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Normed spaces equivalent to inner product spaces and stability of functional equations

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Abstract. Let $(X, \|\cdot\|)$ be a normed space. If $\|\cdot\|_i$ is an equivalent norm coming from an inner product, then the original norm satisfies an approximate parallelogram law. Applying methods and results from the theory of stability of functional equations we study the reverse implication.

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1. Introduction

Let $(X, \|\cdot\|)$ be a real or complex normed space. Suppose that the norm $\|\cdot\|$ is equivalent to a norm $\|\cdot\|_i$ coming from an inner product. More precisely, assume that for some $k \geq 1$ we have

$$\frac{1}{k}\|x\|_i \leq \|x\| \leq k\|x\|_i, \qquad x \in X. \tag{1}$$

Following Joichi [14] we say that a normed space $(X, \|\cdot\|)$ is *equivalent to an inner product space* (X is an e.i.p.-space) iff there exists an inner product in X and a norm $\|\cdot\|_i$ generated by this inner product such that (1) holds with some $k \geq 1$. Although there are numerous characterizations of inner product spaces (cf. e.g. [4,3]), not so many are known for normed spaces merely equivalent to inner product ones. Joichi himself [14, Theorem and Corollary] proved that for a real normed space X the following conditions are equivalent.

(i) X is an e.i.p.-space;
(ii) there exists a constant $k \geq 1$ and a Hilbert space H such that for each finite-dimensional subspace M of X there exists a linear mapping $T_M\colon M \to H$ such that

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$$\frac{1}{k}\|x\| \le \|T_M x\| \le k\|x\|, \qquad x \in M;$$

(that is X is *crudely finitely representable in* H – cf. [10, Theorem 6.2])

(iii) there exists a constant $k \ge 1$ such that for each two finite-dimensional subspaces M and N with $\dim M = \dim N$, there exists a linear mapping T from M onto N such that

$$\frac{1}{k}\|x\| \le \|Tx\| \le k\|x\|, \qquad x \in M.$$

Note that condition (iii) with $k = 1$, i.e., when any two finite and equi-dimensional subspaces M, N are isometrically equivalent, is a necessary and sufficient condition for X to be an inner product space. In fact, it suffices to consider only two-dimensional subspaces M, N (cf. [4, (20.1)]). Other characterizations of e.i.p.-spaces were given e.g. by Lindenstrauss and Tzafriri [22] and Figiel and Pisier [11]. Our aim is to provide some new characterizations of e.i.p.-spaces in possibly simple forms.

Let us write (1) in the following form, with $\varepsilon \ge 0$,

$$\frac{1}{1+\varepsilon}\|x\|_i \le \|x\| \le (1+\varepsilon)\|x\|_i, \qquad x \in X \tag{2}$$

or, equivalently,

$$|\,\|x\| - \|x\|_i\,| \le \varepsilon \min\{\|x\|, \|x\|_i\}, \qquad x \in X. \tag{3}$$

It follows from (2) that

$$\|x+y\|^2 + \|x-y\|^2 \le (1+\varepsilon)^2 \left(\|x+y\|_i^2 + \|x-y\|_i^2\right), \qquad x, y \in X$$

and

$$\frac{1}{2\|x\|^2 + 2\|y\|^2} \le (1+\varepsilon)^2 \frac{1}{2\|x\|_i^2 + 2\|y\|_i^2}, \qquad x, y \in X,\ (x,y) \ne (0,0).$$

Consequently, since the norm $\|\cdot\|_i$ satisfies the parallelogram law,

$$\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} \le (1+\varepsilon)^4 \frac{\|x+y\|_i^2 + \|x-y\|_i^2}{2\|x\|_i^2 + 2\|y\|_i^2} = (1+\varepsilon)^4$$

for $(x,y) \ne (0,0)$. Substituting $x+y$ and $x-y$ in place of x and y, respectively, one obtains

$$\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} \ge \frac{1}{(1+\varepsilon)^4}$$

thus finally

$$\frac{1}{(1+\varepsilon)^4} \le \frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} \le (1+\varepsilon)^4, \quad x, y \in X,\ (x,y) \ne (0,0). \tag{4}$$

Taking $\delta := (1+\varepsilon)^4 - 1$, we write (4) in the form

$$\frac{1}{1+\delta} \le \frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} \le 1+\delta, \quad x, y \in X,\ (x,y) \ne (0,0) \tag{5}$$

or, equivalently,

$$\big|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\big| \le 2\delta(\|x\|^2 + \|y\|^2), \quad x, y \in X. \tag{6}$$

To sum up, we have.

Theorem 1.1. *If a real or complex normed space X is equivalent to an inner product space, i.e., if (2) holds, then the norm satisfies the approximate parallelogram law (6) with $\delta = (1+\varepsilon)^4 - 1$.*

On the other hand, condition (6) is not sufficient for X to be equivalent to an inner product space. We will explain this in the next section by considering an example of a space X which is not equivalent to an inner product one and satisfies condition (6) with δ arbitrarily close to 0.

For $n \ge 2$ each inner product norm satisfies a generalized parallelogram law (cf. [18]):

$$\sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^2 = 2^n \sum_{j=1}^{n} \|x_j\|^2, \qquad x_1, \ldots, x_n \in X.$$

Proceeding similarly as above one can show the following result.

Theorem 1.2. *If a real or complex normed space X is equivalent to an inner product space, i.e., if (2) holds with some $\varepsilon \ge 0$, then the norm satisfies the approximate generalized parallelogram law (with $\delta = (1+\varepsilon)^4 - 1$)*

$$\left| \sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^2 - 2^n \sum_{j=1}^{n} \|x_j\|^2 \right| \le 2^n \delta \sum_{j=1}^{n} \|x_j\|^2, \;\; x_1, \ldots, x_n \in X, \;\; n \ge 2 \tag{7}$$

or, equivalently,

$$\frac{1}{1+\delta} \le \frac{\sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^2}{2^n \sum_{j=1}^{n} \|x_j\|^2} \le 1+\delta, \qquad \begin{array}{l} x_1, \ldots, x_n \in X, \; n \ge 2 \\ (x_1, \ldots, x_n) \ne (0, \ldots, 0). \end{array} \tag{8}$$

2. The von Neumann–Jordan constant

Suppose that the norm in X satisfies, with some $\varepsilon \in [0, 1]$, the inequality

$$\big|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\big| \le 2\varepsilon \left(\|x\|^2 + \|y\|^2\right), \quad x, y \in X \tag{9}$$

or equivalently, with $c := 1 + \varepsilon$,

$$\frac{1}{c} \le \frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} \le c, \quad (0,0) \ne (x,y) \in X^2. \tag{10}$$

The smallest constant c for which the above inequalities hold is called the *von Neumann–Jordan constant* and is denoted by $C_{NJ}(X)$ (this notion has been introduced by Clarkson [7], however, implicitly, it appeared already in [15]). More precisely:

$$C_{NJ}(X) := \inf \left\{ c \geq 1 : \ \frac{1}{c} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} \leq c, \ \begin{array}{c} x, y \in X, \\ (x,y) \neq (0,0) \end{array} \right\}$$
$$= \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} : \ x, y \in X, \ (x,y) \neq (0,0) \right\}.$$

It is visible that $1 \leq C_{NJ}(X) \leq 2$ and $C_{NJ}(X) = 1$ if and only if, X is an inner product space (Jordan–von Neumann theorem). For l_p spaces ($p \geq 1$) we have $C_{NJ}(l_p) = 2^{\frac{2}{\min\{p,p^*\}}-1}$, where $\frac{1}{p} + \frac{1}{p^*} = 1$ (see [7]). Moreover, $C_{NJ}(l_\infty) = 2$ and $C_{NJ}(X) = C_{NJ}(X^*)$. There are many papers devoted to the $C_{NJ}(X)$ constant and geometrical properties of X which can be derived from the value of this constant (cf. for example [9,19,20,25]). Generally, one could say that the smaller the $C_{NJ}(X)$ constant is, the better properties the space X possesses. There are also interesting connections between $C_{NJ}(X)$ and other geometrical constants (see e.g. [2]).

It is convenient to consider an auxiliary constant (defined and discussed in [20]):

$$\tilde{C}_{NJ}(X) := \inf\{C_{NJ}(X, |\cdot|) : \ |\cdot| \text{ norm equivalent with } \|\cdot\|\}.$$

Note that $1 \leq \tilde{C}_{NJ}(X) \leq 2$ and $\tilde{C}_{NJ}(X) \leq C_{NJ}(X)$. As a generalization of $C_{NJ}(X)$, the n-th von Neumann–Jordan constant $C_{NJ}^{(n)}(X)$ was defined in [21].

$$C_{NJ}^{(n)}(X) := \sup \left\{ \frac{\sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^2}{2^n \sum_{j=1}^{n} \|x_j\|^2} : \ x_j \in X, \ \sum_{j=1}^{n} \|x_j\|^2 \neq 0 \right\}.$$

Obviously, $C_{NJ}^{(2)}(X) = C_{NJ}(X)$. X is a Hilbert space if, and only if $C_{NJ}^{(n)}(X) = 1$ for some n (and, equivalently, for all n).

We have shown that inequality (9) is a necessary condition for a normed space to be equivalent to an inner product one. We ask if it is also a sufficient condition. Does there exist $\varepsilon_0 > 0$ such that if the norm in a given space X satisfies (9) with some $\varepsilon < \varepsilon_0$, then it is equivalent to an inner product one? Equivalently: is it true that for some $\alpha \in (1, 2]$ inequality $C_{NJ}(X) < \alpha$ yields that X is equivalent to an inner product space? The answer to this question is negative. There exist normed spaces with von Neumann–Jordan constants arbitrarily close to 1 which are not equivalent to any inner product space. Hashimoto and Nakamura [13] constructed a Banach space X such that $\tilde{C}_{NJ}(X) = 1$ and X is not equivalent to any Hilbert space. The construction is as follows. Let X_n denote the a_n-dimensional space $l_{p_n}^{a_n}$ with $p_n = 2 - \frac{1}{n}$ and

a_n being a suitably chosen increasing sequence of positive integers such that $C_{NJ}^{(a_n)}(l_{p_n}^{a_n}) > n$. Thus $C_{NJ}(X_n) = 2^{\frac{1}{2n-1}}$. The desired space X is defined as an l_2-direct sum $\bigoplus_{n=1}^{\infty} X_n$. It follows from the construction that $\tilde{C}_{NJ}(X) = 1$. On the other hand $\sup_{m\in\mathbb{N}} C_{NJ}^{(m)}(X) = \infty$, which would be impossible if X were equivalent to an inner product space, as it follows from (8). This example explains that condition (9), although necessary, is not sufficient for X to be equivalent to an inner product space. The sufficiency of the stronger (7) is an open problem.

3. An application of a quadratic equation and its stability

Consider the approximate parallelogram law in a more general form. Assume that the norm in a real or complex space X satisfies

$$\big|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\big| \le \Phi(x,y), \qquad x,y \in X \tag{11}$$

with some mapping $\Phi\colon X^2 \to [0,\infty)$, an ask whether (or when) this enforces X to be equivalent (and possibly not necessarily equal) to an inner product space. One can also consider a more general version of condition (7):

$$\left| \sum_{\theta_j=\pm 1} \left\| \sum_{j=1}^{n} \theta_j x_j \right\|^2 - 2^n \sum_{j=1}^{n} \|x_j\|^2 \right| \le \Phi(x_1,\ldots,x_n), \quad x_1,\ldots,x_n \in X, \ n \ge 2. \tag{12}$$

Let $q\colon X \to \mathbb{R}$ satisfy the *quadratic* functional equation

$$q(x+y) + q(x-y) = 2q(x) + 2q(y), \qquad x,y \in X.$$

It is known (cf. [1, §11.1, Proposition 1, p.166]) that there exists a symmetric and bilinear form $B\colon X^2 \to \mathbb{R}$ such that $q(x) = B(x,x)$ for $x \in X$. Moreover, B is unique and given by $B(x,y) = \frac{1}{4}(q(x+y) - q(x-y))$ for $x,y \in X$. There is a vast literature devoted to various kinds of stability of the quadratic functional equation and its various modifications (cf. e.g. the book of Jung [16, Chapter 8]). In particular there are results showing that if a mapping $f\colon X \to \mathbb{R}$ satisfies

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \le \Phi(x,y), \qquad x,y \in X,$$

with a suitable *control* mapping $\Phi\colon X^2 \to [0,\infty)$, then f can be approximated by a quadratic mapping q.

Let us mention some considered forms of Φ.

- $\Phi(x,y) = \varepsilon$ (cf. [6, 24]);
- $\Phi(x,y) = \xi + \varepsilon(\|x\|^p + \|y\|^p), p \ne 2$ (cf. [8]);

- Φ–arbitrary mapping satisfying
$$\sum_{i=1}^{\infty} 2^{-2i}\Phi(2^{i-1}x, 2^{i-1}x) < \infty \quad \text{or} \quad \sum_{i=1}^{\infty} 2^{2(i-1)}\Phi(2^{-i}x, 2^{-i}x) < \infty$$
and
$$2^{-2i}\Phi(2^{i-1}x, 2^{i-1}y) \to 0 \quad \text{or} \quad 2^{2(i-1)}\Phi(2^{-i}x, 2^{-i}y) \to 0$$
(cf. [5]).
- $\Phi(x, y) = \varphi(x, y)(f(x) + f(y))$ – with an arbitrary mapping φ satisfying
$$\sum_{i=1}^{\infty} \varphi(2^i x, 2^i y) < \infty$$
(cf. [12]).
- Φ – arbitrary mapping satisfying $\Phi(2x, 2x) \leq 4L\Phi(x, x)$ with some $0 < L < 1$ and $4^{-n}\Phi(2^n x, 2^n y) \to 0$ as $n \to \infty$ (cf. [17]).

In all the above cases the limit $\lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ (or $\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$) appears to exist and defines the quadratic mapping q which approximates f. Applying the above mentioned results to the mapping $f(x) := \|x\|^2$ and assuming that Φ satisfies one of the above listed properties, (11) yields that $q(x) = \lim_{n\to\infty} \frac{\|2^n x\|^2}{4^n} = \|x\|^2$ is a quadratic function, i.e., the norm $\|\cdot\|$ satisfies the parallelogram law and thus X is an inner product space. It is easy to see that it happens whenever Φ has the property $\lim_{n\to\infty} \Phi(nx, ny)/(n^2) = 0$ or $\lim_{n\to\infty} n^2\Phi(x/n, y/n) = 0$. Indeed, in (11) putting nx and ny (or x/n and y/n) in place of x and y, respectively, and passing to the limit $n \to \infty$, we get that the right hand side of (11) has to be equal to 0.

Since the considered above forms of Φ appear too strong for our aim, we are looking for one which is weaker than the above ones but stronger than (6).

Lemma 3.1. *For a given norm $\|\cdot\|$ the following conditions are equivalent:*
$$\left|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\right| \leq \varepsilon\|x-y\|^2, \ x, y \in X; \tag{13}$$
$$\left|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\right| \leq \varepsilon\|x+y\|^2, \ x, y \in X; \tag{14}$$
$$\left|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\right| \leq 2\varepsilon\|x\|^2, \ x, y \in X; \tag{15}$$
$$\left|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\right| \leq 2\varepsilon\|y\|^2, \ x, y \in X. \tag{16}$$
In other words, each of the conditions (13)–(16) is equivalent to
$$\begin{aligned}&\left|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\right| \\ &\quad \leq \varepsilon \cdot \min\{\|x+y\|^2, \|x-y\|^2, 2\|x\|^2, 2\|y\|^2\},\end{aligned}$$
for all $x, y \in X$.

Proof. The first two inequalities are obviously equivalent (put $-y$ in place of y). Substituting $x+y$ and $x-y$ in place of x and y or y and x, respectively, one obtains the other two inequalities. $\square$

The next result will be used in the proof of the main theorem which follows.

Proposition 3.2. *Let X be a real or complex normed space and let $\psi, \phi, f\colon X \to \mathbb{R}$ be such that*

$$2\psi(y) \le f(x+y) + f(x-y) - 2f(x) \le 2\phi(y), \qquad x, y \in X.$$

Then, there exists $F\colon X \to \mathbb{R}$ such that

$$F(x+y) + F(x-y) = 2F(x) + 2F(y), \qquad x, y \in X$$

and

$$\psi(x) \le F(x) \le \phi(x), \qquad x \in X.$$

Proof. The above result can be derived from more general theorems concerning the stability of functional equations. In particular, Tabor [26, Corollary 2] (motivated by Páles [23]) proved that for a semigroup S and mappings $\psi, \phi, f\colon S \to \mathbb{R}$ satisfying

$$\psi(u) \le \frac{1}{n!}\Delta_u^n f(x) \le \phi(u), \qquad x, u \in S$$

(where $\Delta_u f = \Delta_u^1 f$ is defined by $\Delta_u f(x) = f(x+u) - f(x)$ and $\Delta_u^{n+1} f = \Delta_u(\Delta_u^n f)$), there exists $F\colon S \to \mathbb{R}$ such that

$$\frac{1}{n!}\Delta_u^n F(x) = F(u), \qquad x, u \in S$$

and

$$\psi(u) \le F(u) \le \psi(u), \qquad u \in S.$$

Taking $S = X, n = 2$ and substituting $x-y$ and y in place of x and u, respectively, one gets the above assertion. $\square$

Theorem 3.3. *Suppose that the norm $\|\cdot\|$ in a real or complex space X satisfies, with some $\varepsilon \in [0,1)$:*

$$\big|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\big| \le \varepsilon \|x-y\|^2, \qquad x, y \in X$$

(as a matter of fact, any one of the conditions (13)–(16) can be assumed). Then $(X, \|\cdot\|)$ is equivalent to an inner product space. Precisely, there exists a norm $\|\cdot\|_i$ in X, coming from an inner product, such that

$$|\, \|x\| - \|x\|_i \,| \le \left(\frac{1}{\sqrt{1-\varepsilon}} - 1\right) \min\{\|x\|, \|x\|_i\}, \qquad x \in X. \tag{17}$$

Proof. From (16) we have

$$2(1-\varepsilon)\|y\|^2 \le \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 \le 2(1+\varepsilon)\|y\|^2, \qquad x, y \in X.$$

Let $\psi(x) := (1-\varepsilon)\|x\|^2, \phi(x) := (1+\varepsilon)\|x\|^2$ and $f(x) := \|x\|^2$ for $x \in X$. Then we have

$$2\psi(y) \le f(x+y) + f(x-y) - 2f(x) \le 2\phi(y), \qquad x, y \in X$$

and according to Proposition 3.2 there exists a mapping $q\colon X \to \mathbb{R}$ satisfying

$$q(x+y) + q(x-y) = 2q(x) + 2q(y), \qquad x, y \in X$$

and

$$\psi(x) \le q(x) \le \phi(x), \qquad x \in X.$$

The latter inequalities mean

$$(1-\varepsilon)\|x\|^2 \le q(x) \le (1+\varepsilon)\|x\|^2, \qquad x \in X. \tag{18}$$

It follows from the above that $q(x) > 0$ for $x \in X\backslash\{0\}$ and $q(0) = 0$. The mapping q, as a quadratic one, has to be of the form $q(x) = B(x, x)$ where $B(x, y) = \frac{1}{4}(q(x+y) - q(x-y))$ is a biadditive and symmetric mapping. Moreover, (18) gives that B is locally bounded with respect to each variable – hence B is bi-$\mathbb{R}$-linear. Thus in the case when X is real, B is an inner product in X, generating the norm $\|x\|_i := \sqrt{q(x)}, x \in X$. Now, we consider the case when X is a complex space. Notice that without loss of generality we may assume that

$$q(ix) = q(x), \qquad x \in X. \tag{19}$$

Indeed, one can replace q by $\tilde{q}$ defined by

$$\tilde{q}(x) := \frac{q(x) + q(ix)}{2}, \qquad x \in X.$$

It is easy to observe that $\tilde{q}$ is also a quadratic mapping, it satisfies the estimation (18), i.e.,

$$(1-\varepsilon)\|x\|^2 \le \tilde{q}(x) \le (1+\varepsilon)\|x\|^2, \qquad x \in X$$

and $\tilde{q}(ix) = \tilde{q}(x), x \in X$ (as a quadratic mapping, g is even). Notice that condition (19) yields

$$B(ix, iy) = B(x, y),\ B(x, iy) = -B(ix, y),\ B(ix, x) = 0, \qquad x, y \in X. \tag{20}$$

Indeed, for arbitrary $x, y \in X$ we have $B(ix, iy) = \frac{1}{4}(q(ix+iy) - q(ix-iy)) = \frac{1}{4}(q(x+y) - q(x-y)) = B(x, y)$. Now, $B(x, iy) = B(ix, -y) = -B(ix, y)$ and, putting $y = x, B(x, ix) = -B(ix, x) = -B(x, ix)$, hence $B(ix, x) = B(x, ix) = 0$.

Define $A\colon X^2 \to \mathbb{C}$ by

$$A(x, y) := B(x, y) - iB(ix, y), \qquad x, y \in X.$$

Using (20) and the symmetry of B we see that for arbitrary $x, y \in X$

$$\begin{aligned} A(y,x) &= B(y,x) - iB(iy,x) = B(x,y) - iB(x,iy) = B(x,y) + iB(ix,y) \\ &= \overline{A(x,y)}. \end{aligned}$$

Moreover,

$$\begin{aligned} A(ix,y) &= B(ix,y) - iB(-x,y) = iB(x,y) + B(ix,y) \\ &= i[B(x,y) - iB(ix,y)] \\ &= iA(x,y) \end{aligned}$$

which, with the bi-$\mathbb{R}$-linearity of B and conjugate symmetry of A, gives the sesquilinearity of A. Finally

$$A(x,x) = B(x,x) - iB(ix,x) = B(x,x) = q(x)$$

which proves that A is an inner product in X generating the norm $\|x\|_i := \sqrt{q(x)}$.

Now, we can write (18) as

$$\sqrt{1-\varepsilon}\|x\| \le \|x\|_i \le \sqrt{1+\varepsilon}\|x\|, \qquad x \in X, \tag{21}$$

i.e., the norms $\|\cdot\|$ and $\|\cdot\|_i$ are equivalent. Since $\sqrt{1+\varepsilon} \le \frac{1}{\sqrt{1-\varepsilon}}$, it follows from (21) that

$$\sqrt{1-\varepsilon}\|x\| \le \|x\|_i \le \frac{1}{\sqrt{1-\varepsilon}}\|x\|, \qquad x \in X,$$

which is equivalent to (17). □

Remarks 3.4. Adding (15) and (16) we get that each of (13)–(16) implies

$$\big|\, \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \,\big| \le \varepsilon(\|x\|^2 + \|y\|^2), \quad x, y \in X. \tag{22}$$

The reverse, is obviously not true as it would mean that (22) is sufficient for X to be equivalent to an inner products space. And we have shown that it is not true. Thus (22) is essentially weaker than (13)–(16).

(13) is a sufficient condition for X to be equivalent to an inner product space. But not necessary. The norm $\|x\| = \|(x_1, x_2)\| = |x_1| + |x_2|$ in $\mathbb{R}^2$ is equivalent to the Euclidean one. Inserting, for arbitrary $n \in \mathbb{N}, x = (1, n), y = (-1, n)$ in (13) one gets $8n \le 4\varepsilon$ for all $n \in \mathbb{N}$ — a contradiction.

It can be proved that inequalities (13)–(16) yield that the norm is 2-uniformly convex and 2-uniformly smooth. Hence, the first assertion of the theorem can also be derived from the result of Figiel and Pisier [11]. However, our method also gives estimation (17).

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