Aequat. Math. 87 (2014), 159–163

0001-9054/14/010159-5
published online January 16, 2013
DOI 10.1007/s00010-012-0183-5

Aequationes Mathematicae

On additive involutions and Hamel bases

Karol Baron

Abstract. We provide an example of a discontinuous involutory additive function $a : \mathbb{R} \to \mathbb{R}$ such that $a(H)\backslash H \neq \emptyset$ for every Hamel basis $H \subset \mathbb{R}$ and show that, in fact, the set of all such functions is dense in the topological vector space of all additive functions from $\mathbb{R}$ to $\mathbb{R}$ with the Tychonoff topology induced by $\mathbb{R}^{\mathbb{R}}$.

Mathematics Subject Classification (2000). Primary 39B52.

Keywords. Additive involution, Hamel basis, linear topological space, Tychonoff topology.

1. Introduction

By a Hamel basis of $\mathbb{R}$ we mean (see [2, p. 82]) a basis of the vector space $\mathbb{R}$ over the field $\mathbb{Q}$ of rationals. Inspired by the foot-note on p. 325 of [2] we are interested in discontinuous additive functions $a : \mathbb{R} \to \mathbb{R}$ which are involutory, i.e., $a \circ a = \mathrm{id}_{\mathbb{R}}$, and such that

$$a(H)\backslash H \neq \emptyset \tag{1}$$

for every Hamel basis $H \subset \mathbb{R}$.

2. Existence

The following theorem provides an example of a discontinuous involutory additive function $a : \mathbb{R} \to \mathbb{R}$ such that (1) holds for every set $H \subset \mathbb{R}$ which is linearly independent over $\mathbb{Q}$ and has at least three elements.

Theorem 1. *Assume X is a linear space over the field $\mathbb{Q}$ with $dim X \geq 3$. If H_0 is a basis of X, $h_0 \in H_0$ and $a : X \to X$ is the additive function defined by*

$$a(h_0) = h_0 \quad \textit{and} \quad a(h) = -h \quad \textit{for } h \in H_0\backslash\{h_0\}, \tag{2}$$

then a is involutory,

$$a(x) + x \in \mathbb{Q}h_0 \quad \textit{for} \quad x \in X, \tag{3}$$

and (1) *holds for every linearly independent set $H \subset X$ with* card $H \geq 3$.

Proof. To see that $a(a(x)) = x$ for $x \in X$ it is enough to observe that it holds for $x \in H_0$.

Let $r : X \to \mathbb{Q}$ be the function such that

$$x - r(x)h_0 \in \mathrm{Lin}(H_0 \setminus \{h_0\}) \quad \text{for } x \in X.$$

Then, according to (2),

$$a(x) - r(x)h_0 = a(x - r(x)h_0) = -(x - r(x)h_0)$$

whence

$$a(x) = -x + 2r(x)h_0 \quad \text{for } x \in X. \tag{4}$$

In particular we have (3),

$$\text{if } x \in X \text{ and } a(x) = x, \text{ then } x \in \mathbb{Q}h_0, \tag{5}$$

and

$$\text{if } x \in X \text{ and } r(x) = 0, \text{ then } a(x) = -x. \tag{6}$$

Suppose $H \subset X$ is linearly independent and (1) does not hold, i.e.,

$$a(H) \subset H. \tag{7}$$

It follows from (7) and (6) that $r(h) \neq 0$ for $h \in H$ and making use of (4) we see that

$$h_0 = \frac{a(h) + h}{2r(h)} \quad \text{for } h \in H. \tag{8}$$

We will show that

$$\text{if } h_1, h_2 \in H \setminus \mathbb{Q}h_0 \text{ are different, then } r(h_1) = r(h_2) \text{ and } h_1 + h_2 = 2r(h_1)h_0. \tag{9}$$

To this end fix arbitrarily different elements h_1, h_2 of $H \setminus \mathbb{Q}h_0$. It follows from (8) that

$$\frac{a(h_1) + h_1}{r(h_1)} = \frac{a(h_2) + h_2}{r(h_2)} \tag{10}$$

which jointly with (7) gives

$$\mathrm{card}\{h_1, h_2, a(h_1), a(h_2)\} \leq 3.$$

Hence, taking (5) into account and that a is involutory, we have

$$h_2 = a(h_1)$$

whence, by (4) and (10),

$$h_2 = -h_1 + 2r(h_1)h_0$$

and

$$\frac{h_2+h_1}{r(h_1)} = \frac{a(a(h_1))+h_2}{r(h_2)} = \frac{h_1+h_2}{r(h_2)}$$

which ends the proof of (9).

Since H is linearly independent, (9) implies that $\mathrm{card}H \leq 2$. □

Remark 1. Let a be defined as in Theorem 1.

(i) If $h_1 \in H_0\backslash\{h_0\}$, then h_0+h_1, h_0-h_1 are linearly independent and

$$a(\{h_0+h_1, h_0-h_1\}) = \{h_0+h_1, h_0-h_1\}.$$

(ii) If X is a linear space over a field $\mathbb{K}$ such that $\mathbb{Q}$ is its proper subfield, then

$$a(\alpha h_0) \neq \alpha a(h_0) \quad \text{for } \alpha \in \mathbb{K}\backslash\mathbb{Q}. \tag{11}$$

In particular, if X is a linear topological space, then the function

$$a \mapsto a(\alpha h_0), \quad \alpha \in \mathbb{R}, \tag{12}$$

is discontinuous.

Proof. Part (i) is obvious, (11) follows from (3) and (2), and the continuity of (12) implies $a(\alpha h_0) = \alpha a(h_0)$ for $\alpha \in \mathbb{R}$. □

3. Density

Fix a linear topological space X, $X \neq \{0\}$, and consider the space X^X of all functions from X into X with the usual addition and multiplication by scalars and with the Tychonoff topology. Clearly X^X is a linear topological space. Put

$$\mathcal{A} = \{a \in X^X : a \text{ is additive}\}$$

and consider $\mathcal{A}$ with the topology induced by X^X.

Theorem 2. *The sets*

$$\{a \in \mathcal{A} : a \circ a = \mathrm{id}_X, a\, is\, discontinuous\, and\, (1)\, holds\, for\, every\, uncountable\, set\, H \subset X\, which\, is\, linearly\, independent\, over\, \mathbb{Q}\}, \tag{13}$$

$$\{a \in \mathcal{A} : a \circ a = \mathrm{id}_X, a\, is\, discontinuous\, and\, a(H) = H\, for\, a\, basis\, H\, of\, the\, vector\, space\, X\, over\, the\, field\, \mathbb{Q}\} \tag{14}$$

are dense in $\mathcal{A}$.

For the proof the following lemma from [1] will be used.

Lemma 1. *A subset $\mathcal{D}$ of $\mathcal{A}$ is dense in $\mathcal{A}$ if and only if for any $M \in \mathbb{N}$, for any $h_1, \ldots, h_M, h'_1, \ldots, h'_M \in X$ being linearly independent over $\mathbb{Q}$, and for any neighbourhood $U \subset X$ of zero there exists $a \in \mathcal{D}$ such that*

$$a(h_m) \in U + h'_m \quad for\ m \in \{1, \ldots, M\}.$$

In fact we will apply the following corollary resulting from this lemma.

Corollary 1. *If* $\mathcal{D} \subset \mathcal{A}$ *and for any* $M \in \mathbb{N}$ *and for any* $h_1, \ldots, h_M, {h'}_1, \ldots, {h'}_M \in X$ *being linearly independent over* $\mathbb{Q}$ *there exists* $a \in \mathcal{D}$ *such that*

$$a(h_m) = {h'}_m \quad \textit{for } m \in \{1, \ldots, M\},$$

then $\mathcal{D}$ *is dense in* $\mathcal{A}$.

Proof of the density of the set (13). Fix $M \in \mathbb{N}$ and $h_1^o, \ldots, h_M^o, h_1^{o'}, \ldots, h_M^{o'} \in X$ linearly independent over $\mathbb{Q}$. Let H_0 be a basis of the vector space X over the field $\mathbb{Q}$ such that $h_1^o, \ldots, h_M^o, h_1^{o'}, \ldots, h_M^{o'} \in H_0$ and define the additive function $a : X \to X$ by putting

$$a(h_m^o) = h_m^{o'},\ a(h_m^{o'}) = h_m^o \quad \text{for } m \in \{1, \ldots, M\},$$

and

$$a(h) = -h \quad \text{for } h \in H_0 \setminus \{h_1^o, \ldots, h_M^o, h_1^{o'}, \ldots, h_M^{o'}\}.$$

Clearly, $a \circ a = \mathrm{id}_X$.

Let $r_1, \ldots, r_M, {r'}_1, \ldots, {r'}_M : X \to \mathbb{Q}$ be the functions such that

$$x - \left(\sum_{m=1}^{M} r_m(x) h_m^o + \sum_{m=1}^{M} {r'}_m(x) {h'}_0\right) \in \mathrm{Lin}_{\mathbb{Q}}(H_0 \setminus \{h_1^o, \ldots, h_M^o, h_1^{o'}, \ldots, h_M^{o'}\})$$

for $x \in X$. Then

$$a(x) = -x + \sum_{m=1}^{M} (r_m(x) + {r'}_m(x))(h_m^o + h_m^{o'}) \quad \text{for } x \in X.$$

In particular,

$$a(x) + x \in \mathrm{Lin}_{\mathbb{Q}}\{h_1^o, \ldots, h_M^o, h_1^{o'}, \ldots, h_M^{o'}\} \quad \text{for } x \in X \tag{15}$$

and so a is discontinuous.

Suppose $H \subset X$ is uncountable, linearly independent over $\mathbb{Q}$, and (1) does not hold. Take different $h_1, \ldots, h_{2M+1} \in H$ such that

$$h_k + h_l \notin \mathrm{Lin}_{\mathbb{Q}}\{h_1^o, \ldots, h_M^o, h_1^{o'}, \ldots, h_M^{o'}\} \quad \text{for } k, l \in \{1, \ldots, 2M+1\}. \tag{16}$$

It follows from (15) that

$$a(h_1) + h_1, \ldots, a(h_{2M+1}) + h_{2M+1}$$

are linearly dependent over $\mathbb{Q}$ and so are

$$h_1, \ldots, h_{2M+1}, a(h_1), \ldots, a(h_{2M+1}).$$

Hence, taking (7) into account and that a is injective, we have

$$a(h_k) = h_l$$

for some $k, l \in \{1, \ldots, 2M+1\}$. This and (15) give

$$h_k + h_l = h_k + a(h_k) \in \mathrm{Lin}_{\mathbb{Q}}\{h_1^o, \ldots, h_M^o, h_1^{o'}, \ldots, h_M^{o'}\}$$

which contradicts (16). The contradiction obtained shows that a belongs to the set (13) and applying Corollary 1 we conclude the proof.

Proof of the density of the set (14). Fix $M \in \mathbb{N}$ and $h_1, \ldots, h_M, h'_1, \ldots, h'_M \in X$ linearly independent over $\mathbb{Q}$. Let H be a basis of the vector space X over the field $\mathbb{Q}$ such that $h_1, \ldots, h_M, h'_1, \ldots, h'_M \in H$ and define the additive function $a : X \to X$ by putting

$$a(h_m) = h'_m,\ a(h'_m) = h_m \quad \text{for } m \in \{1, \ldots, M\},$$

and

$$a(h) = h \quad \text{for } h \in H \setminus \{h_1, \ldots, h_M, h'_1, \ldots, h'_M\}.$$

Then $a \circ a = \mathrm{id}_X$,

$$a(x) - x \in \mathrm{Lin}_{\mathbb{Q}}\{h_1, \ldots, h_M, h'_1, \ldots, h'_M\} \quad \text{for } x \in X$$

and $a(H) = H$. Hence a is in the set (14) and applying Corollary 1 we find that this set is dense in $\mathcal{A}$.

Acknowledgement

The research was supported by the Silesian University Mathematics Department (Iterative Functional Equations and Real Analysis program).

References

[1] Baron, K., Volkmann, P.: Dense sets of additive functions, Seminar LV, No. 16. http://www.math.us.edu.pl/smdk (2003)

[2] Kuczma, M.: An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality. In: Gilányi, A. (ed.) 2nd edn. Birkhäuser Verlag, Basel (2009)

Karol Baron
Instytut Matematyki
Uniwersytet Śląski
ul. Bankowa 14
40-007 Katowice
Poland
e-mail: baron@us.edu.pl

Received: November 7, 2012
Revised: December 6, 2012