| **Aequationes Mathematicae**

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On Lipschitzian solutions to an inhomogeneous linear iterative equation

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Dedicated to Professor Roman Ger on his 70th birthday.

Abstract. Based on iteration of random-valued functions we study the problems of existence, uniqueness and continuous dependence of Lipschitzian solutions φ of the equation

$$\varphi(x) = F(x) - \int_\Omega \varphi(f(x,\omega))P(d\omega),$$

where P is a probability measure on a σ-algebra of subsets of Ω.

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1. Introduction

Fix a probability space $(\Omega, \mathcal{A}, P)$ and a complete and separable metric space (X, ρ).

Motivated by the appearance of the equation

$$\varphi(x) = 1 - \int_\Omega \varphi(f(x,\omega))P(d\omega)$$

in the theory of perpetuities and refinement equations, see Sect. 3.4 of the survey paper [3], we add a perturbation to the right-hand-side and consider problems of existence, uniqueness and continuous dependence of Lipschitzian solutions $\varphi \colon X \to \mathbb{R}$ to the equation

$$\varphi(x) = F(x) - \int_\Omega \varphi(f(x,\omega))P(d\omega). \qquad (1)$$

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Concerning the given function f we assume the following hypothesis (H) in which $\mathcal{B}$ stands for the σ-algebra of all Borel subsets of X.

(H) The function $f \colon X \times \Omega \to X$ is measurable with respect to the product σ-algebra $\mathcal{B} \otimes \mathcal{A}$,

$$\int_\Omega \rho\big(f(x,\omega), f(z,\omega)\big) P(d\omega) \leq \lambda \rho(x,z) \quad \text{for all} \quad x, z \in X$$

with a $\lambda \in [0,1)$, and

$$\int_\Omega \rho\big(f(x,\omega), x\big) P(d\omega) < \infty \quad \text{for every} \quad x \in X.$$

2. Tools

As emphasized in [4, Section 0.3] iteration is the fundamental technique for solving functional equations in a single variable, and iterates usually appear in the formulae for solutions. However, as it seems, Lipschitzian solutions are examined rather by the fixed-point method (cf. [4, Section 7.2D]). We use iteration of random-valued functions (see [4, Section 1.4]). For $f \colon X \times \Omega \to X$ we define

$$f^1(x, \omega_1, \omega_2, \dots) = f(x, \omega_1)$$

and

$$f^{n+1}(x, \omega_1, \omega_2, \dots) = f\big(f^n(x, \omega_1, \omega_2, \dots), \omega_{n+1}\big)$$

for all $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \dots)$ from Ω^∞ defined as $\Omega^\mathbb{N}$. Note that if f is measurable with respect to the product σ-algebra $\mathcal{B} \otimes \mathcal{A}$, then f^n is measurable with respect to the product σ-algebra $\mathcal{B} \otimes \mathcal{A}_n$, where $\mathcal{A}_n$ stands for the σ-algebra of all sets of the form

$$\{(\omega_1, \omega_2, \dots) \in \Omega^\infty : (\omega_1, \dots, \omega_n) \in A\}$$

with A from the product σ-algebra $\mathcal{A}^n$.

Let $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$ be the product probability space.

To prove a theorem on the existence and uniqueness of Lipschitzian solutions of Eq. (1) we apply [1, Theorem 3.1] which provides a simple criterion for the convergence in law of $(f^n(x, \cdot))_{n \in \mathbb{N}}$ to a random variable independent of x. We recall it below, but first we give some notions.

By a *distribution* (on X) we mean any probability measure defined on $\mathcal{B}$. A sequence $(\mu_n)_{n \in \mathbb{N}}$ of distributions *converges weakly* to a distribution μ, if

$$\lim_{n \to \infty} \int_X u(y) \mu_n(dy) = \int_X u(y) \mu(dy)$$

for every continuous and bounded function $u\colon X \to \mathbb{R}$. It is well known (see [2, Theorem 11.3.3]) that this convergence is metrizable by the (Fortet–Mourier, Lévy–Prohorov, Wasserstein) metric

$$\|\mu_1 - \mu_2\|_{FM} = \sup\left\{\left|\int_X u(y)\mu_1(dy) - \int_X u(y)\mu_2(dy)\right| : u \in \mathcal{F}\right\},$$

where $\mathcal{F}$ stands for the family of all functions $u\colon X \to \mathbb{R}$ such that

$$|u(x) - u(z)| \leq \rho(x, z) \quad \text{for all} \quad x, z \in X,$$

and

$$\|u\|_\infty = \sup\{|u(x)| : x \in X\} \leq 1.$$

The above mentioned theorem reads as

Theorem 2.1. *Assume* (H) *and let*

$$\pi_n(x, B) = P^\infty\big(\{\omega \in \Omega^\infty : f^n(x, \omega) \in B\}\big) \tag{2}$$

for all $n \in \mathbb{N}$, $x \in X$ *and* $B \in \mathcal{B}$. *Then there exists a distribution* π *on* X *such that for every* $x \in X$ *the sequence* $(\pi_n(x, \cdot))_{n \in \mathbb{N}}$ *converges weakly to* π; *moreover,*

$$\|\pi_n(x, \cdot) - \pi\|_{FM} \leq \frac{\lambda^n}{1 - \lambda} \int_\Omega \rho\big(f(x, \omega), x\big)P(d\omega) \tag{3}$$

for all $n \in \mathbb{N}$ *and* $x \in X$.

3. Existence and uniqueness

Following [1, Section 4] we shall prove the following theorem.

Theorem 3.1. *Assume* (H), *define* $(\pi_n)_{n \in \mathbb{N}}$ *by* (2), *and let* π *denote the weak limit of* $(\pi_n(x, \cdot))_{n \in \mathbb{N}}$ *for every* $x \in X$.

(i) *If* $F\colon X \to \mathbb{R}$ *is continuous and bounded, then any continuous and bounded solution* $\varphi\colon X \to \mathbb{R}$ *of Eq.* (1) *has the form*

$$\varphi(x) = F(x) - \frac{1}{2}\int_X F(y)\pi(dy)$$

$$+ \sum_{n=1}^\infty (-1)^n \left(\int_X F(y)\pi_n(x, dy) - \int_X F(y)\pi(dy)\right) \tag{4}$$

for every $x \in X$, *and*

$$\int_X \varphi(y)\pi(dy) = \frac{1}{2}\int_X F(y)\pi(dy). \tag{5}$$

(ii) *If $F\colon X \to \mathbb{R}$ is bounded and*

$$|F(x) - F(z)| \le L\rho(x,z) \quad \text{for all} \quad x, z \in X \tag{6}$$

with $L \in [0, +\infty)$, then for every $x \in X$ the series occurring in (4) converges absolutely, the function $\varphi\colon X \to \mathbb{R}$ defined by (4) is a Lipschitzian solution of Eq. (1),

$$|\varphi(x) - \varphi(z)| \le \frac{L}{1 - \lambda}\rho(x,z) \quad \text{for all} \quad x, z \in X, \tag{7}$$

and

$$|\varphi(x)| \le \frac{3}{2}\|F\|_\infty + \left(L + \|F\|_\infty\right)\frac{\lambda}{(1 - \lambda)^2}\int_\Omega \rho\big(f(x,\omega),x\big)P(d\omega) \tag{8}$$

for every $x \in X$.

Proof. Fix a continuous and bounded function $F\colon X \to \mathbb{R}$, let $\varphi\colon X \to \mathbb{R}$ be a continuous and bounded solution of Eq. (1), and define $F_0\colon X \to \mathbb{R}$ and $\varphi_0\colon X \to \mathbb{R}$ by

$$F_0(x) = F(x) - \int_X F(y)\pi(dy), \quad \varphi_0(x) = \varphi(x) - \frac{1}{2}\int_X F(y)\pi(dy).$$

Then

$$\int_X F_0(y)\pi(dy) = 0 \quad \text{and} \quad \varphi_0(x) = F_0(x) - \int_\Omega \varphi_0(f(x,\omega))P(d\omega)$$

for every $x \in X$. Hence, by induction,

$$\varphi_0(x) = F_0(x) + \sum_{k=1}^{n-1}(-1)^k\int_{\Omega^\infty} F_0(f^k(x,\omega))P^\infty(d\omega)$$

$$+ (-1)^n\int_{\Omega^\infty} \varphi_0(f^n(x,\omega))P^\infty(d\omega),$$

and then by (2),

$$\varphi_0(x) = F_0(x) + \sum_{k=1}^{n-1}(-1)^k\int_X F_0(y)\pi_k(x,dy) + (-1)^n\int_X \varphi_0(y)\pi_n(x,dy) \tag{9}$$

for all $n \in \mathbb{N}$ and $x \in X$. In particular, the right-hand-side of (9) does not depend on n, whence

$$(-1)^{2n}\int_X F_0(y)\pi_{2n}(x,dy) + (-1)^{2n+1}\int_X \varphi_0(y)\pi_{2n+1}(x,dy)$$

$$= (-1)^{2n}\int_X \varphi_0(y)\pi_{2n}(x,dy)$$

for all $n \in \mathbb{N}$ and $x \in X$, and passing to the limit we get

$$\int_X F_0(y)\pi(dy) - \int_X \varphi_0(y)\pi(dy) = \int_X \varphi_0(y)\pi(dy),$$

and

$$\int_X \varphi_0(y)\pi(dy) = \frac{1}{2}\int_X F_0(y)\pi(dy) = 0.$$

Consequently, (5) holds and making now use of (9) again for every $x \in X$ we have

$$\begin{aligned}
\varphi(x) &= \varphi_0(x) + \frac{1}{2}\int_X F(y)\pi(dy) \\
&= F_0(x) + \sum_{n=1}^{\infty}(-1)^n \int_X F_0(y)\pi_n(x,dy) + \frac{1}{2}\int_X F(y)\pi(dy) \\
&= F(x) - \frac{1}{2}\int_X F(y)\pi(dy) \\
&\quad + \sum_{n=1}^{\infty}(-1)^n \left(\int_X F(y)\pi_n(x,dy) - \int_X F(y)\pi(dy)\right).
\end{aligned}$$

This ends the proof of assertion (i).

To prove assertion (ii), define $M\colon X \to [0,\infty)$ by

$$M(x) = \left(L + \|F\|_\infty\right)\frac{1}{1-\lambda}\int_\Omega \rho\big(f(x,\omega),x\big)P(d\omega) \tag{10}$$

and observe that making use of (6), (3) and (10) we have

$$\begin{aligned}
&\left|\int_X F(y)\pi_n(x,dy) - \int_X F(y)\pi(dy)\right| \\
&\qquad \le \left(L + \|F\|_\infty\right)\|\pi_n(x,\cdot) - \pi\|_{FM} \le M(x)\lambda^n \tag{11}
\end{aligned}$$

for all $n \in \mathbb{N}$ and $x \in X$. It shows that for every $x \in X$ the series occurring in (4) converges absolutely. Define $\varphi\colon X \to \mathbb{R}$ by (4).

It follows from (H) that

$$\int_{\Omega^\infty} \rho\big(f^n(x,\omega), f^n(z,\omega)\big)P^\infty(d\omega) \le \lambda^n \rho(x,z)$$

for all $n \in \mathbb{N}$ and $x, z \in X$, whence, taking also (2) and (6) into account,

$$\begin{aligned}
&\left|\int_X F(y)\pi_n(x,dy) - \int_X F(y)\pi_n(z,dy)\right| \\
&\qquad = \left|\int_{\Omega^\infty} \Big(F(f^n(x,\omega)) - F(f^n(z,\omega))\Big)P^\infty(d\omega)\right| \le L\lambda^n \rho(x,z)
\end{aligned}$$

for all $n \in \mathbb{N}$ and $x, z \in X$, and (7) easily follows.

To get (8) it is enough to apply (4), (11) and (10).

It remains to show that φ solves (1). To this end fix an $x_0 \in X$. An obvious application of (H), (10) and (8) gives

$$M(x) \le \left(L + \|F\|_\infty\right)\frac{1+\lambda}{1-\lambda}\rho(x, x_0) + M(x_0),$$

$$|\varphi(x)| \le \frac{3}{2}\|F\|_\infty + \frac{\lambda}{1-\lambda}M(x)$$

for every $x \in X$. Therefore, putting $c_1 = \left(L + \|F\|_\infty\right)\frac{1+\lambda}{1-\lambda}\max\left\{1, \frac{\lambda}{1-\lambda}\right\}$ and $c_2 = \max\left\{M(x_0), \frac{\lambda}{1-\lambda}M(x_0) + \frac{3}{2}\|F\|_\infty\right\}$ we obtain

$$M(x) \le c_1\rho(x, x_0) + c_2, \quad |\varphi(x)| \le c_1\rho(x, x_0) + c_2 \quad \text{for every} \quad x \in X. \quad (12)$$

Since, according to the Fubini theorem, the function M is Borel, from (12) and (H) the integrability of $M \circ f(x, \cdot)$ and $\varphi \circ f(x, \cdot)$ for every $x \in X$ follows. Finally, making use of (4), (2), (11), the integrability of $M \circ f(x, \cdot)$, the Lebesgue dominated convergence theorem and the equality

$$f^n(f(x, \omega_1), \omega_2, \omega_3, \dots) = f^{n+1}(x, \omega_1, \omega_2, \dots)$$

which holds for all $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \dots) \in \Omega^\infty$, we see that

$$\int_\Omega \varphi(f(x, \omega))P(d\omega)$$

$$= \int_\Omega F(f(x, \omega))P(d\omega) - \frac{1}{2}\int_X F(y)\pi(dy)$$

$$+ \int_\Omega \sum_{n=1}^\infty (-1)^n\left(\int_X F(y)\pi_n(f(x, \omega), dy) - \int_X F(y)\pi(dy)\right)P(d\omega)$$

$$= \int_X F(y)\pi_1(x, dy) - \frac{1}{2}\int_X F(y)\pi(dy)$$

$$+ \sum_{n=1}^\infty (-1)^n\left(\int_\Omega \int_{\Omega^\infty} F\big(f^n(f(x, \omega_1), \omega_2, \omega_3, \dots)\big)\right.$$

$$\left. \times P^\infty\big(d(\omega_2, \omega_3, \dots)\big)P(d\omega_1) - \int_X F(y)\pi(dy)\right)$$

$$= \int_X F(y)\pi_1(x, dy) - \frac{1}{2}\int_X F(y)\pi(dy)$$

$$+ \sum_{n=1}^\infty (-1)^n\left(\int_{\Omega^\infty} F\big(f^{n+1}(x, \omega_1, \omega_2, \dots)\big)P^\infty\big(d(\omega_1, \omega_2, \dots)\big)\right.$$

$$\left. - \int_X F(y)\pi(dy)\right)$$

$$= \int_X F(y)\pi_1(x, dy) - \frac{1}{2}\int_X F(y)\pi(dy)$$

$$+ \sum_{n=1}^{\infty} (-1)^n \left(\int_X F(y)\pi_{n+1}(x, dy) - \int_X F(y)\pi(dy) \right)$$

$$= \frac{1}{2} \int_X F(y)\pi(dy)$$

$$- \sum_{n=1}^{\infty} (-1)^n \left(\int_X F(y)\pi_n(x, dy) - \int_X F(y)\pi(dy) \right)$$

$$= F(x) - \varphi(x)$$

for every $x \in X$. $\hspace{2cm}$ $\square$

4. Continuous dependence

Consider now the Banach space $BL(X)$ of all bounded Lipschitzian functions $F \colon X \to \mathbb{R}$ with the norm (see, e.g., [2, Section 11.2])

$$\|F\|_{BL} = \|F\|_{\infty} + \|F\|_L,$$

where $\|F\|_L$ stands for the smallest Lipschitz constant for F.

Assume (H). According to Theorem 3.1 for every $F \in BL(X)$ the formula

$$\varphi^F(x) = F(x) - \frac{1}{2} \int_X F(y)\pi(dy)$$

$$+ \sum_{n=1}^{\infty} (-1)^n \left(\int_X F(y)\pi_n(x, dy) - \int_X F(y)\pi(dy) \right) \qquad (13)$$

for every $x \in X$, defines a Lipschitzian solution of Eq. (1) and

$$|\varphi^F(x)| \leq \frac{3}{2}\|F\|_{\infty} + \frac{\lambda}{(1-\lambda)^2} \int_{\Omega} \rho\big(f(x, \omega), x\big) P(d\omega) \cdot \|F\|_{BL}$$

for every $x \in X$. Hence, since the operator

$$F \mapsto \varphi^F, \qquad F \in BL(X), \qquad (14)$$

is linear, if $F, G \in BL(X)$, then

$$\big|\varphi^F(x) - \varphi^G(x)\big| \leq \frac{3}{2}\|F - G\|_{\infty} + \frac{\lambda}{(1-\lambda)^2} \int_{\Omega} \rho\big(f(x, \omega), x\big) P(d\omega) \cdot \|F - G\|_{BL}$$

for every $x \in X$. In particular, for every bounded subset B of X there exists $c_B \in [0, \infty)$ such that

$$\big|\varphi^F(x) - \varphi^G(x)\big| \leq c_B \|F - G\|_{BL} \qquad \text{for all} \quad x \in B \quad \text{and} \quad F, G \in BL(X).$$

Finally, if

$$c := \sup \left\{ \int_{\Omega} \rho\big(f(x, \omega), x\big) P(d\omega) : x \in X \right\} < \infty,$$

then for every $F \in BL(X)$ the function φ^F is the only continuous and bounded solution of Eq. (1), it belongs to $BL(X)$ and [cf. (8) and (7)]

$$\|\varphi^F\|_{BL} \le \frac{3}{2}\|F\|_\infty + \frac{\lambda c}{(1-\lambda)^2}\|F\|_{BL} + \frac{\|F\|_L}{1-\lambda}$$
$$\le \left(\max\left\{\frac{3}{2}, \frac{1}{1-\lambda}\right\} + \frac{\lambda c}{(1-\lambda)^2}\right)\|F\|_{BL}.$$

Theorem 4.1. *Assume* (H). *If (X, ρ) is bounded, then operator* (14) *given by* (13) *is a linear homeomorphism of $BL(X)$ onto itself.*

Proof. It remains to show that operator (14) is one-to-one, onto, and the inverse operator is continuous.

The first property follows from the fact that for any $F \in BL(X)$ the function φ^F given by (13) is a solution of Eq. (1): if $\varphi^F = 0$, then $F = 0$. To get the remaining two it is enough to observe that if $\psi \in BL(X)$, then the function $F \colon X \to \mathbb{R}$ given by

$$F(x) = \psi(x) + \int_\Omega \psi(f(x, \omega))P(d\omega)$$

belongs to $BL(X)$ with $\|F\|_\infty \le 2\|\psi\|_\infty$, $\|F\|_L \le \|\psi\|_L + \lambda\|\psi\|_L$, whence $\|F\|_{BL} \le 2\|\psi\|_{BL}$, and, since both ψ and φ^F solve (1), $\varphi^F = \psi$. $\qquad\square$

Remark 4.2. To get the continuity of the inverse operator in the above proof it would be enough to apply the Banach inverse mapping theorem, but by our direct proof we also have the estimate

$$\frac{1}{2}\|F\|_{BL} \le \|\varphi^F\|_{BL} \quad \text{for every} \quad F \in BL(X).$$

Acknowledgments

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