfied

(a) $p + q < 0$,
(b) $p + q > 0$, $q > 0$, $|A| \neq |a|^{p+q}$,
(c) $p + q > 0$, $p > 0$, $|B| \neq |b|^{p+q}$,

then f satisfies the equation

$$f(ax + by) = Af(x) + Bf(y) + A_0, \qquad x, y \in X. \tag{3.10}$$

Proof. On account of the above lemma, it suffices to prove that the function f satisfies (3.7).
If case (a) holds, then it is enough to use Theorem 3.8 for $i_0 = 2$, $j_0 = 1$.
The cases (b) and (c) are symmetric, so we prove only case (b).
If $|a|^{p+q} < |A|$, then setting in Theorem 2.1 $I = \{2\}$ and

$$c_j^k := \begin{cases} 1 & \text{for } j = 1 \\ -\frac{a}{bk} & \text{for } j = 2 \end{cases},$$

we get that f satisfies (3.7), because $lim_{k\to\infty} \frac{1}{|A|}|a(1-\frac{1}{k})|^{p+q} + |\frac{B}{A}||\frac{a}{bk}|^{p+q}) < 1$ and $lim_{k\to\infty} \frac{1}{|A|}|\frac{a}{bk}|^q \|x\|^{p+q} = 0$.
If $|a|^{p+q} > |A|$, then setting in Theorem 2.1 $I = \{1\}$ and

$$c_j^k := \begin{cases} \frac{1}{a} - \frac{1}{ak} & \text{for } j = 1 \\ \\ \frac{1}{bk} & \text{for } j = 2 \end{cases},$$

yields that f satisfies (3.7), since $lim_{k\to\infty}|A|\left|\frac{1}{a}(1-\frac{1}{k})\right|^{p+q} + |B|\left|\frac{1}{bk}\right|^{p+q} < 1$ and $lim_{k\to\infty} \frac{1}{|A|}\left|\frac{1}{a}(1-\frac{1}{k})\right|^p \left|\frac{1}{bk}\right|^q \|x\|^{p+q} = 0$. □

4. Final remarks

Note that our results correspond to the new ones concerning hyperstability. For example it has been proven that the Cauchy and linear equations are θ_1-hyperstable with $k_1 = k_2 < 0$ (in [8] and [17], respectively). In our considerations $k_1, k_2 < 0$ may be different. Corollary 3.12 generalizes [10, Theorem 20], where only the cases (b) and (c) with $p, q \in \mathbb{R}_+$ were considered. For more examples see e.g. [1,2,10,18].

From [11, Theorem 2.2] we obtain the following hyperstability result for a generalization of (1.1), where the constant A is replaced by a function.

Corollary 4.1. *Assume that a functional equation of the form* (1.1) *is θ-hyperstable with some $\theta : X_0^n \to Y$ in the class of functions mapping X into Y. Let $g : X \to Y$, $d : X^n \to Y$ be functions satisfying the inequality*

$$\left\|\sum_{i=1}^m A_i g\left(\sum_{j=1}^n a_{ij}x_j\right) + d(x_1, \ldots, x_n)\right\| \le \theta(x_1, \ldots, x_n), \qquad x_1, \ldots, x_n \in X_0. \tag{4.1}$$

If the functional equation

$$\sum_{i=1}^m A_i f\left(\sum_{j=1}^n a_{ij}x_j\right) + d(x_1, \ldots, x_n) = 0, \qquad x_1, \ldots, x_n \in X \tag{4.2}$$

has a solution, then g is a solution of (4.2) *on X_0.*

The hyperstability results have various interesting consequences. For instance, note that we get at once the following a bit surprising corollary.

Corollary 4.2. *Assume that a functional equation of the form* (1.1) *is* θ_l-*hyperstable with some* $l \in \{1, 2, 3\}$ *in the class of functions mapping* X *into* Y. *Then for every function* $g : X \to Y$, *either* g *satisfies Eq.* (2.6) *or the following condition is fulfilled*

$$\sup_{(x_1,\ldots,x_n)\in X_0^n} \frac{\|\sum_{i=1}^{m} A_i g(\sum_{j=1}^{n} a_{ij}x_j) + A\|}{\theta_l(x_1,\ldots,x_n)} = \infty.$$

Our considerations can be used in further research on θ-hyperstability. It is interesting to investigate the hyperstability of other functional equations of the form (1.1), as well as seek other conditions guaranteeing the θ-hyperstability of specified equations and control functions θ.

References

[1] Bahyrycz, A., Piszczek, M.: Hyperstability of the Jensen functional equation. Acta Math. Hung. **142**, 353–365 (2014)

[2] Bahyrycz, A., Brzdęk, J., Piszczek, M., Sikorska, J.: Hyperstability of the Fréchet equation and a characterization of inner product spaces. J. Funct. Spaces Appl. Art. ID 496361. (2013). doi:10.1155/2013/496361

[3] Bahyrycz, A., Olko, J.: On stability of the general linear equation. Aequationes Math. doi:10.1007/s00010-014-0317-z

[4] Bourgin, D.G.: Approximately isometric and multiplicative transformations on continuous function rings. Duke Math. J. **16**, 385–397 (1949)

[5] Brzdęk, J., Chudziak, J., Páles, Zs.: A fixed point approach to stability of functional equations. Nonlinear Anal. **74**, 6728–6732 (2011)

[6] Brzdęk, J.: Stability of the equation of the p-Wright affine functions. Aequationes Math. **85**, 497–503 (2013)

[7] Brzdęk, J.: Remarks on hyperstability of the Cauchy functional equation. Aequationes Math. **86**, 255–267 (2013)

[8] Brzdęk, J.: Hyperstability of the Cauchy equation on restricted domains. Acta Math. Hung. **141**, 58–67 (2013)

[9] Brzdęk, J.: A hyperstability result for the Cauchy equation. Bull. Aust. Math. Soc. (2013). doi:10.1017/S0004972713000683

[10] Brzdęk, J., Ciepliński, K.: Hyperstability and Superstability. Abstr. Appl. Anal. (2013). doi:10.1155/2013/401756

[11] Brzdęk, J.: Remarks on stability of some inhomogeneous functional equations. Aequationes Math. doi:10.1007/s00010-014-0274-6

[12] Cădariu, L., Radu, V.: Fixed point and the stability of Jensen's functional equation. J. Inequal. Pure Appl. Math. 2. Art. ID 4 (2003)

[13] Daróczy, Z., Lajkó, K., Lovas, R.L., Maksa, G., Páles, Zs.: Functional equations involving means. Acta Math. Hung. **166**, 79–87 (2007)
[14] Gselmann, E.: Hyperstability of a functional equation. Acta Math. Hung. **124**, 179–188 (2009)
[15] Maksa, G.: The stability of the entropy of degree alpha. J. Math. Anal. Appl. **346**, 17–21 (2008)
[16] Maksa, G., Páles, Zs.: Hyperstability of a class of linear functional equations. Acta Math. Acad. Paedag. Nyìregyháziensis **17**, 107–112 (2001)
[17] Piszczek, M.: Remark on hyperstability of the general linear equation. Aequationes Math. **88**, 163–168 (2014)
[18] Piszczek, M., Szczawińska, J.: Hyperstability of the Drygas functional equation. J. Funct. Spaces Appl. Art. ID 912718. (2013)
[19] Popa, D., Raşa, I.: The Fréchet functional equation with application to the stability of certain operators. J. Approx. Theory **164**, 138–144 (2012)

Anna Bahyrycz and Jolanta Olko
Institute of Mathematics
Pedagogical University
Podchorążych 2
30-084 Kraków
Poland
e-mail: bah@up.krakow.pl

Jolanta Olko
Faculty of Applied Mathematics
AGH University of Science and Technology
Mickiewicza 30
30-059 Kraków
Poland
e-mail: olko@agh.edu.pl

Received: February 24, 2015