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From a theorem of R. Ger and T. Kochanek to marginal joints of means

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Dedicated to Professor Roman Ger on the occasion of his 70th birthday

Abstract. We answer in the negative a problem posed in Daróczy (Report on 52nd International Symposium on Functional Equations. Aequat. Math., 2015) by the first author, in connection with a result of Ger and Kochanek (Colloq Math 115:87–99, 2009), and its generalization formulated in Daróczy et al. (Report on 52nd International Symposium on Functional Equations. Aequat. Math., 2015). A further generalization is posed as an open problem. Elaborating an idea of the construction of means presented in Examples 1.2 and 1.4 we come to the notion of marginal joints of means. It provides a pretty wide class of means extending two given means on adjacent intervals.

Mathematics Subject Classification. Primary 26E60; Secondary 39B22.

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Introduction

In the whole paper, given any interval I, a function $M: I \times I \to I$ is called a *mean on* I if

$$\min\{x,y\} \le M(x,y) \le \max\{x,y\}$$

for all $x, y \in I$. If both inequalities are sharp whenever $x \ne y$, then the mean M is called *strict*. Of course, if a mean M is strictly increasing with respect to each variable, then it is strict. A mean M on I is said to be *symmetric* if

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$$M(x, y) = M(y, x)$$

for all $x, y \in I$.

Given any continuous strictly monotonic function $\varphi \colon I \to \mathbb{R}$, the formula

$$M(x, y) = \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right)$$

defines a mean on I; it is called the *quasi-arithmetic* mean with the *generator* φ and denoted also as A^{φ}. Observe that any quasi-arithmetic mean strictly increases with respect to each variable, and thus it is strict.

Our starting point is the following result, proved by Ger and Kochanek [7].

Theorem GK. *Let M be a continuous mean on an interval I, strictly increasing with respect to each variable. If $f \colon I \to \mathbb{R}$ is a non-constant solution of the equation*

$$f(M(x, y)) = \frac{f(x) + f(y)}{2}, \tag{0.1}$$

then the mean M is quasi-arithmetic.

During the 52nd International Symposium on Functional Equations held in Innsbruck in 2014 the following problem was posed by the first author, in connection with the Theorem GK (see [4]).

Problem 0.1. Let M be a mean on an interval I. Is it true that if M is not quasi-arithmetic, then every solution $f \colon I \to \mathbb{R}$ of Eq. (0.1) is constant?

1. Problem 0.1 and its generalizations

We start with a procedure providing means essentially generalizing quasi-arithmetic means. Among them are those giving a negative answer to Problem 0.1 (cf. [5]).

Procedure. Let I be an interval and let $f \colon I \to \mathbb{R}$ be a continuous and (not necessarily strictly) increasing function. For any pair $(x, y) \in I \times I$ put $M(x, y)$ as an arbitrary point of the interval $f^{-1}\left(\left\{\frac{f(x)+f(y)}{2}\right\}\right)$ whenever $f(x) \neq f(y)$ and let $M(x, y)$ be any point of the interval ending in x and y otherwise.

Take any $x, y \in I$, say $x \le y$. If $f(x) \neq f(y)$ then, clearly, equality (0.1) holds. If $f(x) = f(y)$ then $x \le M(x, y) \le y$, whence $f(x) = f(M(x, y)) = f(y)$ by the monotonicity of f, and (0.1) follows again.

To see that M is a mean on I fix any pair $(x, y) \in I \times I$. Assume, for instance, that $x \le y$. If $f(x) = f(y)$ then $x \le M(x, y) \le y$. So consider the case $f(x) \neq f(y)$. This implies $f(x) < f(y)$. If $x > M(x, y)$ then

$$f(x) \ge f(M(x, y)) = \frac{f(x) + f(y)}{2} > f(x),$$

a contradiction. Therefore, $x \le M(x, y)$. Similarly $M(x, y) \le y$.

Observe that if f, in fact, is strictly increasing, then M is the quasi-arithmetic mean generated by f. To obtain a not quasi-arithmetic M we add some requirements. First of all take a continuous increasing $f\colon I \to \mathbb{R}$ which is non-constant and not strictly increasing. Then we can find a non-trivial interval $J \subset I$, a number $c \in \mathbb{R}$ and a point $y_0 \in I$ such that $f(y_0) \neq c$ and

$$f(x) = c, \quad x \in J.$$

Fix a point $m \in f^{-1}\left(\left\{\frac{c+f(y_0)}{2}\right\}\right)$. Then we can additionally require that

$$M(x,y) = m$$

for any $x, y \in I$ such that $f(x) \neq f(y)$ and $\frac{f(x)+f(y)}{2} = \frac{c+f(y_0)}{2}$. Take any different $x_1, x_2 \in J$. Then $f(x_1) = f(x_2) = c \neq f(y_0)$, and thus

$$M(x_1, y_0) = m = M(x_2, y_0).$$

Consequently, M is not quasi-arithmetic being not strictly increasing with respect to the first variable.

Observe that in the Procedure we have a pretty large freedom to define a required mean for any given f. For instance, in some cases we may expect that the mean constructed in the Procedure has nice properties, e.g. is symmetric and increasing (not necessarily strictly) with respect to each variable. This can be seen from the following result.

Example 1.1. Assume that the left endpoint of the interval I is finite and I contains it. Let $f\colon I \to \mathbb{R}$ be a continuous increasing function. For any pair $(x,y) \in I \times I$ put

$$M(x,y) = \begin{cases} \inf f^{-1}\left(\left\{\frac{f(x)+f(y)}{2}\right\}\right), & \text{if } \ f(x) \neq f(y), \\ \min\{x,y\}, & \text{if } \ f(x) = f(y). \end{cases}$$

Clearly M is a mean of the type described in the Procedure. Obviously it is symmetric. Less evident is that it is increasing with respect to each variable. To prove this fix any $y_0 \in I$ and $x_1, x_2 \in I$ with $x_1 < x_2$. Then $f(x_1) \leq f(x_2)$ and exactly one of the following four possibilities can hold:

(a) $f(x_1) = f(x_2) = f(y_0)$,
(b) $f(y_0) = f(x_1) < f(x_2)$,
(c) $f(x_1) < f(x_2) = f(y_0)$,
(d) $f(x_1) \neq f(y_0)$ and $f(x_2) \neq f(y_0)$.

In case (a) we have

$$M(x_1, y_0) = \min\{x_1, y_0\} \leq \min\{x_2, y_0\} = M(x_2, y_0).$$

Assuming (b) we see that

$$f(x_1) = \frac{f(x_1) + f(x_1)}{2} < \frac{f(x_2) + f(y_0)}{2},$$

whence

$$x_1 \leq \inf f^{-1}\left(\left\{\frac{f(x_2)+f(y_0)}{2}\right\}\right) = M\left(x_2, y_0\right),$$

and, consequently,

$$M\left(x_1, y_0\right) = \min\left\{x_1, y_0\right\} \leq x_1 \leq M\left(x_2, y_0\right).$$

If case (c) holds, then $x_2, y_0 \in f^{-1}\left(\left\{f\left(y_0\right)\right\}\right)$, so

$$\inf f^{-1}\left(\left\{f\left(y_0\right)\right\}\right) \leq \min\left\{x_2, y_0\right\} = M\left(x_2, y_0\right),$$

and thus

$$\begin{aligned} M\left(x_1, y_0\right) &= \inf f^{-1}\left(\left\{\frac{f(x_1)+f(y_0)}{2}\right\}\right) \leq \inf f^{-1}\left(\left\{\frac{f(y_0)+f(y_0)}{2}\right\}\right) \\ &= \inf f^{-1}\left(\left\{f\left(y_0\right)\right\}\right) \leq M\left(x_2, y_0\right). \end{aligned}$$

Finally (d) implies

$$\begin{aligned} M\left(x_1, y_0\right) &= \inf f^{-1}\left(\left\{\frac{f(x_1)+f(y_0)}{2}\right\}\right) \leq \inf f^{-1}\left(\left\{\frac{f(x_2)+f(y_0)}{2}\right\}\right) \\ &= M\left(x_2, y_0\right). \end{aligned}$$

Consequently, the function $M\left(\cdot, y_0\right)$ is increasing for any $y_0 \in I$. Since M is symmetric, it follows that also the function $M\left(x_0, \cdot\right)$ is increasing for any $x_0 \in I$.

On the other hand the mean M is not, in general, continuous. To see this take $I = [0, \infty)$ and $f \colon I \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ x, & \text{if } 1 \leq x. \end{cases}$$

Then $f(0) = f(1) = 1$ and $f\left(1+\frac{1}{n}\right) = 1+\frac{1}{n} \neq f(0)$, and thus

$$\begin{aligned} M\left(1+\frac{1}{n}, 0\right) &= \inf f^{-1}\left(\left\{\frac{f(1+\frac{1}{n})+f(0)}{2}\right\}\right) \\ &= \inf f^{-1}\left(\left\{\frac{1+\frac{1}{n}+1}{2}\right\}\right) \\ &= \inf f^{-1}\left(\left\{1+\frac{1}{2n}\right\}\right) = 1+\frac{1}{2n} \end{aligned}$$

and

$$M\left(1, 0\right) = \min\{1, 0\} = 0 \neq 1 = \lim_{n\to\infty} M\left(1+\frac{1}{n}, 0\right).$$

Example 1.2. Another counterexample to Problem 0.1 can be given using quite a different idea than that from the Procedure (see [5]). Define $M\colon (0,\infty)\times(0,\infty)\to(0,\infty)$ and $f\colon (0,\infty)\to\mathbb{R}$ by

$$M(x,y)=\begin{cases}\frac{x+y}{2}, & \text{if } 0<x,y<1,\\ \frac{x+1}{2}, & \text{if } 0<x<1\le y,\\ \frac{1+y}{2}, & \text{if } 1\le y\le x,\\ \min\{x,y\}, & \text{if } 1\le x,y,\end{cases} \tag{1.1}$$

and

$$f(x)=\begin{cases}x, & \text{if } 0<x<1,\\ 1, & \text{if } 1\le x,\end{cases} \tag{1.2}$$

respectively. Obviously, M is a mean which is symmetric, increasing with respect to each variable and continuous, but it is not quasi-arithmetic.

Fix any $(x,y)\in(0,\infty)\times(0,\infty)$. If $x,y<1$ then

$$f\left(M\left(x,y\right)\right)=f\left(\frac{x+y}{2}\right)=\frac{x+y}{2}=\frac{f(x)+f(y)}{2}.$$

In the case when $x<1\le y$ we have

$$f\left(M\left(x,y\right)\right)=f\left(\frac{x+1}{2}\right)=\frac{x+1}{2}=\frac{f(x)+f(y)}{2}.$$

Similarly, equality (0.1) holds when $y<1\le x$. Finally, if $1<x,y$ then

$$f\left(M\left(x,y\right)\right)=f\left(\min\{x,y\}\right)=1=\frac{1+1}{2}=\frac{f(x)+f(y)}{2}.$$

Thus equality (0.1) is satisfied in all the possible cases.

Clearly, since the mean min considered on any non-trivial interval is not strict, the above mean M is not strict, either. Since, at first glance it seemed to us that also the means constructed by the Procedure were not strict, still during the ISFE in Innsbruck we proposed the following modified problem.

Problem 1.1. Let M be a strict mean on an interval I. Is it true that if M is not quasi-arithmetic, then every solution $f\colon I\to\mathbb{R}$ of Eq. (0.1) is constant?

Trying to solve that problem we examined the Procedure in detail once more. Then, rather unexpectedly, we found out that a small modification of the construction described there provides strict means.

Example 1.3. Let $f\colon I\to\mathbb{R}$ be a continuous increasing function. For any point $(x,y)\in I\times I$ let $M(x,y)$ be an arbitrary point of the interval $f^{-1}\left(\left\{\frac{f(x)+f(y)}{2}\right\}\right)$ whenever $f(x)\neq f(y)$ and any point lying strictly between x and y when $x\neq y$ and $f(x)=f(y)$. To prove the strictness of M fix an arbitrary point $(x,y)\in I\times I$ such that $x<y$. If $f(x)=f(y)$ then $x<M(x,y)<y$

by the definition of M. So we may assume that $f(x) \neq f(y)$, which forces (0.1) as in Example 1.1. Now, if either $M(x,y) = x$, or $M(x,y) = y$, then (0.1) implies

$$f\,(x) = \frac{f(x)+f(y)}{2} \quad \text{or} \quad f\,(y) = \frac{f(x)+f(y)}{2},$$

which gives $f(x) = f(y)$, a contradiction. Thus again we come to the inequalities $x < M(x,y) < y$.

Remember, however, that M, in general, is not continuous.

Of course also Example 1.2 can be suitably improved. To see this consider the next example.

Example 1.4. Take any strict mean m on the interval $[1,\infty)$ and consider the mean M on $(0,\infty)$ defined by

$$M(x,y) = \begin{cases} \frac{x+y}{2}, & \text{if } 0 < x, y < 1, \\ \frac{x+1}{2}, & \text{if } 0 < x < 1 \leq y, \\ \frac{1+y}{2}, & \text{if } 0 < y < 1 \leq x, \\ m(x,y), & \text{if } 1 \leq x, y. \end{cases} \tag{1.3}$$

Observe that the mean M is strict and we have got its formula by replacing the mean min on $[1,\infty)$ in (1.1) by the strict mean m. Also now M satisfies Eq. (0.1) with the function $f\colon (0,\infty) \to \mathbb{R}$ defined by formula (1.2).

However, no matter what strict mean m on $[1,\infty)$ is used, unexpectedly the mean M defined by (1.3) cannot be continuous! Indeed, supposing that M is continuous, for any $x \in [1,\infty)$ we have

$$m(x,1) = M(x,1) = \lim_{n\to\infty} M\left(x, 1 - \frac{1}{n}\right) = \lim_{n\to\infty} \frac{1 + \left(1 - \frac{1}{n}\right)}{2} = 1,$$

which is impossible as m is strict.

2. Marginal joints of means

The form of the means defined by formulas (1.1) and (1.3) in Examples 1.2 and 1.4, respectively, suggests the study of means described as follows. Given an interval I and its interior point ξ put

$$I_\xi := \{x \in I \colon x \leq \xi\}, \quad I_\xi^\circ := \{x \in I \colon x < \xi\}$$

and

$${}_\xi I := \{x \in I \colon x \geq \xi\}, \quad {}_\xi I^\circ := \{x \in I \colon x > \xi\}$$

(see Fig. 1).

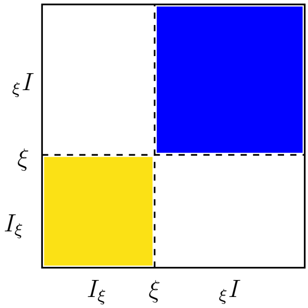

FIGURE 1. Division of I into I_ξ and ξ^I

Then, for every continuous and strictly monotonic function $\varphi\colon I_\xi \to \mathbb{R}$ and for any mean m on the interval $_\xi I$, define the mean M by

$$M(x,y) = \begin{cases} A^\varphi(x,y), & \text{if } (x,y) \in I_\xi \times I_\xi, \\ A^\varphi(x,\xi), & \text{if } (x,y) \in I_\xi^\circ \times {}_\xi I^\circ, \\ A^\varphi(\xi,y), & \text{if } (x,y) \in {}_\xi I^\circ \times I_\xi^\circ, \\ m(x,y), & \text{if } (x,y) \in {}_\xi I \times {}_\xi I. \end{cases} \tag{2.1}$$

Observe that then the function $f\colon I \to \mathbb{R}$, given by

$$f(x) = \begin{cases} \varphi(x), & \text{if } x \in I_\xi^\circ, \\ \varphi(\xi), & \text{if } x \in {}_\xi I, \end{cases}$$

is a continuous, non-constant and (not strictly) monotonic solution of Eq. (0.1).

Following the above idea we propose to consider a more general class of means. Let I be an interval and ξ be its interior point. Assume that we have given two means M and N on the intervals I_ξ and $_\xi I$, respectively. The problem is to find a mean, say $M \oplus N$, on the interval I such that

$$(M \oplus N)\,|_{I_\xi \times I_\xi} = M \quad \text{and} \quad (M \oplus N)\,|_{{}_\xi I \times {}_\xi I} = N.$$

One possible way to do this is as follows. Define the marginal functions $f_1, f_2\colon I_\xi \to I_\xi$ and $g_1, g_2\colon {}_\xi I \to {}_\xi I$ by

$$f_1(x) = M(x,\xi), f_2(y) = M(\xi,y) \quad \text{and} \quad g_1(x) = N(x,\xi),\ g_2(y) = N(\xi,y),$$

respectively. Observe that provided f_1, f_2, g_1, g_2 are continuous and strictly increasing, the sets $f_1(I_\xi), f_2(I_\xi), g_1({}_\xi I)$ and $g_2({}_\xi I)$ are intervals having ξ as one of the endpoints (Fig. 2).

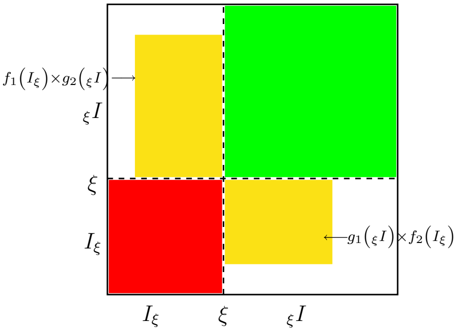

Figure 2. Division of M, N and K

Take any function $K\colon f_1(I_\xi)\times g_2({}_\xi I)\cup g_1({}_\xi I)\times f_2(I_\xi)\to I$ such that

$$\min\{x,y\}\le K(x,y)\le\max\{x,y\} \tag{2.2}$$

for all $(x,y)\in f_1(I_\xi)\times g_2({}_\xi I)\cup g_1({}_\xi I)\times f_2(I_\xi)$ as well as

$$K(x,\xi)=\begin{cases} f_1(x), & \text{if } x\in f_1(I_\xi),\\ g_1(x), & \text{if } x\in g_1({}_\xi I),\end{cases} \tag{2.3}$$

and

$$K(\xi,y)=\begin{cases} f_2(y), & \text{if } y\in f_2(I_\xi),\\ g_2(y), & \text{if } y\in g_2({}_\xi I).\end{cases} \tag{2.4}$$

Any such K will be called a *joining function* for the pair (M,N).

There are many joining functions for (M,N), where M and N are means on I_ξ and ${}_\xi I$, respectively. For instance any $K\colon f_1(I_\xi)\times g_2({}_\xi I)\cup g_1({}_\xi I)\times f_2(I_\xi)\to I$ satisfying (2.3) and (2.4), and the conditions

$$\min\{x,y\}\le K(x,y)\le\xi$$

for all $(x,y)\in f_1(I_\xi)\times g_2({}_\xi I)\cup g_1({}_\xi I)\times f_2(I_\xi)$, or

$$\xi\le K(x,y)\le\max\{x,y\}$$

for all $(x,y)\in f_1(I_\xi)\times g_2({}_\xi I)\cup g_1({}_\xi I)\times f_2(I_\xi)$, is a joining function for the pair (M,N). The most trivial one is that satisfying (2.3) and (2.4) and the condition

$$K(x,y)=\xi,\quad (x,y)\in f_1(I_\xi)\times g_2({}_\xi I)\cup g_1({}_\xi I)\times f_2(I_\xi).$$

Less trivial but very natural joining functions for the pair (M,N) are presented in two examples below.

Example 2.1. The formula

$$K(x,y)=\begin{cases} f_1(x)+g_2(y)-\xi, & \text{if } (x,y)\in f_1(I_\xi)\times g_2({}_\xi I),\\ g_1(x)+f_2(y)-\xi, & \text{if } (x,y)\in g_1({}_\xi I)\times f_2(I_\xi),\end{cases} \tag{2.5}$$

where f_1, f_2, g_1, g_2 are the marginal functions for (M, N), defines a joining function for this pair. To see inequalities (2.2) fix a point $(x, y) \in f_1(I_\xi) \times g_2({}_\xi I)$. Then $x \le f_1(x) \le \xi \le g_2(y) \le y$, whence

$$\begin{aligned}\min\{x, y\} = x \le f_1(x) \le f_1(x) + g_2(y) - \xi = K(x, y)\\ \le g_2(y) \le y = \max\{x, y\},\end{aligned}$$

that is (2.2) holds. Similarly we get (2.2) for all $(x, y) \in g_1({}_\xi I) \times f_2(I_\xi)$. Equalities (2.3) and (2.4) are obvious. An additional important advantage of the above formula is that if f_1, f_2, g_1, g_2 are continuous and strictly increasing, then K is strictly increasing with respect to each variable and continuous.

Given a continuous strictly monotonic function φ mapping an interval into $\mathbb{R}$ and a number $p \in (0, 1)$ we denote by A_p^φ the quasi-arithmetic mean, generated by φ and weighted by $(p, 1-p)$, i.e. the mean on that interval defined by

$$A_p^\varphi(x, y) = \varphi^{-1}\left(p\varphi(x) + (1-p)\varphi(y)\right).$$

Example 2.2. Let $\varphi\colon I_\xi \to \mathbb{R}$ and $\psi\colon {}_\xi I \to \mathbb{R}$ be continuous strictly increasing functions vanishing at ξ and let $p, q \in (0, 1)$. We find a joining function for the pair (M, N) where $M = A_p^\varphi$ and $N = A_q^\psi$. Then

$$f_1(x) = A_p^\varphi(x, \xi) = \varphi^{-1}\left(p\varphi(x)\right),\ f_2(y) = A_p^\varphi(\xi, y) = \varphi^{-1}\left((1-p)\varphi(y)\right)$$

for all $x, y \in I_\xi$ and

$$g_1(x) = A_q^\psi(x, \xi) = \psi^{-1}\left(q\psi(x)\right),\ g_2(y) = A_q^\psi(\xi, y) = \psi^{-1}\left((1-q)\psi(y)\right)$$

for all $x, y \in {}_\xi I$.

For any $(x, y) \in f_1(I_\xi) \times g_2({}_\xi I)$ put

$$K(x, y) = \begin{cases} \varphi^{-1}\left(p\left(\varphi(x) + \psi(y)\right)\right), & \text{if } \varphi(x) + \psi(y) < 0,\\ \psi^{-1}\left((1-q)\left(\varphi(x) + \psi(y)\right)\right), & \text{if } \varphi(x) + \psi(y) \ge 0.\end{cases} \tag{2.6}$$

Similarly, having $(x, y) \in g_1({}_\xi I) \times f_2(I_\xi)$ we put

$$K(x, y) = \begin{cases} \varphi^{-1}\left((1-p)\left(\psi(x) + \varphi(y)\right)\right), & \text{if } \psi(x) + \varphi(y) < 0,\\ \psi^{-1}\left(q\left(\psi(x) + \varphi(y)\right)\right), & \text{if } \psi(x) + \varphi(y) \ge 0.\end{cases} \tag{2.7}$$

Evidently the function $K\colon f_1(I_\xi) \times g_2({}_\xi I) \cup g_1({}_\xi I) \times f_2(I_\xi) \to I$ is strictly increasing with respect to each variable and continuous.

Take any $(x, y) \in f_1(I_\xi) \times g_2({}_\xi I)$. Then $x \le \xi \le y$, whence $\varphi(x) \le \varphi(\xi) = 0 = \psi(\xi) \le \psi(y)$. Clearly,

$$K(x, \xi) = \varphi^{-1}\left(p\varphi(x)\right) = f_1(x)$$

and

$$K(\xi, y) = \psi^{-1}\left((1-q)\psi(y)\right) = g_2(y).$$

Moreover, if $\varphi(x) + \psi(y) < 0$ then

$$\min\{x, y\} = x \le \varphi^{-1}\left(p\varphi(x)\right) \le \varphi^{-1}\left(p\left(\varphi(x) + \psi(y)\right)\right) = K(x, y)$$

$$\leq \varphi^{-1}(p \cdot 0) = \xi \leq y = \max\{x, y\},$$

and if $\varphi(x) + \psi(y) \geq 0$ then

$$\begin{aligned}\min\{x, y\} = x \leq \xi = \psi^{-1}\left((1-q)\cdot 0\right) &\leq \psi^{-1}\left((1-q)\left(\varphi(x)+\psi(y)\right)\right) \\ = K(x,y) &\leq \psi^{-1}((1-q)\psi(y)) \leq y = \max\{x, y\}.\end{aligned}$$

For $(x, y) \in g_1\left({}_\xi I\right) \times f_2\left(I_\xi\right)$ we proceed analogously. Consequently, K is a joining function for the pair $\left(A_p^\varphi, A_q^\psi\right)$, strictly increasing with respect to each variable and continuous.

Fix means M and N on the intervals I_ξ and ${}_\xi I$, respectively, and a joining function K for the pair (M, N). Let f_1, f_2, g_1, g_2 be the marginal functions and put

$$\begin{aligned}
D_{12}^- &:= \left\{(x,y) \in I_\xi^\circ \times {}_\xi I^\circ \colon K\left(f_1(x), g_2(y)\right) < \xi\right\}, \\
D_{12} &:= \left\{(x,y) \in I_\xi^\circ \times {}_\xi I^\circ \colon\ K\left(f_1(x), g_2(y)\right) = \xi\right\}, \\
D_{12}^+ &:= \left\{(x,y) \in I_\xi^\circ \times {}_\xi I^\circ \colon K\left(f_1(x), g_2(y)\right) > \xi\right\}, \\
D_{21}^- &:= \left\{(x,y) \in {}_\xi I^\circ \times I_\xi^\circ \colon K\left(g_1(x), f_2(y)\right) < \xi\right\}, \\
D_{22} &:= \left\{(x,y) \in {}_\xi I^\circ \times I_\xi^\circ \colon\ K\left(g_1(x), f_2(y)\right) = \xi\right\}, \\
D_{21}^+ &:= \left\{(x,y) \in {}_\xi I^\circ \times I_\xi^\circ \colon K\left(g_1(x), f_2(y)\right) > \xi\right\}.
\end{aligned}$$

Obviously, the above sets are pairwise disjoint and

$$D_{12}^- \cup D_{12} \cup D_{12}^+ = I_\xi^\circ \times {}_\xi I^\circ \quad \text{and} \quad D_{21}^- \cup D_{21} \cup D_{21}^+ = {}_\xi I^\circ \times I_\xi^\circ.$$

Some of those six sets may be empty (cf. Example 2.3). However, under some rather natural assumptions like continuity and strict increase, imposed on M, N and K, they are non-empty and have a really simple and nice structure, which can be seen from the next result.

Theorem 2.1. *Let M and N be means on the intervals I_ξ and ${}_\xi I$, respectively, and K be a joining function for the pair (M, N). Assume that the functions M, N and K are strictly increasing with respect to each variable and continuous. Then there are an interval $J \subset I$ containing ξ in its interior and a continuous strictly decreasing function $\omega\colon J \to I$ such that the graph $\mathrm{Gr}\,\omega$ of ω, i.e. the curve $\mathrm{Gr}\,\omega = \{(x,y) \in I \times I \colon y = \omega(x)\}$, joints two sides of the square $I \times I$,*

$$\mathrm{Gr}\,\omega = D_{12} \cup \{(\xi, \xi)\} \cup D_{21},$$

and the strict epigraph $\mathrm{epi_s}\omega = \{(x,y) \in J \times I \colon y > \omega(x)\}$ and the strict hypograph $\mathrm{hyp_s}\omega = \{(x,y) \in J \times I \colon y < \omega(x)\}$ of ω have the decompositions

$$\mathrm{epi_s}\omega = D_{12}^+ \cup \left({}_\xi J \times {}_\xi I \setminus \{(\xi,\xi)\}\right) \cup D_{21}^+, \tag{2.8}$$

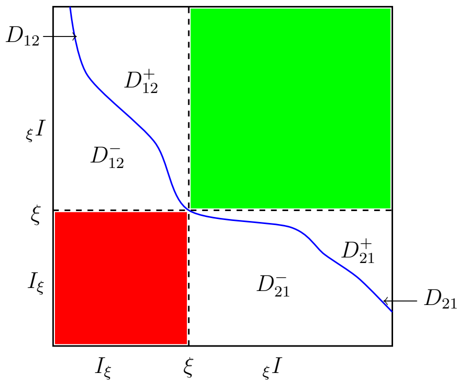

FIGURE 3. $\mathrm{Gr}\omega = D_{12} \bigcup \{(\xi,\xi)\} \bigcup D_{21}$

and

$$\mathrm{hyp_s}\omega = D_{12}^{-} \cup (J_{\xi} \times I_{\xi} \setminus \{(\xi,\xi)\}) \cup D_{21}^{-},$$

respectively (cf. Fig. 3).

If, in addition,

$$M(x,\xi) = M(\xi,x), \quad x \in I_{\xi}, \quad \textit{and} \quad N(x,\xi) = N(\xi,x), \quad x \in {}_{\xi}I, \tag{2.9}$$

and the function K is symmetric:

$$K(x,y) = K(y,x), \quad (x,y) \in f\left(I_{\xi}\right) \times g\left({}_{\xi}I\right) \cup g\left({}_{\xi}I\right) \times f\left(I_{\xi}\right),$$

where $f\colon I_{\xi} \to I_{\xi}$ and $g\colon {}_{\xi}I \to {}_{\xi}I$ are given by

$$f(x) = M(x,\xi) \quad \textit{and} \quad g(x) = N(x,\xi),$$

respectively, then $\omega(J) = J$ and

$$\omega\left(\omega\left(x\right)\right) = x, \quad x \in J,$$

that is ω is an involution.

Proof. Put

$$J := J_{\xi} \cup {}_{\xi}J,$$

where

$$J_{\xi} = \{x \in I_{\xi} \colon K\left(f_1(x), g_2(y)\right) = \xi \text{ with some } y \in {}_{\xi}I\} \tag{2.10}$$

and

$${}_{\xi}J = \{x \in {}_{\xi}I \colon K\left(g_1(x), f_2(y)\right) = \xi \text{ with some } y \in I_{\xi}\}. \tag{2.11}$$

Fix elements $x_1, x_2 \in J_\xi$, $x_1 < x_2$, and take any $x \in (x_1, x_2)$. Let $y_1, y_2 \in {}_\xi I$ satisfy $K(f_1(x_1), g_2(y_1)) = \xi = K(f_1(x_2), g_2(y_2))$. Then, as the functions $f_1, f_2, K(\cdot, g_2(y_1))$ and $K(\cdot, g_2(y_2))$ are strictly increasing, we have

$$\begin{aligned} K(f_1(x), g_2(y_2)) &< K(f_1(x_2), g_2(y_2)) = \xi \\ &= K(f_1(x_1), g_2(y_1)) < K(f_1(x), g_2(y_1)), \end{aligned}$$

and the continuity of the function $K(f_1(x), g_2(\cdot))$ implies that

$$K(f_1(x), g_2(y)) = \xi$$

for a y lying between y_1 and y_2, that is in ${}_\xi I$. Therefore $x \in J_\xi$. This proves that J_ξ is an interval. Similarly, one can check that so is ${}_\xi J$. Since

$$K(f_1(\xi), g_2(\xi)) = K(\xi, \xi) = \xi = K(g_1(\xi), f_2(\xi)), \tag{2.12}$$

we have $\xi \in J_\xi \cap {}_\xi J$, and thus J is an interval containing ξ. Moreover, the continuity and monotonicity assumptions imposed on K and, consequently, on the functions f_1, f_2, g_1, g_2, together with (2.12) give $\xi \in \operatorname{int} J$.

Observe that for every $x \in J_\xi$ the element $y \in {}_\xi I$ satisfying the equality $K(f_1(x), g_2(y)) = \xi$ is unique. Analogously, for every $x \in {}_\xi J$ the point $y \in I_\xi$ such that $K(g_1(x), f_2(y)) = \xi$ is determined uniquely. Therefore the formula

$$\omega(x) = y \Longleftrightarrow \begin{cases} K(f_1(x), g_2(y)) = \xi, & \text{if } x \in J_\xi, \\ K(g_1(x), f_2(y)) = \xi, & \text{if } x \in {}_\xi J, \end{cases}$$

defines a function $\omega\colon J \to I$. Of course

$$\omega(J_\xi) \subset {}_\xi I \quad \text{and} \quad \omega({}_\xi J) \subset I_\xi; \tag{2.13}$$

in particular, $\omega(\xi) = \xi$.

If $x_1, x_2 \in J_\xi$ and $x_1 < x_2$, then

$$\begin{aligned} K(f_1(x_2), g_2(\omega(x_2))) = \xi &= K(f_1(x_1), g_2(\omega(x_1))) \\ &< K(f_1(x_2), g_2(\omega(x_1))), \end{aligned}$$

whence $g_2(\omega(x_2)) < g_2(\omega(x_1))$, i.e. $\omega(x_2) < \omega(x_1)$. Thus $\omega|_{J_\xi}$ is strictly decreasing. Similarly, so is $\omega|_{{}_\xi J}$ and, by (2.13), also ω.

Fix any point $x_0 \in J_\xi$, an increasing sequence $(x_n)_{n\in\mathbb{N}}$ tending to x_0 and let $y_0 := \lim_{n\to\infty} \omega(x_n)$. Then $y_0 \in {}_\xi I$ and

$$K(f_1(x_0), g_2(x_0)) = \lim_{n\to\infty} K(f_1(x_n), g_2(\omega(x_n))) = \xi,$$

which implies that $y_0 = \omega(x_0)$, that is $\lim_{n\to\infty} \omega(x_n) = \omega(x_0)$. This means that ω is left continuous at x_0. Analogously one can prove the right continuity of ω at x_0 whenever $x_0 < \xi$. Thus $\omega|_{J_\xi}$ is continuous. The same argument gives the continuity of $\omega|_{{}_\xi J}$.

To prove that the graph of the decreasing ω joints two sides of the square $I \times I$ it is enough to show that

(i) $\inf J = \inf I$ or $\sup \omega(J) = \sup I$
and

(ii) $\sup J = \sup I$ or $\inf \omega(J) = \inf I$.

Suppose to the contrary that the sentence (i) is false. Then

$$\inf J > \inf I \quad \text{and} \quad \sup \omega(J) < \sup I.$$

Take any decreasing sequence $(x_n)_{n\in\mathbb{N}}$ tending to $x_0 := \inf J$. Then $x_0 \in I_\xi$ and $\lim_{n\to\infty} \omega(x_n) = \omega(x_0+) \le \sup \omega(J) < \sup I$, and thus we can find a $y_0 \in {}_\xi I$ such that $y_0 > \omega(x_0+)$. We have

$$K(f_1(x_n), g_2(\omega(x_n))) = \xi, \quad n \in \mathbb{N},$$

whence

$$K(f_1(x_0), g_2(y_0)) > K(f_1(x_0), g_2(\omega(x_0+))) = \xi > K(f_1(x_0), \xi),$$

which implies that for an $x < x_0 = \inf J$ we get

$$K(f_1(x), g_2(y_0)) > \xi > K(f_1(x), \xi) = K(f_1(x), g_2(\xi)).$$

Thus there is a $y \in {}_\xi I$ such that $K(f_1(x), g_2(y)) = \xi$ which means that $x \in J$; a contradiction. Therefore (i) holds true. Similarly, we can prove (ii).

Now we examine equality (2.8). First fix any $(x, y) \in \mathrm{epi}_\mathrm{s}\omega$. Then $x \in J, y \in I$ and $y > \omega(x)$. If $x < \xi$ then $x < \xi < \omega(x) < y$, and thus $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ$ and

$$K(f_1(x), g_2(y)) > K(f_1(x), g_2(\omega(x))) = \xi,$$

that is $(x, y) \in D_{12}^+$. If $x \ge \xi$ and $y \ge \xi$ then $(x, y) \in {}_\xi J \times {}_\xi I \setminus \{(\xi, \xi)\}$ by $\omega(\xi) = \xi$. Finally, if $x \ge \xi$ and $y < \xi$ then $\omega(x) < y < \xi$, whence $(x, y) \in {}_\xi I^\circ \times I_\xi^\circ$ and

$$K(g_1(x), f_2(y)) > K(g_1(x), f_2(\omega(x))) = \xi,$$

which means that $(x, y) \in D_{21}^+$. We prove the reverse inclusion in (2.8). If $(x, y) \in D_{21}^+$ then $x < \xi < y$ and

$$K(f_1(x), g_2(y)) > \xi = K(\xi, \xi) > K(f_1(x), \xi) = K(f_1(x), g_2(\xi))$$

which implies that ξ is a value of $K(f_1(x), g_2(\cdot))$ at a point of the interval $(\xi, y) \subset {}_\xi I$; thus $x \in J_\xi$ and

$$K(f_1(x), g_2(y)) > \xi = K(f_1(x), g_2(\omega(x))),$$

whence $y > \omega(x)$ and, consequently, $(x, y) \in \mathrm{epi}_\mathrm{s}\omega$. We argue similarly when $(x, y) \in D_{21}^+$. Lastly, if $(x, y) \in {}_\xi J \times {}_\xi I \setminus \{(\xi, \xi)\}$ then $x \ge \xi$ and $y \ge \xi$, which gives $y \ge \xi \ge \omega(x)$; the equality $y = \omega(x)$ would imply $y = \xi = \omega(x)$, that is $(x, y) = (\xi, \xi)$, and thus we have again $y > \omega(x)$.

Finally, assume condition (2.9) and the symmetry of K. For any $y \in {}_\xi I$ we have

$$y \in \omega(J_\xi) \iff \bigvee_{x\in J_\xi} y = \omega(x) \iff \bigvee_{x\in J_\xi} K(f(x), g(y)) = \xi$$

$$\Longleftrightarrow \bigvee_{x\in J_\xi} K\left(g\left(y\right), f\left(x\right)\right) = \xi \Longleftrightarrow y \in {}_\xi J.$$

This means that $\omega\left(J_\xi\right) = {}_\xi J$. Similarly, $\omega\left({}_\xi J\right) = J_\xi$, and thus $\omega(J) = J$. Moreover,

$$K\left(f(x), g(y)\right) = K\left(g(y), f(x)\right), \quad x \in J_\xi, y \in {}_\xi J,$$

which gives

$$y = \omega(x) \Longleftrightarrow x = \omega(y), \quad x \in J_\xi, y \in {}_\xi J,$$

and proves that $\omega^{-1} = \omega$, that is ω is an involution. □

Passing to the definition of the marginal joint fix any means M and N on the intervals I_ξ and ${}_\xi I$, respectively, and an arbitrary joining function K for the pair (M, N). Assume that

$$\begin{array}{l} f_1 \text{ is continuous strictly increasing provided } D_{12}^- \neq \emptyset, \\ f_2 \text{ is continuous strictly increasing provided } D_{21}^- \neq \emptyset, \\ g_1 \text{ is continuous strictly increasing provided } D_{21}^+ \neq \emptyset, \\ g_2 \text{ is continuous strictly increasing provided } D_{12}^+ \neq \emptyset. \end{array} \tag{2.14}$$

Assume that $D_{12}^- \neq \emptyset$ and take any $(x, y) \in D_{12}^-$. Then $(x, y) \in I_\xi^\circ \times {}_\xi I^\circ$ and $K\left(f_1(x), g_2(y)\right) < \xi$, whence $\left(f_1(x), g_2(y)\right) \in I_\xi \times {}_\xi I$ and, by the first inequality (2.2),

$$f_1(x) \leq K\left(f_1(x), g_2(y)\right) < \xi = f_1(\xi).$$

Hence, by the continuity and strict increase of f_1,

$$K\left(f_1(x), g_2(y)\right) \in f_1\left([x, \xi)\right)$$

and we may calculate the value $f_1^{-1}\left(K\left(f_1(x), g_2(y)\right)\right)$, which is an element of the interval $[x, \xi)$, and thus also of $[x, y]$. This and similar arguments concerning the sets $D_{12}^-, D_{21}^+, D_{12}^+$ show that the formula below define a function $M \oplus_K N\colon I \times I \to I$:

$$\left(M \oplus_K N\right)(x, y) = \begin{cases} M(x, y), & \text{if } (x, y) \in I_\xi \times I_\xi, \\ f_1^{-1}\left(K\left(f_1(x), g_2(y)\right)\right), & \text{if } (x, y) \in D_{12}^-, \\ \xi, & \text{if } (x, y) \in D_{12}, \\ g_2^{-1}\left(K\left(f_1(x), g_2(y)\right)\right), & \text{if } (x, y) \in D_{12}^+, \\ f_2^{-1}\left(K\left(g_1(x), f_2(y)\right)\right), & \text{if } (x, y) \in D_{21}^-, \\ \xi, & \text{if } (x, y) \in D_{21}, \\ g_1^{-1}\left(K\left(g_1(x), f_2(y)\right)\right), & \text{if } (x, y) \in D_{21}^+, \\ N(x, y), & \text{if } (x, y) \in {}_\xi I \times {}_\xi I. \end{cases} \tag{2.15}$$

Clearly, $M \oplus_K N$ is a mean on I extending the means M and N to the square $I \times I$. We call it the *marginal K-joint of the means M and N*.

To illustrate the notion of the joint of means consider the following example.

Example 2.3. Fix an interval I and its interior point ξ. Let $\varphi\colon I_\xi \to \mathbb{R}$ be any continuous and strictly monotonic function and let m be any mean on ${}_\xi I$. Consider the quasi-arithmetic mean A^φ with the generator φ:

$$A^\varphi(x,y) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(y)}{2}\right), \qquad x,y \in I_\xi.$$

Define the function $K\colon f_1\left(I_\xi\right) \times g_2\left({}_\xi I\right) \cup g_1\left({}_\xi I\right) \times f_2\left(I_\xi\right) \to I$ by

$$K(x,y) = \begin{cases} A^\varphi(x,\xi), & \text{if } \ x < \xi \le y, \\ A^\varphi(\xi,y), & \text{if } \ y < \xi \le x, \\ m(x,\xi), & \text{if } \ y = \xi \le x, \\ m(\xi,y), & \text{if } \ x = \xi \le y. \end{cases}$$

It is evident that K is a joining function for the pair (A^φ, m). Then the functions $f_1, f_2\colon I_\xi \to I_\xi$ and $g_1, g_2\colon {}_\xi I \to {}_\xi I$ are given by

$$f_1(x) = f_2(x) = A^\varphi(x,\xi) = \varphi^{-1}\left(\frac{\varphi(x)+\varphi(\xi)}{2}\right)$$

and

$$g_1(x) = m(x,\xi) \quad \text{and} \quad g_2(x) = m(\xi,x).$$

Observe that

$$K(x,y) = A^\varphi(x,\xi) < \xi, \quad (x,y) \in \left(f_1\left(I_\xi\right) \setminus \{\xi\}\right) \times \left(g_2\left({}_\xi I\right) \setminus \{\xi\}\right),$$

and

$$K(x,y) = A^\varphi(\xi,y) < \xi, \quad (x,y) \in \left(g_1\left({}_\xi I\right) \setminus \{\xi\}\right) \times \left(f_2\left(I_\xi\right) \setminus \{\xi\}\right),$$

so

$$D_{12} = D_{12}^+ = D_{21} = D_{21}^+ = \emptyset.$$

If $(x,y) \in I_\xi^\circ \times {}_\xi I^\circ$ then $(x,y) \in D_{12}^-$, whence

$$\begin{aligned}\left(A^\varphi \oplus_K m\right)(x,y) &= f_1^{-1}\left(K(f_1(x), g_2(y))\right) = f_1^{-1}\left(A^\varphi\left(f_1(x), \xi\right)\right) \\ &= f_1^{-1}\left(f_1\left(f_1(x)\right)\right) = f_1(x) = A^\varphi(x,\xi).\end{aligned}$$

Similarly, if $(x,y) \in {}_\xi I^\circ \times I_\xi^\circ$ then $(x,y) \in D_{21}^-$, and thus

$$\begin{aligned}\left(A^\varphi \oplus_K m\right)(x,y) &= f_2^{-1}\left(K(g_1(x), f_2(y))\right) = f_2^{-1}\left(A^\varphi\left(\xi, f_2(y)\right)\right) \\ &= f_2^{-1}\left(f_2\left(f_2(y)\right)\right) = f_2(y) = A^\varphi(\xi,y).\end{aligned}$$

Consequently, the joint $A^\varphi \oplus_K m$ has the form

$$A^\varphi \oplus_K m(x,y) = \begin{cases} A^\varphi(x,y), & \text{if } (x,y) \in I_\xi \times I_\xi, \\ A^\varphi(x,\xi), & \text{if } (x,y) \in I_\xi^\circ \times {}_\xi I^\circ, \\ A^\varphi(\xi,y), & \text{if } (x,y) \in {}_\xi I^\circ \times I_\xi^\circ, \\ m(x,y), & \text{if } (x,y) \in {}_\xi I \times {}_\xi I. \end{cases}$$

Therefore, $A^\varphi \oplus_K m$ is the mean defined by formula (2.1) at the beginning of that section.

Now we describe marginal joints obtained with the use of the joining functions defined in Examples 2.1 and 2.2.

Example 2.4. Let M and N be means on the intervals I_ξ and ${}_\xi I$, respectively, such that the marginal functions f_1, f_2, g_1, g_2 for the pair (M,N) satisfy assumptions (2.14). Consider the joining function K defined by (2.5). Then, using (2.15), we come to the form of the K-joint $M \oplus_K N$:

$$(M \oplus_K N)(x,y) = \begin{cases} M(x,y), & \text{if } (x,y) \in I_\xi \times I_\xi, \\ f_1^{-1}\left(f_1^2(x) + g_2^2(y) - \xi\right), & \text{if } (x,y) \in D_{12}^-, \\ \xi, & \text{if } (x,y) \in D_{12}, \\ g_2^{-1}\left(f_1^2(x) + g_2^2(y) - \xi\right), & \text{if } (x,y) \in D_{12}^+, \\ f_2^{-1}\left(g_1^2(x) + f_2^2(y) - \xi)\right), & \text{if } (x,y) \in D_{21}^-, \\ \xi, & \text{if } (x,y) \in D_{21}, \\ g_1^{-1}\left(g_1^2(x) + f_2^2(y) - \xi\right), & \text{if } (x,y) \in D_{21}^+, \\ N(x,y), & \text{if } (x,y) \in {}_\xi I \times {}_\xi I. \end{cases}$$

A standard computation shows also that the function ω is defined on an interval containing ξ in its interior by the formula

$$\omega(x) = \begin{cases} g_2^{-2}\left(2\xi - f_1^2(x)\right), & \text{if } (x,y) \in I_\xi, \\ \\ f_2^{-2}\left(2\xi - g_1^2(x)\right), & \text{if } (x,y) \in {}_\xi I. \end{cases}$$

Assume for instance that $I = (0,\infty), \xi = 1$, A is the arithmetic mean on $(0,1]$ and G is the geometric mean on $[1,\infty)$:

$$A(x,y) = \frac{x+y}{2}, \quad x,y \in (0,1], \quad \text{and} \quad G(x,y) = \sqrt{xy}, \quad x,y \in [1,\infty).$$

Then

$$f_1(x) = f_2(x) = \frac{x+1}{2}, \quad x \in (0,1],$$

and

$$g_1(x) = g_2(x) = \sqrt{x}, \quad x \in [1,\infty).$$

The joining function $K\colon (0,1]\times[1,\infty)\cup[1,\infty)\times(0,1] \to (0,\infty)$, given by (2.5), has the form

$$K(x,y) = \begin{cases} \frac{x-1}{2} + \sqrt{y}, & \text{if } 0 < x \le 1 \le y, \\ \sqrt{x} + \frac{y-1}{2}, & \text{if } 0 < y \le 1 \le x, \end{cases}$$

and, consequently,

$$(A \oplus_K G)(x,y) = \begin{cases} \frac{x+y}{2}, & \text{if } 0 < x, y \le 1, \\ \frac{x+3}{2} + 2y^{\frac{1}{4}} - 3, & \text{if } 0 < x < 1 < y < \left(\frac{5-x}{4}\right)^4, \\ \left(\frac{x+3}{4} + y^{\frac{1}{4}} - 1\right)^4, & \text{if } 0 < x < 1 < \left(\frac{5-x}{4}\right)^4 \le y, \\ 2x^{\frac{1}{4}} + \frac{y+3}{2} - 3 & \text{if } 0 < y < 1 < x < \left(\frac{5-x}{4}\right)^4, \\ \left(x^{\frac{1}{4}} + \frac{y+3}{4} - 1\right)^4, & \text{if } 0 < y < 1 < \left(\frac{5-x}{4}\right)^4 \le x, \\ \sqrt{xy}, & \text{if } 1 \le x, y. \end{cases}$$

Now the function ω is an involution mapping the interval $J = (0,(5/4)^4)$ onto itself by the formula

$$\omega(x) = \begin{cases} \left(\frac{5-x}{4}\right)^4, & \text{if } 0 < x \le 1, \\ 5 - 4x^{\frac{1}{4}}, & \text{if } 1 \le x. \end{cases}$$

Example 2.5. Let $\varphi\colon I_\xi \to \mathbb{R}$ and $\psi\colon {}_\xi I \to \mathbb{R}$ be continuous strictly increasing functions vanishing at ξ and let $p, q \in (0,1)$. To find the K-joint $A_p^\varphi \oplus_K A_q^\psi$, where K is the joining function defined in Example 2.2, we use formulas (2.6) and (2.7) and the definition (2.15). After easy calculations we come to the formula

$$
\left(A_p^{\varphi} \oplus_K A_q^{\psi}\right)(x,y)
= \begin{cases}
\varphi^{-1}\left(p\varphi(x)+(1-p)\varphi(y)\right), & \text{if } \quad (x,y) \in I_\xi \times I_\xi, \\[2ex]
\varphi^{-1}\left(p\varphi(x)+(1-q)\psi(y)\right), & \text{if } \quad (x,y) \in I_\xi^\circ \times {}_\xi I^\circ \\
& \text{and } \ p\varphi(x)+(1-q)\psi(y) < 0, \\[2ex]
\psi^{-1}\left(p\varphi(x)+(1-q)\psi(y)\right), & \text{if } \quad (x,y) \in I_\xi^\circ \times {}_\xi I^\circ \\
& \text{and } \ p\varphi(x)+(1-q)\psi(y) \geq 0, \\[2ex]
\varphi^{-1}\left(q\psi(x)+(1-p)\varphi(y)\right), & \text{if } \quad (x,y) \in {}_\xi I^\circ \times I_\xi^\circ \\
& \text{and } \ q\psi(x)+(1-p)\varphi(y) < 0, \\[2ex]
\psi^{-1}\left(q\psi(x)+(1-p)\varphi(y)\right), & \text{if } \quad (x,y) \in {}_\xi I^\circ \times I_\xi^\circ \\
& \text{and } \ q\psi(x)+(1-p)\varphi(y) \geq 0, \\[2ex]
\psi^{-1}\left(q\psi(x)+(1-q)\psi(y)\right), & \text{if } \quad (x,y) \in {}_\xi I \times {}_\xi I.
\end{cases} \tag{2.16}
$$

The function ω is defined on an interval J containing ξ in its interior by the formula

$$
\omega(x) = \begin{cases}
\psi^{-1}\left(-\frac{p}{1-q}\varphi(x)\right), & \text{if } \ x < 0, \\
\varphi^{-1}\left(-\frac{q}{1-p}\psi(x)\right), & \text{if } \ x \geq 0.
\end{cases}
$$

One can check that ω is an involution if and only if $p+q=1$.

Taking $I=(0,\infty)$, $\xi=1$, $p=q=\frac{1}{2}$ and defining $\varphi\colon (0,1] \to \mathbb{R}$ and $\psi\colon [1,\infty) \to \mathbb{R}$ by

$$
\varphi(x) = x-1 \quad \text{and} \quad \psi(x) = \log x,
$$

respectively, we see that $\varphi(\xi)=\psi(\xi)=0$, and $A_p^{\varphi}=A$ and $A_q^{\psi}=G$. Take any $(x,y) \in (0,\infty)\times(0,\infty)$ such that $x \leq y$. Then one can check that the formula for $A \oplus_K G$ reduces to the following one (cf. Fig. 4):

$$
(A \oplus_K G)(x,y) = \begin{cases}
\frac{x+y}{2}, & \text{if } \ 0 < x, y \leq 1, \\
\frac{x+\log y-1}{2}+1, & \text{if } \ 0 < x < 1 < y \text{ and } x+\log y < 1, \\
\exp\left(\frac{x+\log y-1}{2}\right), & \text{if } \ 0 < x < 1 < y \text{ and } x+\log y \geq 1, \\
\sqrt{xy}, & \text{if } \ 1 \leq x, y.
\end{cases}
$$

Since $p=q=\frac{1}{2}$ it follows from (2.16) that the mean $A \oplus_K G$ is symmetric, so having the formula above we are done also with the case when $y \leq x$.

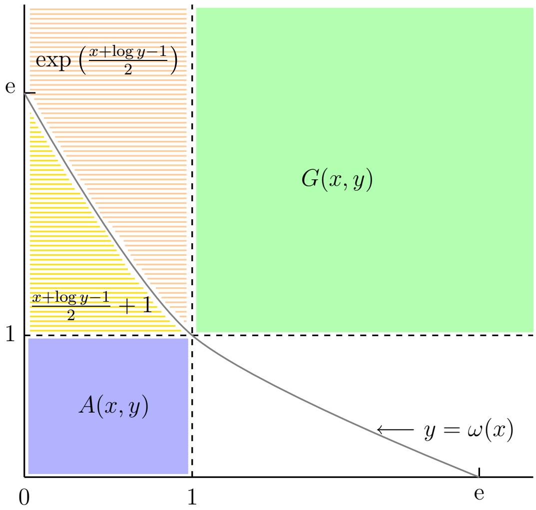

FIGURE 4. Form of $A\bigoplus_K G$

Moreover, as $p+q=1$, the function ω is an involution: ω maps $(0,e)$ onto itself and

$$\omega(x)=\begin{cases}\exp(1-x), & \text{if } \ x\in(0,1),\\ 1-\log x, & \text{if } \ x\in[1,\mathrm{e}).\end{cases}$$

Observe that the joints $A\oplus_K G$ obtained in Examples 2.4 and 2.5 are different.

We may expect that if means M,N and a joining function K for the pair (M,N) have nice properties, then so does the joint $M\oplus_K N$. The following result is an example of such a situation.

Theorem 2.2. *Let M and N be means on the intervals I_ξ and ${}_\xi I$, respectively, and K be a joining function for the pair (M,N). If the functions M,N and K are strictly increasing with respect to each variable and continuous, then so is the joint $M\oplus_K N$. If, in addition, the means M,N and the function K are symmetric, then also $M\oplus_K N$ is symmetric.*

Proof. The assumptions imposed on the joining function K imply that the marginal functions $f_1,f_2\colon I_\xi\to I_\xi$ and $g_1,g_2\colon {}_\xi I\to {}_\xi I$ are continuous and strictly increasing. Let $J\subset I$ and $\omega\colon J\to I$ be an interval and a function, respectively, satisfying the assertion of Theorem 2.1. Define intervals J_ξ and ${}_\xi J$ by (2.10) and (2.11), respectively.

Fix a point $y\in I$. We prove that the function $(M\oplus_K N)(\cdot,y)$ is strictly increasing and continuous. First assume that $y\in I_\xi^\circ$. If $y\notin\omega({}_\xi J)$ then, by formula (2.15)

$$(M \oplus_K N)(x,y) = \begin{cases} M(x,y), & \text{if } x \in I_\xi, \\ f_2^{-1}\left(K\left(g_1(x), f_2(y)\right)\right), & \text{if } x \in {}_\xi I^\circ, \end{cases}$$

so it is enough to observe that

$$\begin{aligned} \lim_{x\to\xi+} f_2^{-1}\left(K\left(g_1(x), f_2(y)\right)\right) &= f_2^{-1}\left(K\left(\xi, f_2(y)\right)\right) = f_2^{-1}\left(f_2\left(f_2(y)\right)\right) \\ &= f_2(y) = M(\xi, y) \end{aligned}$$

and to take into account the continuity and strict increase of the functions involved. If $y \in \omega\left({}_\xi J\right)$ then $y = \omega\left(\omega^{-1}(y)\right)$ and, by (2.15), we have

$$(M \oplus_K N)(x,y) = \begin{cases} M(x,y), & \text{if } x \in I_\xi, \\ f_2^{-1}\left(K\left(g_1(x), f_2(y)\right)\right), & \text{if } x \in {}_\xi I^\circ,\ x < \omega^{-1}(y), \\ \xi, & \text{if } x = \omega^{-1}(y), \\ g_1^{-1}\left(K\left(g_1(x), f_2(y)\right)\right), & \text{if } x \in {}_\xi I^\circ,\ x > \omega^{-1}(y), \end{cases}$$

and it is enough to observe that

$$\begin{aligned} \lim_{x\to\omega^{-1}(y-)} f_2^{-1}\left(K\left(g_1(x), f_2(y)\right)\right) &= f_2^{-1}\left(K\left(g_1\left(\omega^{-1}(y)\right), f_2(y)\right)\right) \\ &= f_2^{-1}(\xi) = \xi, \\ \lim_{x\to\omega^{-1}(y+)} g_1^{-1}\left(K\left(g_1(x), f_2(y)\right)\right) &= g_1^{-1}\left(K\left(g_1\left(\omega^{-1}(y)\right), f_2(y)\right)\right) \\ &= g_1^{-1}(\xi) = \xi, \end{aligned}$$

and, as in the previous case, $\lim_{x\to\xi+} f_2^{-1}\left(K\left(g_1(x), f_2(y)\right)\right) = M(\xi, y)$. In the case $y = \xi$ we have

$$(M \oplus_K N)(x,y) = \left\{ \begin{matrix} M(x,\xi), & \text{if } x \in I_\xi, \\ N(x,\xi), & \text{if } x \in {}_\xi I, \end{matrix} \right\} = \begin{cases} f_1(x), & \text{if } x \in I_\xi, \\ g_1(x), & \text{if } x \in {}_\xi I, \end{cases}$$

and we are done. The case $y \in {}_\xi I^\circ$ can be treated analogously to the one when $y \in I_\xi^\circ$. Therefore, we infer that the function $M \oplus_K N$ is continuous and strictly increasing with respect to the first variable.

To prove its continuity and strict increase with respect to the second variable we proceed similarly; the only essential difference is now that for any fixed $x \in I_\xi^\circ$ we have

$$(M \oplus_K N)(x,y) = \begin{cases} M(x,y), & \text{if } y \in I_\xi, \\ f_1^{-1}\left(K\left(f_1(x), g_2(y)\right)\right), & \text{if } y \in {}_\xi I^\circ,\ y < \omega(x), \\ \xi, & \text{if } x \in J_\xi,\ y = \omega(x), \\ g_2^{-1}\left(K\left(f_1(x), g_2(y)\right)\right), & \text{if } x \in J_\xi,\ y > \omega(x). \end{cases}$$

Consequently, the function $M \oplus_K N$ is continuous and strictly increasing with respect to each variable, and thus continuous.

Finally assume additionally that the means M, N and the function K are symmetric. Then $f_1 = f_2$, $g_1 = g_2$ and, according to Theorem 2.1, we see that ω is an involution. Put $f := f_1 = f_2$ and $g := g_1 = g_2$. Fix any point $(x,y) \in I \times I$. If $(x,y) \in I_\xi \times I_\xi$ then so is (y,x), whence

$$(M \oplus_K N)(y,x) = M(y,x) = M(x,y) = (M \oplus_K N)(x,y)$$

by the symmetry of M. If $(x,y) \in D_{12}^-$ then Theorem 2.1 implies that $y < \omega(x)$, whence $x < \omega^{-1}(y) = \omega(y)$ and, again by Theorem 2.1, we have $(y,x) \in D_{21}^-$, and thus

$$\begin{aligned}(M \oplus_K N)(y,x) &= f_2^{-1}\left(K\left(g_1(y), f_2(x)\right)\right) = f^{-1}\left(K\left(g(y), f(x)\right)\right) \\ &= f^{-1}\left(K\left(f(x), g(y)\right)\right) = f_1^{-1}\left(K\left(f_1(x), g_2(y)\right)\right) \\ &= (M \oplus_K N)(x,y).\end{aligned}$$

If $(x,y) \in D_{12}$ then $y = \omega(x)$, and thus $x = \omega^{-1}(y) = \omega(y)$, $(y,x) \in D_{21}$ and

$$(M \oplus_K N)(y,x) = \xi = (M \oplus_K N)(x,y).$$

In the remaining cases we proceed quite analogously. □

3. Open problems

In view of the negative answers to Problems 0.1 and 1.1, given in Section 1, the following modified question (see [5]) seems to be of interest and should be considered. Summing up, we still cannot answer the following modified question (see [5]).

Problem 3.1. Let M be a continuous strict mean on an interval I. Is it true that if M is not quasi-arithmetic, then every solution $f\colon I \to \mathbb{R}$ of Eq. (0.1) is constant?

Observe that the situation is very subtle. If Problem 3.1 has a negative answer, that is Eq. (0.1) with a continuous strict and not quasi-arithmetic mean M of I has a non-constant solution $f\colon I \to \mathbb{R}$, then:

(i) the mean M cannot be strictly increasing with respect to each variable; this follows immediately from the Theorem GK;
(ii) the function f cannot be continuous; this is a direct consequence of the following theorem of Zoltán Daróczy and Zsolt Páles, which is a particular case of a result published in [6] in 2003 (see also [3]).

Theorem DP. *Let M be a continuous strict mean on an interval I. Then every continuous non-constant solution $f\colon I \to \mathbb{R}$ of Eq.* (0.1) *is strictly increasing.*

Indeed, supposing $f\colon I \to \mathbb{R}$ to be a continuous solution of Eq. (0.1) we deduce that f is strictly increasing, whence

$$M(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right), \quad x,y \in I,$$

that is M is quasi-arithmetic with the generator f, a contradiction.

It seems that the notion of marginal joints is of interest and need further research. The following open problems serve as examples of those questions which could be studied at the very beginning. Fix an interval I and its interior point ξ.

Problem 3.2. Given means M and N on the intervals I_ξ and ${}_\xi I$, respectively, find the joining functions K for the pair (M,N) such that:

(i) the joint $M \oplus_K N$ is homogeneous (or conditionally homogeneous) provided M and N are homogeneous (or conditionally homogeneous),
(ii) the joint $M \oplus_K N$ is translative (or conditionally translative) provided M and N are translative (or conditionally translative),
(iii) the joint $M \oplus_K N$ is continuous provided M and N are continuous.
(For the definitions of *homogeneity* and *translativity* of means see, for instance, the monograph [1], whereas for *conditional homogeneity* and *conditional translativity* consult [2].)

Problem 3.3. For which joints $M \oplus_K N$ on I do there exist non-constant solutions $f\colon I \to \mathbb{R}$ of the equation

$$f\left(\left(M \oplus_K N\right)(x,y)\right) = \frac{f(x)+f(y)}{2}\,?$$

Problem 3.4. Is it possible, to produce the K-joints if the joining function K is defined only in a neighbourhood of the point (ξ,ξ)?

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