

Some generalization of the quadratic and Wilson's functional equation

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Abstract. We find the solutions $f, g, h: G \rightarrow X$, $\varphi: G \rightarrow \mathbb{K}$ of each of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|\varphi(y)g(x) + |K|h(y), \quad x, y \in G,$$

where $(G, +)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G , X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

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1. Introduction

The generalization of the quadratic functional equation

$$f(x + y) + f(x + \sigma y) = 2f(x) + 2f(y), \quad x, y \in G,$$

where σ is automorphism of an abelian group G such that $\sigma^2 = id_G$, $f, g: G \rightarrow \mathbb{C}$, was investigated by Stetkær [13].

In another work [14] he solved the functional equation

$$\frac{1}{N} \sum_{n=0}^{N-1} f(z + \omega^n \zeta) = g(z) + h(\zeta), \quad z, \zeta \in \mathbb{C},$$

where $N \in \{2, 3, \dots\}$, ω is the primitive N th root of unity, $f, g, h: \mathbb{C} \rightarrow \mathbb{C}$ are continuous.

Lukasik [8] showed the solution of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|g(x) + |K|h(y), \quad x, y \in S,$$

where $(S, +)$ is an abelian semigroup, K is a finite subgroup of the automorphism group of S , $(H, +)$ is an abelian group.

The functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|g(x)h(y), \quad x, y \in G,$$

where $(G, +)$ is an abelian group, K is a finite subgroup of the automorphism group on G , $f, g, h: G \rightarrow \mathbb{C}$, was studied by Förg-Rob and Schwaiger [5], Gajda [6], Stetkær [11, 12], Badora [2].

2. Main result

Throughout the present paper, we assume that X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(G, +)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G , $L := \text{card } K$.

We give the complete solution of the following functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y)g(x) + Lh(y), \quad x, y \in S.$$

In this work we use the following two theorems and one lemma:

Theorem 1 (Shinya [9, Corollary 3.12], Kannappan [7], Czerwik [4], Sinopoulos [10], Chojnacki [3]). *Let $(G, +)$ be an abelian, locally compact, Hausdorff topological group, K be a compact Hausdorff topological transformation group of G acting by automorphisms on G . Let further $d\lambda$ be a normalized Haar measure on K . If $\varphi \in C(G)$ is a nonzero solution of*

$$\int_{\lambda \in K} \varphi(x + \lambda y)d\lambda = \varphi(x)\varphi(y), \quad x, y \in G,$$

then there exists a continuous homomorphism $\chi: G \rightarrow \mathbb{C}^*$ such that

$$\varphi(x) = \int_{\lambda \in K} \chi(\lambda x)d\lambda, \quad x \in G.$$

If φ is bounded, then χ may be taken as a unitary character.

Lemma 1 [1, Lemma 14.1]. *Let Ω be a nonempty set, $n \in \mathbb{N}$. Functions $f_1, \dots, f_n: \Omega \rightarrow \mathbb{C}$ are linearly dependent if and only if for all $x_1, \dots, x_n \in \Omega$*

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_n) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_n) \end{vmatrix} = 0.$$

Theorem 2 [8, Theorem 5]. *Let $(S, +)$ be an abelian semigroup, K be a finite subgroup of the automorphism group of S , $L := \text{card } K$, $(H, +)$ be an abelian group uniquely divisible by $L!$. Then the function $f: S \rightarrow H$ satisfies the equation*

$$\sum_{\lambda \in K} f(x + \lambda y) = Lf(x), \quad x, y \in S$$

if and only if there exist k -additive, symmetric mappings $A_k: S^k \rightarrow H$, $k \in \{1, \dots, L-1\}$ and $A_0 \in H$ such that

$$f(x) = A_0 + A_1(x) + \dots + A_{L-1}(x, \dots, x), \quad x \in S,$$

$$\sum_{\lambda \in K} A_k(x, \dots, x, \underbrace{\lambda y, \dots, \lambda y}_i) = 0, \quad x, y \in S, \quad 1 \leq i \leq k \leq L-1.$$

First we show some lemmas.

Lemma 2. *Let $m: G \rightarrow \mathbb{C}^*$ be a homomorphism, $K_1 \subset K$ be a set such that $\{m \circ \lambda : \lambda \in K_1\}$ is linearly independent. If for some $\mu \in K$*

$$m(\mu x) = \sum_{\lambda \in K_1} a_\lambda m(\lambda x), \quad x \in G, \tag{1}$$

then $m \circ \mu = m \circ \lambda$ for some $\lambda \in K_1$.

Proof. Let

$$m(\mu x) = \sum_{\lambda \in K_1} a_\lambda m(\lambda x), \quad x \in G.$$

Therefore, for $x, y \in G$, we have

$$\sum_{\lambda \in K_1} a_\lambda m(\lambda y) m(\lambda x) = m(\mu(y+x)) = m(\mu y) m(\mu x) = m(\mu y) \sum_{\lambda \in K_1} a_\lambda m(\lambda x),$$

which means that

$$0 = \sum_{\lambda \in K_1} a_\lambda [m(\lambda y) - m(\mu y)] m(\lambda x).$$

From the linear independence we obtain

$$0 = a_\lambda [m(\lambda y) - m(\mu y)], \quad y \in G, \quad \lambda \in K_1,$$

since there exists $\lambda \in K_1$ such that $a_\lambda \neq 0$,

$$m(\lambda y) = m(\mu y), \quad y \in G.$$

□

Lemma 3. *Let $m: G \rightarrow \mathbb{C}^*$ be a homomorphism. Then the set $K_0 := \{\lambda \in K : m \circ \lambda = m\}$ is a subgroup of K .*

Lemma 4. *Let $K_0 := \{\lambda \in K : m \circ \lambda = m\}$, $K_1 \subset K$ be a minimal set such that $K = K_0 \circ K_1$. Then K_1 is a maximal set such that the set $\{m \circ \lambda : \lambda \in K_1\}$ is linearly independent.*

Proof. In view of Lemma 2, if $\{m \circ \lambda : \lambda \in K_1\}$ is linearly dependent, then $m \circ \lambda = m \circ \mu$ for some $\lambda, \mu \in K_1$, so we get $\lambda \circ \mu^{-1} \in K_0$, which means that $\lambda \in K_0 \circ \mu$, which contradicts the definition of K_1 .

On the other hand each $\lambda \in K$ has the form $\lambda = \lambda_0 \circ \lambda_1$, where $\lambda_0 \in K_0$, $\lambda_1 \in K_1$, hence $m \circ \lambda = m \circ \lambda_0 \lambda_1 = m \circ \lambda_1$, which gives us the maximality of K_1 . □

Theorem 3. *Let $\varphi : G \rightarrow \mathbb{C}$, $\varphi \neq 0$, satisfy the equation*

$$\sum_{\lambda \in K} \varphi(x + \lambda y) = L\varphi(x)\varphi(y), \quad x, y \in G. \tag{2}$$

Then there exist a homomorphism $m : G \rightarrow \mathbb{C}^$, $\beta_\lambda \in \mathbb{C}^*$, $b_\lambda \in G$, $\lambda \in K_1$, such that*

$$\varphi(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G, \tag{3}$$

$$\sum_{\lambda \in K_1} \beta_\lambda m(\mu b_\lambda) = \begin{cases} |K_1|, & \mu \in K_0 \\ 0, & \mu \notin K_0 \end{cases} \tag{4}$$

$$m(x) = \sum_{\lambda \in K_1} \beta_\lambda \varphi(x + b_\lambda), \quad x \in G, \tag{5}$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}$, $K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.

Proof. If we take a discrete topology on group $(G, +)$ and a counting measure on K divided by L , then the assumptions of Theorem 1 are fulfilled. Hence there exists a homomorphism $m : G \rightarrow \mathbb{C}^*$ satisfying (3).

Let K_0 and K_1 be as in the statement of this theorem. From the linear independence of the set $\{m \circ \lambda : \lambda \in K_1\}$ and Lemma 1 there exist $b_\lambda \in G$, $\lambda \in K_1$ such that the matrix $[m(\lambda b_\mu)]_{\lambda, \mu \in K_1}$ has a nonzero determinant. Hence, there exist $\beta_\lambda \in \mathbb{C}^*$, $\lambda \in K_1$, which satisfy (4).

We notice that

$$\begin{aligned} m(x) &= \frac{1}{|K_0|} \sum_{\mu \in K_0} m(\mu x) = \frac{1}{|K_0| \cdot |K_1|} \sum_{\mu \in K} m(\mu x) \sum_{\lambda \in K_1} \beta_\lambda m(\mu b_\lambda) \\ &= \sum_{\lambda \in K_1} \beta_\lambda \frac{1}{L} \sum_{\mu \in K} m(\mu(x + b_\lambda)) = \sum_{\lambda \in K_1} \beta_\lambda \varphi(x + b_\lambda), \quad x \in G, \end{aligned}$$

which proves (5). □

Now we prove a generalization of Wilson’s functional equation.

Theorem 4. Assume that X is complex. Functions $f: G \rightarrow X, f \neq 0, \varphi: G \rightarrow \mathbb{C}$ satisfy the equation

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y)f(x), \quad x, y \in G, \tag{6}$$

if and only if there exist a homomorphism $m: G \rightarrow \mathbb{C}^*, A_0^\lambda \in X, k$ -additive, symmetric mappings $A_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$\varphi(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G, \tag{7}$$

$$f(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right], \quad x \in G, \tag{8}$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1, \tag{9}$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}, K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.

Proof. It is easy to check that if functions f and φ satisfy conditions (7), (8), (9), then they satisfy Eq. (6).

Assume that f and φ satisfy Eq. (6). Since $f \neq 0, \varphi \neq 0$. Note that for $x, y, z \in G$ we have

$$\begin{aligned} L \sum_{\mu \in K} \varphi(y + \mu z)f(x) &= \sum_{\mu \in K} \sum_{\lambda \in K} f(x + \lambda(y + \mu z)) = \sum_{\mu \in K} \sum_{\lambda \in K} f(x + \lambda y + \mu z) \\ &= \sum_{\lambda \in K} L\varphi(z)f(x + \lambda y) = L^2\varphi(z)\varphi(y)f(x). \end{aligned}$$

Taking $x \in G$ such that $f(x) \neq 0$ we obtain that φ satisfies Eq. (2). In view of Theorem 3 we get (7). Now we show that

$$|K_1| \sum_{\sigma \in K_0} f(x + \sigma y) = \sum_{\rho \in K} m(\rho y) \sum_{\nu \in K_1} \beta_\nu f(x + \rho^{-1}b_\nu), \quad x, y \in G. \tag{10}$$

Indeed, we have the following sequence of identities

$$\begin{aligned} |K_1| \sum_{\sigma \in K_0} f(x + \sigma y) &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \beta_\mu m(\lambda b_\mu) f(x + \lambda y) \\ &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \beta_\mu \beta_\nu \varphi(\lambda b_\mu + b_\nu) f(x + \lambda y) \\ &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \sum_{\rho \in K} \frac{1}{L} \beta_\mu \beta_\nu f(x + \lambda y + \rho \lambda b_\mu + \rho b_\nu) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\rho \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \beta_\mu \beta_\nu \varphi(y + \rho b_\mu) f(x + \rho b_\nu) \\
&= \sum_{\rho \in K} \sum_{\nu \in K_1} \beta_\nu m(\rho^{-1}y) f(x + \rho b_\nu) \\
&= \sum_{\rho \in K} m(\rho y) \sum_{\nu \in K_1} \beta_\nu f(x + \rho^{-1}b_\nu), \quad x, y \in G.
\end{aligned}$$

For each $\tau \in K_1$ we define $g_\tau : G \rightarrow X$ by the formula

$$g_\tau(x) := \frac{1}{Lm(\tau x)} \sum_{\nu \in K_1} \beta_\nu \sum_{\sigma \in K_0} f(x + \sigma \tau^{-1}b_\nu), \quad x \in G.$$

From equality (10) we obtain

$$\begin{aligned}
&Lm(\tau x + \tau y) \sum_{\sigma \in K_0} g_\tau(x + \sigma y) \\
&= \sum_{\sigma \in K_0} Lm(\tau(x + \sigma y)) g_\tau(x + \sigma y) \\
&= \sum_{\sigma \in K_0} \sum_{\nu \in K_1} \beta_\nu \sum_{\rho \in K_0} f(x + \sigma y + \rho \tau^{-1}b_\nu) \\
&= \sum_{\nu \in K_1} \beta_\nu \sum_{\rho \in K_0} \sum_{\sigma \in K_0} f(x + \sigma y + \sigma \rho \tau^{-1}b_\nu) \\
&= \sum_{\nu \in K_1} \beta_\nu \sum_{\rho \in K_0} \frac{1}{|K_1|} \sum_{\mu \in K} m(\mu(y + \rho \tau^{-1}b_\nu)) \sum_{\sigma \in K_1} \beta_\sigma f(x + \mu^{-1}b_\sigma) \\
&= \sum_{\mu \in K} m(\mu y) \sum_{\nu \in K_1} \beta_\nu \sum_{\rho \in K_0} \frac{1}{|K_1|} m(\mu \rho \tau^{-1}b_\nu) \sum_{\sigma \in K_1} \beta_\sigma f(x + \mu^{-1}b_\sigma) \\
&= \sum_{\mu \in K} m(\mu y) \frac{|K_0|}{|K_1|} \sum_{\nu \in K_1} \beta_\nu m(\mu \tau^{-1}b_\nu) \sum_{\sigma \in K_1} \beta_\sigma f(x + \mu^{-1}b_\sigma) \\
&= |K_0| \sum_{\mu \in K_0} m(\mu \tau y) \sum_{\sigma \in K_1} \beta_\sigma f(x + \tau^{-1} \mu^{-1}b_\sigma) \\
&= |K_0| m(\tau y) \sum_{\sigma \in K_1} \beta_\sigma \sum_{\mu \in K_0} f(x + \mu \tau^{-1}b_\sigma) \\
&= |K_0| \cdot Lm(\tau y) m(\tau x) g_\tau(x), \quad x \in G.
\end{aligned}$$

Hence

$$\sum_{\sigma \in K_0} g_\tau(x + \sigma y) = |K_0| g_\tau(x), \quad x \in G, \quad \tau \in K_1. \quad (11)$$

In view of Theorem 2, for each $\lambda \in K_1$ there exist $A_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda: S^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}$ such that

$$g_\lambda(x) = A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad 1 \leq i \leq k \leq |K_0| - 1.$$

We observe that

$$\begin{aligned} L \sum_{\lambda \in K_1} m(\lambda x) g_\lambda(x) &= \sum_{\lambda \in K_1} \sum_{\nu \in K_1} \beta_\nu \sum_{\sigma \in K_0} f(x + \sigma \lambda^{-1} b_\nu) \\ &= \sum_{\nu \in K_1} \beta_\nu \sum_{\lambda \in K} f(x + \lambda b_\nu) = \sum_{\nu \in K_1} \beta_\nu L\varphi(b_\nu) f(x) \\ &= Lm(0) f(x) = Lf(x), \quad x \in G, \end{aligned}$$

which ends the proof. □

Theorem 5. Assume that X is real. Functions $f: G \rightarrow X, f \neq 0, \varphi: G \rightarrow \mathbb{R}$ satisfy the equation

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y) f(x), \quad x, y \in G, \tag{12}$$

if and only if there exist a homomorphism $m: G \rightarrow \mathbb{C}^*, A_0^\lambda \in X, B_0^\lambda \in X, k$ -additive, symmetric mappings $A_k^\lambda, B_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$\varphi(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G, \tag{13}$$

$$\begin{aligned} f(x) = \sum_{\lambda \in K_1} \left(\operatorname{Re} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] \right. \\ \left. - \operatorname{Im} m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \right), \quad x \in G, \end{aligned} \tag{14}$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k \leq |K_0| - 1, \tag{15}$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad \lambda \in K_1, \quad 1 \leq i \leq k \leq |K_0| - 1, \tag{16}$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}, K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.

Proof. It is easy to check that if functions f and φ satisfy conditions (13), (14), (15), then they satisfy Eq. (12).

Assume that f and φ satisfy Eq. (12). Since $f \neq 0, \varphi \neq 0$. We observe that for $x, y, z \in G$ we have

$$\begin{aligned} L \sum_{\mu \in K} \varphi(y + \mu z) f(x) &= \sum_{\mu \in K} \sum_{\lambda \in K} f(x + \lambda(y + \mu z)) = \sum_{\mu \in K} \sum_{\lambda \in K} f(x + \lambda y + \mu z) \\ &= \sum_{\lambda \in K} L\varphi(z) f(x + \lambda y) = L^2 \varphi(z) \varphi(y) f(x). \end{aligned}$$

Taking $x \in G$ such that $f(x) \neq 0$ we obtain that φ satisfies Eq. (2). In view of Theorem 3 we get (13). From equalities (4) and (5) we have

$$|K_1| \sum_{\sigma \in K_0} f(x + \sigma y) = \sum_{\rho \in K_1} \sum_{\nu \in K_1} \operatorname{Re} (\beta_\nu m(\rho y)) \sum_{\sigma \in K_0} f(x + \sigma \rho^{-1} b_\nu), \quad x, y \in G, \tag{17}$$

$$0 = \sum_{\rho \in K_1} \sum_{\nu \in K_1} \operatorname{Im} (\beta_\nu m(\rho y)) \sum_{\sigma \in K_0} f(x + \sigma \rho^{-1} b_\nu), \quad x, y \in G. \tag{18}$$

Indeed, for $x, y \in G$ we have

$$\begin{aligned} |K_1| \sum_{\sigma \in K_0} f(x + \sigma y) &= \sum_{\lambda \in K} \operatorname{Re} \left(\sum_{\mu \in K_1} \beta_\mu m(\lambda b_\mu) \right) f(x + \lambda y) \\ &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \operatorname{Re} (\beta_\mu \beta_\nu) \varphi(\lambda b_\mu + b_\nu) f(x + \lambda y) \\ &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \sum_{\rho \in K} \frac{1}{L} \operatorname{Re} (\beta_\mu \beta_\nu) f(x + \lambda y + \rho \lambda b_\mu + \rho b_\nu) \\ &= \sum_{\rho \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \operatorname{Re} (\beta_\mu \beta_\nu) \varphi(y + \rho b_\mu) f(x + \rho b_\nu) \\ &= \sum_{\rho \in K} \sum_{\nu \in K_1} \operatorname{Re} (\beta_\nu m(\rho^{-1} y)) f(x + \rho b_\nu) \\ &= \sum_{\rho \in K_1} \sum_{\nu \in K_1} \operatorname{Re} (\beta_\nu m(\rho y)) \sum_{\sigma \in K_0} f(x + \sigma \rho^{-1} b_\nu), \end{aligned}$$

and

$$\begin{aligned} 0 &= \sum_{\lambda \in K} \operatorname{Im} \left(\sum_{\mu \in K_1} \beta_\mu m(\lambda b_\mu) \right) f(x + \lambda y) \\ &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \operatorname{Im} (\beta_\mu \beta_\nu) \varphi(\lambda b_\mu + b_\nu) f(x + \lambda y) \\ &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \sum_{\rho \in K} \frac{1}{L} \operatorname{Im} (\beta_\mu \beta_\nu) f(x + \lambda y + \rho \lambda b_\mu + \rho b_\nu) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\rho \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \operatorname{Im} (\beta_\mu \beta_\nu) \varphi(y + \rho b_\mu) f(x + \rho b_\nu) \\
 &= \sum_{\rho \in K} \sum_{\nu \in K_1} \operatorname{Im} (\beta_\nu m(\rho^{-1}y)) f(x + \rho b_\nu) \\
 &= \sum_{\rho \in K_1} \sum_{\nu \in K_1} \operatorname{Im} (\beta_\nu m(\rho y)) \sum_{\sigma \in K_0} f(x + \sigma \rho^{-1} b_\nu),
 \end{aligned}$$

For each $\tau \in K_1$ we define $g_\tau, h_\tau: G \rightarrow X$ by the formula

$$\begin{aligned}
 g_\tau(x) &:= \sum_{\nu \in K_1} \operatorname{Re} \frac{\beta_\nu}{Lm(\tau x)} \sum_{\sigma \in K_0} f(x + \sigma \tau^{-1} b_\nu), \quad x \in G, \\
 h_\tau(x) &:= \sum_{\nu \in K_1} \operatorname{Im} \frac{\beta_\nu}{Lm(\tau x)} \sum_{\sigma \in K_0} f(x + \sigma \tau^{-1} b_\nu), \quad x \in G.
 \end{aligned}$$

From equalities (17) and (18) we obtain

$$\sum_{\sigma \in K_0} g_\tau(x + \sigma y) = |K_0| g_\tau(x), \quad x, y \in G, \quad \tau \in K_1, \tag{19}$$

$$\sum_{\sigma \in K_0} h_\tau(x + \sigma y) = |K_0| h_\tau(x), \quad x, y \in G, \quad \tau \in K_1. \tag{20}$$

Indeed, we have the following sequence of identities

$$\begin{aligned}
 &|K_1| \sum_{\lambda \in K_0} g_\tau(x + \lambda y) \\
 &= |K_1| \sum_{\lambda \in K_0} \sum_{\nu \in K_1} \operatorname{Re} \frac{\beta_\nu}{Lm(\tau(x + \lambda y))} \sum_{\sigma \in K_0} f(x + \lambda y + \sigma \tau^{-1} b_\nu) \\
 &= \sum_{\nu \in K_1} \operatorname{Re} \frac{\beta_\nu}{Lm(\tau(x + y))} \sum_{\sigma \in K_0} |K_1| \sum_{\lambda \in K_0} f(x + \lambda(y + \sigma \tau^{-1} b_\nu)) \\
 &= \sum_{\nu \in K_1} \operatorname{Re} \frac{\beta_\nu}{Lm(\tau(x + y))} \sum_{\sigma \in K_0} \sum_{\rho \in K_1} \sum_{\mu \in K_1} \operatorname{Re} (\beta_\mu m(\rho(y + \sigma \tau^{-1} b_\nu))) \\
 &\quad \cdot \sum_{\lambda \in K_0} f(x + \lambda \rho^{-1} b_\mu) \\
 &= \sum_{\nu \in K_1} \operatorname{Re} \frac{|K_0| \beta_\nu}{Lm(\tau(x + y))} \sum_{\rho \in K_1} \sum_{\mu \in K_1} \operatorname{Re} (\beta_\mu m(\rho y) m(\rho \tau^{-1} b_\nu)) \\
 &\quad \cdot \sum_{\lambda \in K_0} f(x + \lambda \rho^{-1} b_\mu) - \sum_{\nu \in K_1} \operatorname{Im} \frac{|K_0| \beta_\nu}{Lm(\tau(x + y))} \\
 &\quad \cdot \sum_{\rho \in K_1} \sum_{\mu \in K_1} \operatorname{Im} (\beta_\mu m(\rho y) m(\rho \tau^{-1} b_\nu)) \sum_{\lambda \in K_0} f(x + \lambda \rho^{-1} b_\mu)
 \end{aligned}$$

$$\begin{aligned}
&= |K_0| \sum_{\rho \in K_1} \sum_{\mu \in K_1} \operatorname{Re} \left(\frac{\beta_\mu m(\rho y)}{Lm(\tau(x+y))} \sum_{\nu \in K_1} \beta_\nu m(\rho \tau^{-1} b_\nu) \right) \sum_{\lambda \in K_0} f(x + \lambda \rho^{-1} b_\mu) \\
&= |K_0| \cdot |K_1| \sum_{\mu \in K_1} \operatorname{Re} \left(\frac{\beta_\mu m(\tau y)}{Lm(\tau(x+y))} \right) \sum_{\lambda \in K_0} f(x + \lambda \tau^{-1} b_\mu) \\
&= |K_0| \cdot |K_1| \sum_{\mu \in K_1} \operatorname{Re} \frac{\beta_\mu}{Lm(\tau x)} \sum_{\lambda \in K_0} f(x + \lambda \tau^{-1} b_\mu) \\
&= |K_0| \cdot |K_1| g_\tau(x), \quad x \in G.
\end{aligned}$$

and

$$\begin{aligned}
&|K_1| \sum_{\lambda \in K_0} h_\tau(x + \lambda y) \\
&= |K_1| \sum_{\lambda \in K_0} \sum_{\nu \in K_1} \operatorname{Im} \frac{\beta_\nu}{Lm(\tau(x + \lambda y))} \sum_{\sigma \in K_0} f(x + \lambda y + \sigma \tau^{-1} b_\nu) \\
&= \sum_{\nu \in K_1} \operatorname{Im} \frac{\beta_\nu}{Lm(\tau(x + y))} \sum_{\sigma \in K_0} |K_1| \sum_{\lambda \in K_0} f(x + \lambda(y + \sigma \tau^{-1} b_\nu)) \\
&= \sum_{\nu \in K_1} \operatorname{Im} \frac{\beta_\nu}{Lm(\tau(x + y))} \sum_{\sigma \in K_0} \sum_{\rho \in K_1} \sum_{\mu \in K_1} \operatorname{Re} (\beta_\mu m(\rho(y + \sigma \tau^{-1} b_\nu))) \\
&\quad \cdot \sum_{\lambda \in K_0} f(x + \lambda \rho^{-1} b_\mu) \\
&= \sum_{\nu \in K_1} \operatorname{Im} \frac{|K_0| \beta_\nu}{Lm(\tau(x + y))} \sum_{\rho \in K_1} \sum_{\mu \in K_1} \operatorname{Re} (\beta_\mu m(\rho y) m(\rho \tau^{-1} b_\nu)) \\
&\quad \cdot \sum_{\lambda \in K_0} f(x + \lambda \rho^{-1} b_\mu) + \sum_{\nu \in K_1} \operatorname{Re} \frac{|K_0| \beta_\nu}{Lm(\tau(x + y))} \\
&\quad \cdot \sum_{\rho \in K_1} \sum_{\mu \in K_1} \operatorname{Im} (\beta_\mu m(\rho y) m(\rho \tau^{-1} b_\nu)) \sum_{\lambda \in K_0} f(x + \lambda \rho^{-1} b_\mu) \\
&= |K_0| \sum_{\rho \in K_1} \sum_{\mu \in K_1} \operatorname{Im} \left(\frac{\beta_\mu m(\rho y)}{Lm(\tau(x+y))} \sum_{\nu \in K_1} \beta_\nu m(\rho \tau^{-1} b_\nu) \right) \sum_{\lambda \in K_0} f(x + \lambda \rho^{-1} b_\mu) \\
&= |K_0| \cdot |K_1| \sum_{\mu \in K_1} \operatorname{Im} \left(\frac{\beta_\mu m(\tau y)}{Lm(\tau(x+y))} \right) \sum_{\lambda \in K_0} f(x + \lambda \tau^{-1} b_\mu) \\
&= |K_0| \cdot |K_1| \sum_{\mu \in K_1} \operatorname{Im} \frac{\beta_\mu}{Lm(\tau x)} \sum_{\lambda \in K_0} f(x + \lambda \tau^{-1} b_\mu) \\
&= |K_0| \cdot |K_1| h_\tau(x), \quad x \in G.
\end{aligned}$$

In view of Theorem 2, for each $\lambda \in K_1$ there exist $A_0^\lambda, B_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda, B_k^\lambda : S^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}$ such that

$$g_\lambda(x) = A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad 1 \leq i \leq k \leq |K_0| - 1,$$

$$h_\lambda(x) = B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x), \quad x \in G,$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \quad 1 \leq i \leq k \leq |K_0| - 1.$$

We observe that

$$\begin{aligned} & \sum_{\tau \in K_1} \operatorname{Re} m(\tau x) g_\tau(x) - \sum_{\tau \in K_1} \operatorname{Im} m(\tau x) h_\tau(x) \\ &= \sum_{\tau \in K_1} \left[\operatorname{Re} m(\tau x) \sum_{\nu \in K_1} \operatorname{Re} \frac{\beta_\nu}{Lm(\tau x)} \sum_{\sigma \in K_0} f(x + \sigma \tau^{-1} b_\nu) \right. \\ & \quad \left. - \operatorname{Im} m(\tau x) \sum_{\nu \in K_1} \operatorname{Im} \frac{\beta_\nu}{Lm(\tau x)} \sum_{\sigma \in K_0} f(x + \sigma \tau^{-1} b_\nu) \right] \\ &= \sum_{\tau \in K_1} \sum_{\nu \in K_1} \operatorname{Re} \left(m(\tau x) \frac{\beta_\nu}{Lm(\tau x)} \right) \sum_{\sigma \in K_0} f(x + \sigma \tau^{-1} b_\nu) \\ &= \sum_{\tau \in K_1} \sum_{\sigma \in K_0} \sum_{\nu \in K_1} \operatorname{Re} \left(\frac{\beta_\nu}{L} \right) f(x + \sigma \tau^{-1} b_\nu) \\ &= \sum_{\lambda \in K} \sum_{\nu \in K_1} \operatorname{Re} \left(\frac{\beta_\nu}{L} \right) f(x + \lambda b_\nu) \\ &= \sum_{\nu \in K_1} \operatorname{Re} \beta_\nu \varphi(b_\nu) f(x) = m(0) f(x) = f(x), \quad x \in G, \end{aligned}$$

which ends the proof. □

Corollary 1. *Functions $f, g : G \rightarrow X, \varphi : G \rightarrow \mathbb{K}, f \neq 0$, satisfy the equality*

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y)g(x), \quad x, y \in G, \tag{21}$$

if and only if there exists a homomorphism $m : G \rightarrow \mathbb{C}^$ such that*

$$\varphi(x) = \varphi(0) \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G,$$

and (i) if X is real, then there exist $A_0^\lambda, B_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda, B_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$f(x) = \varphi(0) \sum_{\lambda \in K_1} \left(\operatorname{Re} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \operatorname{Im} m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \right), \quad x \in G,$$

$$g(x) = \sum_{\lambda \in K_1} \left(\operatorname{Re} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \operatorname{Im} m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \right), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

(ii) if X is complex, then there exist $A_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$f(x) = \varphi(0) \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right], \quad x \in G,$$

$$g(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right], \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}, K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.

Proof. Putting $y = 0$ in (21) we have

$$Lf(x) = L\varphi(0)g(x), \quad x \in G.$$

Since $f \neq 0, g \neq 0, \varphi(0) \neq 0$ and

$$\varphi(0) \sum_{\lambda \in K} g(x + \lambda y) = L\varphi(y)g(x), \quad x, y \in G,$$

hence for $\varphi_0 := \frac{\varphi}{\varphi(0)}$ we have

$$\sum_{\lambda \in K} g(x + \lambda y) = L\varphi_0(y)g(x), \quad x, y \in G.$$

In view of Theorems 4 and 5 accordingly we obtain (ii) and (i) of the theorem. □

Theorem 6. *Functions $f: G \rightarrow X, \varphi: G \rightarrow \mathbb{K}, f \neq 0, \varphi \neq \text{const}$, satisfy the equality*

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y)f(x) + \sum_{\lambda \in K} f(\lambda y), \quad x, y \in G, \tag{22}$$

if and only if there exists a homomorphism $m: G \rightarrow \mathbb{C}^$ such that*

$$\varphi(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G,$$

and

(i) *if X is real, then there exist $A_0^\lambda, B_0^\lambda \in X, k$ -additive, symmetric mappings $A_k^\lambda, B_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that*

$$\begin{aligned} f(x) = & \sum_{\lambda \in K_1} \left(\operatorname{Re} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] \right. \\ & \left. - \operatorname{Im} m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \right) - \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G, \\ \sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = & 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1, \\ \sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = & 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1, \end{aligned}$$

(ii) *if X is complex, then there exist $A_0^\lambda \in X, k$ -additive, symmetric mappings $A_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that*

$$\begin{aligned} f(x) = & \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G, \\ \sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = & 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1, \end{aligned}$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}, K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.

Moreover

$$\sum_{\lambda \in K} f(\lambda x) = L(\varphi(x) - 1) \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G. \tag{23}$$

Proof. Putting $x = 0$ in (22) we have

$$\sum_{\lambda \in K} f(\lambda y) = L\varphi(y)f(0) + \sum_{\lambda \in K} f(\lambda y), \quad y \in G.$$

Since $\varphi \neq \text{const}$, $f(0) = 0$. Putting $y = 0$ in (22) we get

$$Lf(x) = L\varphi(0)f(x) + Lf(0), \quad x \in G,$$

hence we obtain $\varphi(0) = 1$. We observe that

$$\begin{aligned} L\varphi(x) \sum_{\lambda \in K} f(\lambda y) + L \sum_{\lambda \in K} f(\lambda x) &= \sum_{\mu \in K} \sum_{\lambda \in K} f(\lambda y + \mu x) = \sum_{\mu \in K} \sum_{\lambda \in K} f(\mu x + \lambda y) \\ &= L\varphi(y) \sum_{\mu \in K} f(\mu x) + L \sum_{\lambda \in K} f(\lambda y), \quad x, y \in G. \end{aligned}$$

Hence we have

$$L(\varphi(y) - 1) \sum_{\mu \in K} f(\mu x) = L(\varphi(x) - 1) \sum_{\lambda \in K} f(\lambda y), \quad x, y \in G,$$

which means that

$$\sum_{\lambda \in K} f(\lambda y) = L(\varphi(y) - 1)A_0, \quad y \in G, \tag{24}$$

for some $A_0 \in X$. Therefore we obtain that $f \neq \text{const}$. Putting the above equality in (22) we get

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y)f(x) + L(\varphi(y) - 1)A_0, \quad x, y \in G,$$

hence, for the function $g: G \rightarrow X$ given by the formula $g := f + A_0$, we obtain

$$\sum_{\lambda \in K} g(x + \lambda y) = L\varphi(y)g(x), \quad x, y \in G.$$

In view of Theorems 4 and 5 there exists a homomorphism $m: G \rightarrow \mathbb{C}^*$ such that

$$\varphi(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G,$$

and

(i) if X is real, then there exist $A_0^\lambda, B_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda, B_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$g(x) = \sum_{\lambda \in K_1} \left(\operatorname{Re} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \operatorname{Im} m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \right), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

(ii) if X is complex, then there exist $A_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$g(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right], \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

Since $f(0) = 0$ and $g = f + A_0$, we have

$$A_0 = g(0) = \sum_{\lambda \in K_1} A_0^\lambda,$$

which together with (24) gives equality (23) and the form of the function f . □

Corollary 2. *Functions $f: G \rightarrow X, \varphi: G \rightarrow \mathbb{K}, f \neq 0, \varphi \neq \text{const}$, satisfy the equation*

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y)f(x) + Lf(y), \quad x, y \in G, \tag{25}$$

if and only if there exist a homomorphism $m: G \rightarrow \mathbb{C}^*$, and $A \in X$ such that

$$\varphi(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G,$$

$$f(x) = (\varphi(x) - 1)A, \quad x \in G.$$

Proof. It is easy to check that if functions f and φ satisfy the above conditions, then they satisfy Eq. (25).

Assume that f and φ satisfy Eq. (25). Putting $x = 0$ in (25) we have

$$\sum_{\lambda \in K} f(\lambda y) = L\varphi(y)f(0) + Lf(y), \quad y \in G.$$

Putting $y = 0$ in (25) we get

$$Lf(x) = L\varphi(0)f(x) + Lf(0), \quad x \in G.$$

If $\varphi(0) = 0$, then $f = f(0)$ and from the above equalities $f = 0$ which gives a contradiction. Hence, since $\varphi \neq \text{const}$, $f(0) = 0$, $\varphi(0) = 1$ and $\sum_{\lambda \in K} f(\lambda y) = Lf(y)$. In particular f and φ satisfy Eq. (22). In view of Theorem 6 we have

$$Lf(x) = \sum_{\mu \in K} f(\mu x) = L(\varphi(x) - 1)A, \quad x, y \in G,$$

for some $A \in X$. □

Theorem 7. *Functions $f, g, h: G \rightarrow X, \varphi: G \rightarrow \mathbb{K}, f \neq 0, \varphi \neq \text{const}, \varphi(0) \neq 0$, satisfy the equation*

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y)g(x) + Lh(y), \quad x, y \in G, \quad (26)$$

if and only if there exists a homomorphism $m: G \rightarrow \mathbb{C}^*, A, B \in X$ such that

$$\varphi(x) = \varphi(0) \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G,$$

and

(i) if X is real, then there exist $A_0^\lambda, B_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda, B_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$f(x) = \varphi(0) \sum_{\lambda \in K_1} \left(\operatorname{Re} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \operatorname{Im} m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \right) + A - \varphi(0) \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G,$$

$$g(x) = \sum_{\lambda \in K_1} \left(\operatorname{Re} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \operatorname{Im} m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \right) + B - \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G,$$

$$h(x) = \varphi(x) \left(\sum_{\lambda \in K_1} A_0^\lambda - B \right) + \left(A - \varphi(0) \sum_{\lambda \in K_1} A_0^\lambda \right), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

(ii) if X is complex, then there exist $A_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$f(x) = \varphi(0) \left(\sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \sum_{\lambda \in K_1} A_0^\lambda \right) + A, \quad x \in G,$$

$$g(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] + B - \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G,$$

$$h(x) = \varphi(x) \left(\sum_{\lambda \in K_1} A_0^\lambda - B \right) + \left(A - \varphi(0) \sum_{\lambda \in K_1} A_0^\lambda \right), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

where $K_0 := \{\lambda \in K : m \circ \lambda = m\}, K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.

Proof. Putting $x = 0$ in (26) we have

$$\sum_{\lambda \in K} f(\lambda y) = L\varphi(y)g(0) + Lh(y), \quad y \in G.$$

Putting $y = 0$ in (26) we get

$$Lf(x) = L\varphi(0)g(x) + Lh(0), \quad x \in G.$$

Hence we have

$$\begin{aligned} \varphi(0) \sum_{\lambda \in K} g(x + \lambda y) &= \sum_{\lambda \in K} f(x + \lambda y) - Lh(0) \\ &= L\varphi(y)g(x) + Lh(y) - Lh(0) \\ &= L\varphi(y)g(x) + \sum_{\lambda \in K} f(\lambda y) - L\varphi(y)g(0) - Lh(0) \\ &= L\varphi(y)(g(x) - g(0)) + \varphi(0) \sum_{\lambda \in K} g(\lambda y), \quad x, y \in G, \end{aligned}$$

which gives us that functions $g_0 = g - g(0), \varphi_0 = \frac{\varphi}{\varphi(0)}$ satisfy the equation

$$\sum_{\lambda \in K} g_0(x + \lambda y) = L\varphi_0(y)g_0(x) + \sum_{\lambda \in K} g_0(\lambda y), \quad x, y \in G.$$

In view of Theorem 6 there exists a homomorphism $m: G \rightarrow \mathbb{C}^*$ such that

$$\varphi_0(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G,$$

and

(i) if X is real, then there exist $A_0^\lambda, B_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda, B_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$g_0(x) = \sum_{\lambda \in K_1} \left(\operatorname{Re} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \operatorname{Im} m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \right) - \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

(ii) if X is complex, then there exist $A_0^\lambda \in X$, k -additive, symmetric mappings $A_k^\lambda: G^k \rightarrow X, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$g_0(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_i) = 0, \quad x, y \in G, \lambda \in K_1, 1 \leq i \leq k \leq |K_0| - 1,$$

Moreover

$$\sum_{\lambda \in K} g_0(\lambda x) = L(\varphi_0(x) - 1) \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G.$$

Hence, putting $B := g(0)$, we obtain the form of φ and g . Since

$$Lf(x) = L\varphi(0)g(x) + Lh(0) = L\varphi(0)g_0(x) + Lf(0), \quad x \in G,$$

and

$$Lh(x) = \sum_{\lambda \in K} f(\lambda x) - L\varphi(x)g(0) = \varphi(0) \sum_{\lambda \in K} g(\lambda x) + Lh(0) - L\varphi(x)g(0)$$

$$= \varphi(0) \sum_{\lambda \in K} g_0(\lambda x) + Lf(0) - L\varphi(x)g(0), \quad x \in G,$$

from the form of g_0 we obtain the form of f and h in the real and the complex case of the space X . □

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