Aequat. Math. 85 (2013), 497–503

0001-9054/13/030497-7
published online August 17, 2012
DOI 10.1007/s00010-012-0152-z

Aequationes Mathematicae

Stability of the equation of the p-Wright affine functions

Janusz Brzdęk

Abstract. We prove some stability results for the equation of the p-Wright affine functions.

Mathematics Subject Classification (2010). Primary 39B82; Secondary 39B62.

Keywords. Hyers–Ulam stability, p-Wright convexity, affine function.

1. Introduction and preliminaries

Let $0 < p < 1$ be a fixed real number. We say that a function f mapping a real nonempty interval I into the set of reals $\mathbb{R}$ is p-Wright convex provided (see, e.g., [4,9,16,22])

$$f(px + (1-p)y) + f((1-p)x + py) \le f(x) + f(y), \qquad x, y \in I.$$

If f satisfies the functional equation

$$f(px + (1-p)y) + f((1-p)x + py) = f(x) + f(y), \tag{1.1}$$

then we say that it is p-Wright affine (see [4]).

Note that for $p = 1/2$ Eq. (1.1) becomes the Jensen's functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

For $p = 1/3$ Eq. (1.1) takes the form

$$f(2x + y) + f(x + 2y) = f(3x) + f(3y), \tag{1.2}$$

which has been investigated by Najati and Park [18]; in particular, they proved some results on its stability and applied them in the investigation of the generalized (σ, τ)-Jordan derivations on Banach algebras. The cases of more arbitrary p were studied in [4,5,15] (see also [9,13]).

We prove some results concerning the Hyers–Ulam stability and superstability of (1.1). For more information and numerous references on the stability

Birkhäuser

of functional equations we refer to, e.g., [10,14,17,21]; for some examples of various recent outcomes showing new directions in this area of research see, e.g., [3,7,8,11,12,19,20].

The method of the proof of the main result corresponds to some observations in [6,7,20] and the main tool in it is a fixed point result that can be derived from [1, Theorem 1] (see also [2, Theorem 2]). To present it we need the following four hypotheses ($\mathbb{R}_+$ denotes the set of nonnegative reals).

(H1) X is a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ ($\mathbb{C}$ denotes the set of complex numbers) and Y is a Banach space.

(H2) $f_1, \ldots, f_k : X \to X$ and $L_1, \ldots, L_k : X \to \mathbb{R}_+$ are given maps.

(H3) $\mathcal{T} : Y^X \to Y^X$ is an operator satisfying the inequality

$$\left\|(\mathcal{T}\xi)(x) - (\mathcal{T}\mu)(x)\right\| \leq \sum_{i=1}^{k} L_i(x) \left\|\xi(f_i(x)) - \mu(f_i(x))\right\|$$

for every $\xi, \mu \in Y^X$, $x \in X$.

(H4) $\Lambda : {\mathbb{R}_+}^X \to {\mathbb{R}_+}^X$ is a linear operator defined by

$$(\Lambda\delta)(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^X, x \in X.$$

Now we are in a position to present the above mentioned fixed point theorem.

Theorem 1.1. *Assume that hypotheses* (H1)–(H4) *are satisfied. Suppose that there are functions* $\varepsilon : X \to \mathbb{R}_+$ *and* $\varphi : X \to Y$ *such that*

$$\left\|(\mathcal{T}\varphi)(x) - \varphi(x)\right\| \leq \varepsilon(x), \qquad x \in X, \tag{1.3}$$

and

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty, \qquad x \in X. \tag{1.4}$$

Then there exists a unique fixed point ψ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \qquad x \in X. \tag{1.5}$$

Moreover

$$\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n \varphi)(x), \qquad x \in X. \tag{1.6}$$

2. Stability

The next theorem is the main result in this paper and concerns the stability of Eq. (1.1); it corresponds in particular to some results in [18].

Theorem 2.1. *Let* (H1) *be valid,* $p \in \mathbb{F}$, $A, k \in (0, \infty)$, $|p|^k + |1-p|^k < 1$, *and* $g : X \to Y$ *satisfy*

$$\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \\ \leq A(\|x\|^k + \|y\|^k), \qquad x, y \in X. \tag{2.1}$$

Then there exists a unique solution $G : X \to Y$ *of Eq.* (1.1) *such that*

$$\|g(x) - G(x)\| \leq \frac{A\|x\|^k}{1 - |p|^k - |1-p|^k}, \qquad x \in X \tag{2.2}$$

and G *is given by:*

$$G(x) := g(0) + \lim_{n\to\infty} (\mathcal{T}^n g_0)(x), \qquad x \in X, \tag{2.3}$$

where g_0 *and* $\mathcal{T}$ *are defined by* (2.6) *and* (2.7).

Moreover, G *is the unique solution of Eq.* (1.1) *such that there exists a constant* $M \in (0, \infty)$ *with*

$$\|g(x) - G(x)\| \leq M\|x\|^k, \qquad x \in X. \tag{2.4}$$

Proof. Note that (2.1) with $y = 0$ gives

$$\|g(px) + g((1-p)x) - g(x) - g(0)\| \leq A\|x\|^k, \qquad x \in X. \tag{2.5}$$

Write

$$g_0(x) = g(x) - g(0), \qquad x \in X \tag{2.6}$$

and

$$\mathcal{T}\xi(x) := \xi(px) + \xi((1-p)x), \qquad x \in X, \xi \in Y^X. \tag{2.7}$$

Then (2.5) implies the inequality

$$\|g_0(px) + g_0((1-p)x) - g_0(x)\| \leq A\|x\|^k, \qquad x \in X,$$

which means that

$$\|\mathcal{T}g_0(x) - g_0(x)\| \leq A\|x\|^k, \qquad x \in X. \tag{2.8}$$

Further note that (H3) holds with $k = 2$, $f_1(x) = px$, $f_2(x) = (1-p)x$, $L_i(x) = 1$ for $i = 1, 2$, $x \in X$. Define Λ as in (H4). Clearly, with $\varepsilon(x) := A\|x\|^k$ for $x \in X$, we have

$$\begin{aligned}\varepsilon^*(x) &:= \sum_{n=0}^{\infty}(\Lambda^n\varepsilon)(x)\\ &\le A\|x\|^k\sum_{n=0}^{\infty}\left(|p|^k+|1-p|^k\right)^n\\ &=\frac{A\|x\|^k}{1-|p|^k-|1-p|^k}, \qquad x\in X.\end{aligned}$$

Hence, according to Theorem 1.1, there exists a unique solution $G_0 : X \to Y$ of the equation

$$G_0(x) = G_0(px) + G_0((1-p)x) \tag{2.9}$$

such that

$$\|g_0(x) - G_0(x)\| \le \frac{A\|x\|^k}{1-|p|^k-|1-p|^k}, \qquad x \in X;$$

moreover

$$G_0(x) := \lim_{n\to\infty}(\mathcal{T}^n g_0)(x), \qquad x \in X.$$

Now we show that, for every $x, y \in X$, $n \in \mathbb{N}_0$ (nonnegative integers),

$$\begin{aligned}&\|\mathcal{T}^n g_0(px+(1-p)y) + \mathcal{T}^n g_0((1-p)x+py) - \mathcal{T}^n g_0(x) - \mathcal{T}^n g_0(y)\|\\ &\quad\le A(|p|^k+|1-p|^k)^n(\|x\|^k+\|y\|^k).\end{aligned} \tag{2.10}$$

Clearly, the case $n = 0$ is just (2.1). Next, fix $m \in \mathbb{N}_0$ and assume that (2.10) holds for every $x, y \in X$ with $n = m$. Then

$$\begin{aligned}&\big\|\mathcal{T}^{m+1} g_0(px+(1-p)y) + \mathcal{T}^{m+1}g_0((1-p)x+py)\\ &\qquad - \mathcal{T}^{m+1}g_0(x) - \mathcal{T}^{m+1}g_0(y)\big\|\\ &\quad= \big\|\mathcal{T}^m g_0(p(px+(1-p)y)) + \mathcal{T}^m g_0((1-p)(px+(1-p)y))\\ &\qquad + \mathcal{T}^m g_0(p((1-p)x+py)) + \mathcal{T}^m g_0((1-p)((1-p)x+py))\\ &\qquad - \mathcal{T}^m g_0(px) - \mathcal{T}^m g_0((1-p)x) - \mathcal{T}^m g_0(py) - \mathcal{T}^m g_0((1-p)y)\big\|\\ &\quad\le \big\|\mathcal{T}^m g_0(ppx+(1-p)py)) + \mathcal{T}^m g_0((1-p)px+ppy))\\ &\qquad - \mathcal{T}^m g_0(px) - \mathcal{T}^m g_0(py)\big\|\\ &\qquad + \big\|\mathcal{T}^m g_0(p(1-p)x+(1-p)(1-p)y))\\ &\qquad + \mathcal{T}^m g_0((1-p)(1-p)x+p(1-p)y))\\ &\qquad - \mathcal{T}^m g_0((1-p)x) - \mathcal{T}^m g_0((1-p)y)\big\|\\ &\quad\le A(|p|^k+|1-p|^k)^m(\|px\|^k+\|py\|^k)\\ &\qquad + (|p|^k+|1-p|^k)^m(\|(1-p)x\|^k+\|(1-p)y\|^k)\\ &\quad= (|p|^k+|1-p|^k)^{m+1}(\|x\|^k+\|y\|^k), \qquad x, y \in X.\end{aligned}$$

Thus, by induction we have shown that (2.10) holds for every $x, y \in X$ and $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (2.10), we obtain that

$$G_0(px + (1-p)y) + G_0((1-p)x + py) = G_0(x) + G_0(y), \qquad x, y \in X.$$

Write $G(x) := G_0(x) + g(0)$ for $x \in X$. Then it is easily seen that

$$G(px + (1-p)y) + G((1-p)x + py) = G(x) + G(y), \qquad x, y \in X$$

and (2.2) holds. It remains to show the statement concerning the uniqueness of G.

So suppose that $M_0 \in (0, \infty)$ and $G_1 : X \to Y$ is a solution to (1.1) with

$$\|g(x) - G_1(x)\| \le M_0 \|x\|^k, \qquad x \in X.$$

Note that $G(0) = g(0) = G_1(0)$,

$$G_1(px) + G_1((1-p)x) = G_1(x) + G_1(0), \qquad x \in X, \tag{2.11}$$

$$G(px) + G((1-p)x) = G(x) + G(0), \qquad x \in X, \tag{2.12}$$

and, by (2.2),

$$\begin{aligned} \|G(x) - G_1(x)\| &\le \frac{(M+A)\|x\|^k}{1 - |p|^k - |1-p|^k} \\ &= (M+A)\|x\|^k \sum_{n=0}^{\infty} \left(|p|^k + |1-p|^k\right)^n, \quad x \in X, \end{aligned} \tag{2.13}$$

where $M := M_0(1 - |p|^k - |1-p|^k)$.

We show that, for each $j \in \mathbb{N}_0$,

$$\|G(x) - G_1(x)\| \le (M+A)\|x\|^k \sum_{n=j}^{\infty} \left(|p|^k + |1-p|^k\right)^n, \quad x \in X. \tag{2.14}$$

The case $j = 0$ is exactly (2.13). So fix $l \in \mathbb{N}_0$ and assume that (2.14) holds for $j = l$. Then, in view of (2.11) and (2.12),

$$\begin{aligned} \|G(x) - G_1(x)\| &= \|G(px) + G((1-p)x) - G_1(px) - G_1((1-p)x)\| \\ &\le \|G(px) - G_1(px)\| + \|G((1-p)x) - G_1((1-p)x)\| \\ &\le (M+A)(\|px\|^k + \|(1-p)x\|^k) \sum_{n=l}^{\infty} \left(|p|^k + |1-p|^k\right)^n \\ &\le (M+A)\|x\|^k \sum_{n=l+1}^{\infty} \left(|p|^k + |1-p|^k\right)^n, \qquad x \in X. \end{aligned}$$

Thus we have shown (2.14). Now, letting $j \to \infty$ in (2.14) we get $G_1 = G$. $\square$

3. A complementary observation on superstability

The following very simple observation on the superstability of Eq. (1.1) complements Theorem 2.1.

Theorem 3.1. *Let* (H1) *be valid,* $p \in \mathbb{F}$, $A, k \in (0, \infty)$, $|p|^{2k} + |1-p|^{2k} < 1$, *and* $g : X \to Y$ *satisfy*

$$\|g(px + (1-p)y) + g((1-p)x + py) \ -g(x) - g(y)\| \leq A\|x\|^k \|y\|^k \quad (3.1)$$

for every $x, y \in X$. *Then* g *is a solution to* (1.1).

Proof. Note that (3.1) with $y = 0$ gives

$$g(x) = g(px) + g((1-p)x) - g(0), \qquad x \in X. \quad (3.2)$$

We show that, for every $x, y \in X$, $n \in \mathbb{N}_0$,

$$\begin{aligned} &\|g(px + (1-p)y) + \ g((1-p)x + py) - g(x) - g(y)\| \\ &\quad \leq A(|p|^{2k} + |1-p|^{2k})^n \|x\|^k \|y\|^k. \end{aligned} \quad (3.3)$$

Clearly the case $n = 0$ is just (3.1). Next, fix $m \in \mathbb{N}_0$ and assume that (3.3) holds for every $x, y \in X$ with $n = m$. Then, by (3.2),

$$\begin{aligned} &\|g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y)\| \\ &\quad = \big\|g(p(px + (1-p)y)) + g((1-p)(px + (1-p)y)) \\ &\qquad + g(p((1-p)x + py)) + g((1-p)((1-p)x + py)) \\ &\qquad - g(px) - g((1-p)x) - g(py) - g((1-p)y)\big\| \\ &\quad \leq A(|p|^{2k} + |1-p|^{2k})^m \|px\|^k \|py\|^k \\ &\qquad + A(|p|^{2k} + |1-p|^{2k})^m \|(1-p)x\|^k \|(1-p)y\|^k \\ &\quad = A(|p|^{2k} + |1-p|^{2k})^{(m+1)} \|x\|^k \|y\|^k \end{aligned}$$

for every $x, y \in X$.

Thus, by induction we have shown that (3.3) holds for every $x, y \in X$ and $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (3.3), we obtain that g is a solution to (1.1), because $|p|^k + |1-p|^k < 1$. □

References

[1] Brzdęk, J., Chudziak, J., Páles, Zs.: A fixed point approach to stability of functional equations. Nonlinear Anal. **74**, 6728–6732 (2011)

[2] Brzdęk, J., Ciepliński, K.: A fixed point approach to the stability of functional equations in non-Archimedean metric spaces. Nonlinear Anal. **74**, 6861–6867 (2011)

[3] Chudziak, J.: Approximate solutions of the Gołąb-Schinzel functional equation. J. Approx. Theory **136**, 21–25 (2005)
[4] Daróczy, Z., Lajkó, K., Lovas, R.L., Maksa, Gy., Páles, Zs.: Functional equations involving means. Acta Math. Hungar. **166**, 79–87 (2007)
[5] Daróczy, Z., Maksa, Gy., Páles, Zs.: Functional equations involving means and their Gauss composition. Proc. Am. Math. Soc. **134**, 521–530 (2006)
[6] Forti, G.-L.: Comments on the core of the direct method for proving Hyers–Ulam stability of functional equations. J. Math. Anal. Appl. **295**, 127–133 (2004)
[7] Forti, G.-L.: Elementary remarks on Ulam-Hyers stability of linear functional equations. J. Math. Anal. Appl. **328**, 109–118 (2007)
[8] Forti, G.-L., Sikorska, J.: Variations on the Drygas equation and its stability. Nonlinear Anal. **74**, 343–350 (2011)
[9] Gilányi, A., Páles, Zs.: On Dinghas-type derivatives and convex functions of higher order. Real Anal. Exchange **27**, 485–493 (2001/2002)
[10] Hyers, D.H., Isac, G., Rassias, Th.M.: Stability of Functional Equations in Several Variables. Birkhäuser, Boston (1998)
[11] Jabłoński, W.: Stability of homogeneity almost everywhere. Acta Math. Hungar. **117**, 219–229 (2007)
[12] Jabłoński, W., Reich, L.: Stability of the translation equation in rings of formal power series and partial extensibility of one-parameter groups of truncated formal power series. Sitzungsber. Abt. II **215**, 127–137 (2006) Österreichischen Akademie der Wissenschaften (2007)
[13] Jarczyk, W., Sablik, M.: Duplicating the cube and functional equations. Results Math. **26**, 324–335 (1994)
[14] Jung, S.-M.: Hyers–Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis. Springer Optimization and Its Applications, vol. 48. Springer, New York (2011)
[15] Lajkó, K.: On a functional equation of Alsina and García-Roig. Publ. Math. Debrecen **52**, 507–515 (1998)
[16] Maksa, Gy., Nikodem, K., Páles, Zs.: Results on t-Wright convexity. C. R. Math. Rep. Acad. Sci. Canada **13**, 274–278 (1991)
[17] Moszner, Z.: On the stability of functional equations. Aequationes Math. **77**, 33–88 (2009)
[18] Najati, A., Park, C.: Stability of homomorphisms and generalized derivations on Banach algebras. J. Inequal. Appl. **2009**, 1–12 (2009)
[19] Paneah, B.: A new approach to the stability of linear functional operators. Aequationes Math. **78**, 45–61 (2009)
[20] Sikorska, J.: On a direct method for proving the Hyers–Ulam stability of functional equations. J. Math. Anal. Appl. **372**, 99–109 (2010)
[21] Ulam, S.M.: Problems in Modern Mathematics. Science Editions. Wiley, New York (1964)
[22] Wright, E.M.: An inequality for convex functions. Am. Math. Monthly **61**, 620–622 (1954)

Janusz Brzdęk
Department of Mathematics
Pedagogical University
Podchorążych 2
30–084 Kraków
Poland
e-mail: jbrzdek@up.krakow.pl

Received: April 23, 2012
Revised: June 25, 2012