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On the Schröder equation and iterative sequences of C^r diffeomorphisms in $\mathbb{R}^N$ space

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Abstract. Let $U \subset \mathbb{R}^N$ be a neighbourhood of the origin and a function $F\colon U \to U$ be of class C^r, $r \geq 2$, $F(0) = 0$. Denote by F^n the n-th iterate of F and let $0 < |s_1| \leq \cdots \leq |s_N| < 1$, where $s_1, \ldots, s_N$ are the eigenvalues of $dF(0)$. Assume that the Schröder equation $\varphi(F(x)) = S\varphi(x)$, where $S := dF(0)$ has a C^2 solution φ such that $d\varphi(0) = id$. If $\frac{log|s_1|}{log|s_N|} < 2$ then the sequence $\{S^{-n}F^n(x)\}$ converges for every point x from the basin of attraction of F to a C^2 solution φ of (1). If $2 \leq \frac{log|s_1|}{log|s_N|}$ then this sequence can be diverging. In this case we give some sufficient conditions for the convergence and divergence of the sequence $\{S^{-n}F^n(x)\}$. Moreover, we show that if F is of class C^r and $r > \left[\frac{log|s_1|}{log|s_N|}\right] := p \geq 2$ then every C^r solution of the Schröder equation such that $d\varphi(0) = id$ is given by the formula

$$\varphi(x) = \lim_{n\to\infty} (S^{-n}F^n(x) + \sum_{k=2}^{p} S^{-n}L_k(F^n(x))),$$

where $L_k\colon \mathbb{R}^N \to \mathbb{R}^N$ are some homogeneous polynomials of degree k, which are determined by the differentials $d^{(j)}F(0)$ for $1 < j \leq p$.

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1. Introduction

Let U be an open set in the N dimensional real space. The n-th iterate of a self-map $F\colon U \to U$ is defined by $F^n(x) = F^{n-1}(F(x))$ and $F^0 = id$, the identity map. Assume that F is a C^r- diffeomorphism, $r \geq 2$, $0 \in U$ and 0 is the unique fixed point of F. Denote by $d^{(k)}F(0)$ the differential of F of k-th order at 0. The aim of the present paper is to investigate the convergence and divergence of the sequence $\{(dF(0))^{-n}F^n(x)\}$. The limit of this sequence, if it exists, is strictly connected with C^r solutions of the Schröder functional equation

$$\varphi(F(x)) = S\varphi(x), \quad x \in U, \tag{1}$$

where $\varphi : U \to \mathbb{R}^N$ and $S = dF(0)$, which is the fundamental equation of linearization of diffeomorphisms in the $\mathbb{R}^N$ space. The idea goes back to Poincaré [16] and was further developed by Sternberg [19], Hartman [8], Chaperon [3], Kuczma [11] and [13]. In a one dimensional real case $S \in \mathbb{R}$ and it is easy to show that if $0 < |S| < 1$ and Eq. (1) possesses a solution φ differentiable at 0 with $\varphi'(0) \neq 0$ then

$$\varphi(x) = \lim_{n\to\infty} \frac{F^n(x)}{S^n}$$

(see for e.g., Szekeres [20], Kuczma [11] and for analytic functions Koenigs [9]). This limit exists also with some weaker assumptions on F (see e.g., Seneta [17], Dubic [6]). Intuitively the existence of the limit $\lim_{n\to\infty} \frac{F^n(x)}{S^n} \in (0, \infty)$ may be interpreted as follows: the iterative sequence $\{F^n(x)\}$ tends to zero with the speed of convergence of the geometric sequence $\{S^n\}$. We generalize this idea on the $\mathbb{R}^N$ space. Another approach to the problem of the speed of convergence of the sequence $\{F^n(x)\}$ in a Banach space and its relation to the Schröder equation, where S is a real positive number, one can find in Walorski [21].

In this note we consider the following problem: does the above given formula but with matrix S still hold in the $\mathbb{R}^N$ space? That is: if the equality

$$\varphi(x) = \lim_{n\to\infty} S^{-n}F^n(x) = \lim_{n\to\infty} (dF(0))^{-n}F^n(x) \tag{2}$$

is true for C^r solutions of the Schröder equation? The answer is negative. The aim of this paper is to give some sufficient conditions for the truthfulness of formula (2) and also to give conditions which imply the falsehood of it. We also give a general formula expressing the diffeomorphic solution of Eq. (1).

A similar problem but in the $\mathbb{C}^N$ space was considered by Berteloot in [2]. He dealt with the mappings of the form $F = M + \sum_{p\geq k} H_p$, where H_p are homogeneous polynomials in $\mathbb{C}^N$ of degree p and M is an automorphism of $\mathbb{C}^N$. He showed that if $k \geq k_0$ for a specially chosen k_0 then the sequence $\{M^{-n}F^n\}$ converges to a locally biholomorphic solution φ of (1). Recall that the problem of holomorphic linearization in $\mathbb{C}^N$ was initiated by Poincaré and Dulac in [16] and [7]. Since then numerous papers on this subject have been published and we refer to [1] for more details, some discussions and further references. We shall concentrate on the shape of diffeomorphic solutions of (1) in the $\mathbb{R}^{\mathbb{N}}$ space. Our approach to the problem is different from those which are useful for analytic functions. To obtain a desired formula we apply the ideas of Kuczma from [13]. Moreover, we assume for a given function F as minimal regularity assumptions as possible, which assure our formula.

2. Auxiliary facts

Let $r \in \mathbb{N}$ and $r \geq 1$. Assume the following general hypotheses:

(i) $U \subset \mathbb{R}^N$ is a neighbourhood of the origin and $F\colon U \to U$ is a C^r diffeomorphism, $F(0) = 0$, $|s_1| \leq \cdots \leq |s_N| < 1$, where $s_1, s_2, \ldots, s_N$ are the characteristic roots of the matrix $S := dF(0)$.

(H_r) $s_1^{q_1} \cdot \ldots \cdot s_N^{q_N} \neq s_i$ for $i = 1, 2, \ldots, N$, $q_1, \ldots, q_N \in \mathbb{N}_0$, where $2 \leq \sum_{j=1}^N q_j \leq r$.

If F of class C^r with $F(0) = 0$ satisfies (H_r) we will say that F *has no resonances*.

Note that S is a nonsingular matrix and if φ is a solution of (1) in U then $\varphi(0) = 0$, since 1 is not an eigenvalue of S.

Suppose that the Schröder Eq. (1) has a C^r solution such that $d\varphi(0) = E$, where E is the identity matrix. Differentiating (1) k-times and setting $x = 0$ we get

$$d^{(k)}(\varphi \circ F - S\varphi)(0) = 0 \quad \text{for } k = 2, \ldots, r. \tag{3}$$

This means that the multidimensional matrices $d^{(k)}\varphi(0)$, $k = 2, 3, \ldots, r$ have to satisfy the system of Eq. (3).

Define

$$F_r(x) = Sx + \sum_{k=2}^{r} \frac{d^{(k)}F(0)}{k!}(x, x, \ldots, x)$$

and

$$\eta(x) := x + \sum_{k=2}^{r} \frac{d^{(k)}\varphi(0)}{k!}(x, x, \ldots, x).$$

Note that $d^{(k)}F_r(0) = d^{(k)}F(0)$ and $d^{(k)}\eta(0) = d^{(k)}\varphi(0)$. Hence we get the following

Remark 1. The system (3) is equivalent to the system of equations

$$d^{(k)}(\eta \circ F_r)(0) = Sd^{(k)}\eta(0), \quad k = 2, \ldots, r, \tag{4}$$

where η is a polynomial function of degree r and the unknown quantities are multidimensional matrices $d^{(k)}\eta(0)$.

The elements of $d^{(k)}\eta(0)$ are N^k-tuples of elements of $\mathbb{R}^N$ and we can write $d^{(k)}\eta(0) = \{v_{i_1 \ldots i_k}\}, i_1 \ldots i_k = 1 \ldots N$ and $\{v_i\} = E$. Moreover, $v_{i_1 \ldots i_k} \in \mathbb{R}^N$ satisfy the consistency conditions

$$\{v_{i_{l_1}\dots i_{l_k}}\} = \{v_{i_1\dots i_k}\}$$

for every permutation $l_1, \dots, l_k$ of the sequence $1, \dots, k$. We can write system (4) in the following form

$$\sum_{s=1}^{k} \sum_{j_1,\dots,j_s}^{N} \beta_{i_1\dots i_k}^{j_1\dots j_s} v_{j_1\dots j_s} = S v_{i_1\dots i_k}, \quad k = 2, \dots, r, \tag{5}$$

where $\beta_{i_1\dots i_k}^{j_1\dots j_s}$ are real numbers determined by the left hand side of equality (4) (for detailed evaluation see [13] p. 77).

Systems $\mathbf{v}_k = \{v_{i_1\dots i_k}\}$, $k = 1, \dots, r$ with $\mathbf{v}_1 = E$ fulfilling equation (4) (or in other symbols equation (5)) and the consistency condition will be called *admissible*.

Obviously the admissible system of multidimensional matrices $\mathbf{v}_k$, $k = 1, \dots, r$ determines uniquely the polynomial function η satisfying (4) such that $\mathbf{v}_k = d^{(k)}\eta(0)$, $k = 1, \dots, r$.

This observation allows us to introduce the following

Definition. A polynomial function $\eta_r \colon \mathbb{R}^N \to \mathbb{R}^N$ such that $\eta_r(x) = x + L(x)$, where $2 \le \deg L \le r$ or $L \equiv 0$ and η_r satisfies (4) is said to be *a formal solution of* (1) *of order* r.

Note that, in general, a solution of system (4) may exist or not, and even if it exists, then there is no uniqueness attached to the solution. The existence and the uniqueness of a solution guarantees the condition for the lack of resonance (see Sternberg [19], Smajdor [18], Hartman [8] and Abate [1]). We can note this fact as follow

Lemma 1. *If F fulfils* (*i*) *and the characteristic roots $s_1, s_2, \dots, s_N$ of matrix $S = dF(0)$ satisfy* (H_r)*, then there exists a unique polynomial function η_r : $\mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ of r-order satisfying system* (4)*.*

If Eq.(1) *has a C^r solution φ such that $\varphi(0) = 0$ and $d\varphi(0) = E$, then $d^{(k)}\varphi(0) = d^{(k)}\eta_r(0)$, $k = 2, \dots, r$.*

Note that every C^r-solution φ of Eq. (1) determines uniquely a formal solution of r-order. This solution has the following form:

$$\eta_r(x) = x + \sum_{k=2}^{r} L_k(x),$$

where $L_k(x) = A_k(x, x, \dots, x)$, $A_k \in L^k(\mathbb{R}^N, \mathbb{R}^N)$ are k-linear mappings and $A_k(h_1, \dots, h_k) = \frac{d^{(k)}\varphi(0)}{k!}(h_1, \dots, h_k)$ as well as $L_k(x) = \frac{d^{(k)}\varphi(0)}{k!}(x, \dots, x)$.

We begin our considerations recalling the Kuczma generalization of a well-known Sternberg and Hartman theorem on linearization (see [8,19]).

Denote by $C^r_\delta(U)$ the set of all functions $F : U \to \mathbb{R}^N$ of class C^r such that $F(0) = 0$ and

$$d^{(r)}F(x) = d^{(r)}F(0) + O(||x||^\delta), \quad ||x|| \to 0, \quad 0 \le \delta \le 1.$$

Proposition 1 (Kuczma [12]). *Suppose that F satisfies hypothesis (i) and $F \in C^r_\delta(U)$, $r \geq 1$ and $0 \leq \delta \leq 1$. If*

$$|s_N|^{r+\delta} < |s_1|,$$

then for every system of admissible multidimensional matrices $\mathbf{v}_k = \{v_{i_1 \dots i_K}\}$, $k = 1, \dots, r$ Eq. (1) has a unique solution $\varphi \colon V \to \mathbb{R}^N$ in a neighbourhood V of the origin, such that $d^{(k)}\varphi(0) = \mathbf{v}_k$, $k = 1, \dots, r$ and $\varphi \in C^r_\delta(V)$.

Let us note that if furthermore F has no resonance then Eq. (1) has a unique solution in the class $C^r_\delta(V)$ such that $d\varphi(0) = E$, where E is the identity matrix.

Definition. Every C^r solution φ of (1) such that $\varphi(0) = 0$ and $d\varphi(0) = E$ is said to be *a C^r Schröder function of F.*

Note that in the literature this name is used in various but similar meanings (see e.g., Dubuc [5,6]).

3. General results

We start with a theorem which gives a shape of C^r solutions of Eq. (1). This is a continuation and application of Kuczma's ideas from paper [13].

Theorem 1. *Let F satisfy (i), $r \geq 1$, $F \in C^r_\delta(U)$ for a $0 \leq \delta \leq 1$ and*

$$|s_N|^{r+\delta} < |s_1|.$$

If $r = 1$ then in a neighbourhood V of the origin there exists the limit

$$\varphi(x) := \lim_{n\to\infty} S^{-n}F^n(x).$$

If $r \geq 2$ and (1) has a C^r-solution $\gamma \neq 0$ then there exists the limit

$$\varphi(x) := \lim_{n\to\infty} \left(S^{-n}F^n(x) + \sum_{k=2}^{r} S^{-n}L_k(F^n(x))\right), \tag{6}$$

in a neighbourhood V of the origin, where $L_k(x) = \frac{d^{(k)}\gamma(0)}{k!}(x \dots x)$ are homogeneous polynomial functions of k-order. The convergence is uniform in V. Moreover, $\gamma = \varphi \in C^r_\delta(V)$.

Proof. To prove our theorem we recall briefly the idea of M. Kuczma's proof of Proposition 3 from the paper [13] (see also the version for $r \geq 2$ in [14] pp. 336–338).

Let $\varepsilon > 0$ be such that $(|s_N| + \varepsilon)^{r+\delta}(|s_1|^{-1} + \varepsilon) < 1$. By the Ostrowski lemma (see e.g. [14,15] p. 19 and [10], §4) there exists a nonsingular matrix A such that

$$||A^{-1}SA|| < |s_N| + \varepsilon, \quad ||A^{-1}S^{-1}A|| < |s_1|^{-1} + \varepsilon,$$

where $||\ ||$ denotes the Euclidean norm in the $\mathbb{R}^N$ space. Hence we get

$$||A^{-1}SA||^{r+\delta}||A^{-1}S^{-1}A|| < 1.$$

Now, after putting $F^*(x) = A^{-1}F(Ax)$ and $\varphi^*(x) = A^{-1}\varphi(Ax)$, Eq. (1) is equivalent to $\varphi^*(F^*(x)) = A^{-1}SA\varphi^*(x)$. Thus we may assume that

$$\|S\|^{r+\delta}\|S^{-1}\| < 1.$$

Let $\theta \in (||S||,\ ||S^{-1}||^{\frac{-1}{r+\delta}})$. Obviously

$$\theta^{r+\delta}||S^{-1}|| < 1 \quad \text{and} \quad ||S|| < \theta < 1. \tag{7}$$

Consequently we can find a neighbourhood $V := \{x \in \mathbb{R}^N : ||x|| \le a\},\ \ a > 0$ of the origin such that

$$0 < ||F(x)|| < \theta||x|| \quad \text{in } V\backslash\{0\} \quad \text{and} \quad ||dF(x)|| < \theta \quad \text{in } V. \tag{8}$$

Let γ be a C^r solution of (1) such that $d\gamma(0) = id$. Define the space (Ξ, ϱ)

$$\Xi := \{\sigma\colon V \to \mathbb{R}^N : \sigma \in C^r(V),\ \sigma(0) = 0,\ d\sigma(0) = id,$$
$$d^{(k)}\sigma(0) = d^{(k)}\gamma(0),\ k = 2, \ldots, r\},$$

for $r \ge 2$ and

$$\Xi := \{\sigma\colon V \to \mathbb{R}^N : \sigma \in C^r(V), \sigma(0) = 0,\ d\sigma(0) = id\}$$

for $r = 1$.

In Ξ we introduce the metric

$$\varrho(\sigma_1, \sigma_2) := \sup_{\|x\| \le b} \|x\|^{-\delta}\|d^{(r)}\sigma_1(x) - d^{(r)}\sigma_2(x)\|_r,$$

where the norm $\|\mathbf{v}\|_r$ is defined for multidimensional matrices in $\mathbb{R}^N$ $\mathbf{v} = \{v_{k_1\ldots k_r}\}$ by

$$\|\mathbf{v}\|_r = \left(\sum_{k_1=1}^{N} \cdots \sum_{k_r=1}^{N} \|v_{k_1\ldots k_r}\|^2\right)^{1/2}.$$

Using (7), (8) and repeating the same arguments as in [13] and [14] p. 337 we get that (Ξ, ϱ) is a complete metric space and the transformation $\mathcal{T}$ given by the formula

$$\mathcal{T}\sigma := S^{-1} \circ \sigma \circ F$$

maps space Ξ into itself and satisfies the Lipschitz condition with a constant $l < 1$. Thus, by the Banach theorem, there exists a unique solution $\varphi \in \Xi$ of (1) and this solution is given by the formula

$$\varphi = \lim_{n\to\infty} \mathcal{T}^n\sigma_0 = \lim_{n\to\infty} S^{-n} \circ \sigma_0 \circ F^n \qquad \text{for every} \quad \sigma_0 \in \Xi.$$

If $r = 1$ then $\sigma_0 = id \in \Xi$ and consequently formula (2) holds. For $r \geq 2$ $\sigma_0 = id + \sum_{k=2}^{r} \frac{d^{(k)}\gamma(0)}{n!} \in \Xi$, since $d^{(k)}\sigma(0) = d^{(k)}\gamma(0)$. Hence we get formula (6).

In general, $id \notin \Xi$ and even, for every $n \in \mathbb{N}$, $\mathcal{T}^n id = S^{-n}F^n \notin \Xi$. □

As a consequence of Theorem 1, Proposition 1 and Lemma 1 we get the following partial answer to our problem.

Theorem 2. *Let F of class C^2 satisfy hypothesis (i) and*

$$|s_N|^2 < |s_1| < 1.$$

Then in a neighbourhood of the origin equation (1) has a unique C^2 solution φ such that $\varphi(0) = 0$ and $d\varphi(0) = E$. It is given by the formula

$$\varphi(x) = \lim_{n\to\infty} S^{-n}F^n(x).$$

Proof. Note that the eigenvalues of $dF(0)$ satisfy hypothesis (H_2). In fact (H_2) reads as follows $s_i s_j \neq s_k$ for all $i, j, k = 1, \ldots, N$. Since $|s_1| \leq |s_i| \leq |s_N|$ we have $|s_i s_j| \leq |s_N|^2 < |s_1| \leq |s_k|$. By Lemma 1 and Proposition 1 the Schröder Eq. (1) has a unique C^2 solution φ such that $\varphi(0) = 0$ and $d\varphi(0) = E$. Since φ belongs to class C_1^1 it follows by Theorem 1, where $r = 1$ and $\delta = 1$, that φ is given by formula (2). □

In the case of holomorphic functions the similar thesis one can also deduce from the theorem of Berteloot mentioned in Sect.1 (see Th. 4.2 in [2]).

In the case $r \geq 3$ as a consequence of Theorem 1 we obtain the following

Theorem 3. *Let $r \geq 3$, hypothesis (i) be satisfied, $|s_N|^r < |s_1|$ and (1) have a formal solution*

$$\eta(x) = x + \sum_{k=2}^{r} L_k(x),$$

where $L_k(x)$ are polynomial functions of k-th order. Then there exists a unique C^r solution φ of (1) such that $d^{(k)}\varphi(0) = d^{(k)}\eta(0)$ for $k = 1, \ldots, r$ in a neighbourhood V of the origin and it is given by the formula

$$\varphi(x) := \lim_{n\to\infty} (S^{-n}F^n(x) + \sum_{k=2}^{r-1} S^{-n}L_k(F^n(x))). \tag{9}$$

Proof. It follows by Proposition 1 for $\delta = 0$ that Eq. (1) has a unique C^r-solution φ such that $d^{(k)}\varphi(0) = d^{(k)}\eta(0)$ for $k = 1, \ldots, r$. Since $\varphi \in C_1^{r-1}$ we get, by Theorem 1, that φ is given by formula (9). □

By Theorem 3 and Lemma 1 we get the following supplement to the Sternberg theorem

Corollary 1. *If $r \geq 3$, F satisfies hypotheses (i), (H_r) $[(H_{r-1})]$* and $|s_N|^r < |s_1|$ then Eq. (1) has a unique solution φ of class C^r $[C_1^{r-1}]$ such that $d\varphi(0) = E$ and φ is given by formula (9).

4. Particular results

If for $r \geq 3$ we assume hypothesis (H_r) then the determination of the N-dimensional polynomials $L_k(x_1, \ldots, x_N)$ is very strenuous. In this section we consider the problem when the part of formula (9) with factors L_k disappears and (9) has a simpler form (2). To this aim we shall deal with the asymptotic behavior of the sequence $\{S^{-n}F^n(x)\}$ in the case, where $r \geq 3$ and $|s_1| < |s_N|^2$.

Let us start with the following

Lemma 2. *Let φ be a C^1 solution of* (1) *such that $\varphi(0) = 0$, $d\varphi(0) = E$. If $\varphi(x) = (\varphi^1(x), \ldots, \varphi^N(x))$, then for every $1 \leq i \leq N$ there exists $x_i \in U$ such that $\varphi^i(x_i) \neq 0$.*

Proof. By the assumptions φ is invertible in a neighbourhood V of zero. Suppose that there exists i such that $\varphi^i(x) = 0$ for $x \in V$, i.e. $\varphi(x) = (\varphi^1(x), \ldots, 0, \ldots, \varphi^N(x))$. This function cannot be invertible since $\tilde{\varphi} = (\varphi^1, \ldots, \varphi^{i-1}, \varphi^{i+1}, \ldots, \varphi^N) \colon \mathbb{R}^N \to \mathbb{R}^{N-1}$ is not invertible because of reduction of the dimension. Hence there exists $x_1 \neq x_2$ for which $\tilde{\varphi}(x_1) = \tilde{\varphi}(x_2)$, so $\varphi(x_1) = \varphi(x_2)$, but this is a contradiction. $\square$

Remark 2. Let J be a Jordan form of matrix S and P be a nonsingular matrix such that $PSP^{-1} = J$. Then φ is a C^r Schröder function of F if and only if $\psi = P \circ \varphi \circ P^{-1}$ is a C^r Schröder function of $G := P \circ F \circ P^{-1}$.

Proof. Let $\varphi \circ F = S \circ \varphi$. Then $\varphi \circ F = P^{-1}JP \circ \varphi$ and

$$(P \circ \varphi \circ P^{-1}) \circ (P \circ F \circ P^{-1}) = J \circ (P \circ \varphi \circ P^{-1}),$$

so $\psi \circ G = J \circ \psi$.

In particular, if φ satisfies the assumptions of Theorem 3, then

$$\varphi(x) = P^{-1} \lim_{n\to\infty} (J^{-n}G^n(Px) + \sum_{k=2}^{r-1} J^{-n}M_k(J^n(Px))),$$

where $\nu(x) = x + \sum_{k=2}^{r-1} M_k(x)$ is a formal solution of the equation $\psi(G(x)) = J\psi(x)$ of $(r-1)$-order. Moreover, if φ is given by formula (9), then $M_k = P \circ L_k \circ P^{-1}$. Hence we may further assume that S has a Jordan form. $\square$

Assume that F satisfies the assumptions of Theorem 3 and the matrix $dF(0) = S$ is diagonalizable. Thus, in view of Remark 2, we may assume that S has the diagonal form, i.e.,

$$S = \begin{bmatrix} s_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_N \end{bmatrix} =: [s_1, \ldots, s_N],$$

where $0 < |s_1| \leq \cdots \leq |s_N| < 1$.

Let φ be a C^r solution of (1) such that $\varphi(0) = 0$ and $d\varphi(0) = E$. By (9) we have

$$\varphi(x) = \lim_{n\to\infty}[S^{-n}F^n(x) + \sum_{k=2}^{r-1} S^{-n}L_k(S^n(S^{-n}F^n(x)))]$$

in a neighbourhood of the origin V. Put

$$y_n(x) := S^{-n}F^n(x), \qquad y_n(x) =: (y_n^1(x), \dots, y_n^N(x)),$$

and

$$z_n(x) := \sum_{k=2}^{r-1} S^{-n}L_k(S^n y_n(x)), \qquad z_n(x) =: (z_n^1(x), \dots, z_n^N(x)).$$

Hence

$$\varphi(x) = \lim_{n\to\infty}[y_n(x) + \sum_{k=2}^{r-1} S^{-n}L_k(S^n y_n(x))]$$

and

$$\varphi^j(x) = \lim_{n\to\infty}[y_n^j(x) + \sum_{k=2}^{r-1}(S^{-n}L_k(S^n y_n(x)))^j], \quad \text{for } j = 1, \dots, N. \tag{10}$$

Write also

$$L_k(x) =: (L_k^1(x), \dots, L_k^N(x)).$$

We know that $L_k(x)$ is a homogenous polynomial of k-order. The jth-coordinate of $L_k(x)$ we can write in the following form

$$L_k^j(x) = \sum_{i_1,\dots,i_k \in \{1,2,\dots,N\}} a^j_{i_1,\dots,i_k} x^{i_1} \dots, x^{i_k}, \tag{11}$$

where $x = (x^1, \dots, x^N)$. Since

$$S^n y_n(x) = \begin{bmatrix} s_1^n y_n^1(x) \\ \vdots \\ s_N^n y_n^N(x) \end{bmatrix},$$

we have

$$\begin{aligned} L_k^j(S^n y_n(x)) &= \sum_{i_1,\dots,i_k \in \{1,2,\dots,N\}} a^j_{i_1,\dots,i_k}(s_{i_1}^n y_n^{i_1}(x)) \dots, (s_{i_k}^n y_n^{i_k}(x)) \\ &= \sum_{i_1,\dots,i_k \in \{1,2,\dots,N\}} a^j_{i_1,\dots,i_k}(s_{i_1} \dots s_{i_k})^n y_n^{i_1}(x) \dots y_n^{i_k}(x). \end{aligned}$$

On the other hand

$$S^{-n}L_k(S^n y_n(x)) = \begin{bmatrix} s_1^{-n} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_N^{-n} \end{bmatrix} \begin{bmatrix} L_k^1(S^n y_n(x)) \\ \vdots \\ L_k^N(S^n y_n(x)) \end{bmatrix} = \begin{bmatrix} s_1^{-n}L_k^1(S^n y_n(x)) \\ \vdots \\ s_N^{-n}L_k^N(S^n y_n(x)) \end{bmatrix}.$$

Hence we infer that the j-th coordinate of $z_n(x)$ is expressed by the formula

$$z_n^j(x) = \sum_{k=2}^{r-1} \sum_{i_1,\dots,i_k \in \{1,2,\dots,N\}} a^j_{i_1,\dots,i_k}(s_j^{-1}s_{i_1}\dots s_{i_k})^n y_n^{i_1}(x)\dots y_n^{i_k}(x). \quad (12)$$

By (10) we have

$$\varphi^j(x) = \lim_{n\to\infty} u_n^j(x), \qquad j = 1,\dots,N, \quad (13)$$

where

$$u_n^j(x) = y_n^j(x) + z_n^j(x). \quad (14)$$

The above mentioned observations let us to prove the following

Theorem 4. *Let $r \geq 3$ and hypothesis (i) be satisfied, φ be a C^r Schröder function of F, matrix S be diagonal, $|s_N|^r < |s_1|$ and $|s_1| \leq |s_N|^2 < |s_p|$ for a $p \geq 2$. If the sequence $\{S^{-n}F^n(x_0)\}$ is bounded, then for every $j \geq p$ the j-th coordinate of the sequence $\{S^{-n}F^n(x_0)\}$ converges to the j-th coordinate of the C^r- Schröder function φ at x_0, i.e.*

$$\lim_{n\to\infty}(S^{-n}F^n)^j(x_0) = \varphi^j(x_0), \qquad p \leq j.$$

Proof. Fix $j \geq p$. Note that

$$|s_j^{-1}s_{i_1}\dots s_{i_k}| < 1 \quad \text{for} \quad i_1,\dots,i_k \in \{1,\dots,N\},\ 2 \leq k \leq r-1.$$

In fact,

$$|s_j^{-1}s_{i_1}\dots s_{i_k}| \leq |s_j|^{-1}|s_N|^k \leq |s_j|^{-1}|s_N|^2 \leq |s_p|^{-1}|s_N|^2 < 1,$$

since $k \geq 2$ and $|s_p| \leq |s_j| \leq |s_N| < 1$. Thus we get that

$$\lim_{n\to\infty}(s_j^{-1}s_{i_1}\dots s_{i_k})^n = 0, \quad \text{for every } i_1,\dots,i_k \in \{1,\dots,N\},\ k \leq j.$$

By the assumption the sequences $\{y_n^{i_1}(x_0),\dots,y_n^{i_k}(x_0))\}$ are bounded, thus (11) implies that $\lim_{n\to\infty} z_n^j(x_0) = 0$. Consequently, by (12) and (13),

$$\varphi^j(x_0) = \lim_{n\to\infty} y_n^j(x_0).$$

□

Note that for $p = N$ the inequality $|s_N|^2 < |s_p|$ always holds. Hence we get

Remark 3. If in the previous theorem we assume only the inequality $|s_N|^r < |s_1|$ then the N-th coordinates of every bounded subsequence $\{S^{-n_\nu}F^{n_\nu}(x_0)\}$ converges to the N-th coordinate of the C^r Schröder function φ at x_0.

We have the following criterion of the convergence of the sequence $\{S^{-n}F^n(x)\}$.

Theorem 5. *Let $r \geq 3$ and hypotheses (i), (H_r) be satisfied, the matrix S be diagonal, $|s_N|^r < |s_1|$ and $|s_{p-1}| < |s_N|^2 < |s_p|$ for $p \geq 2$ and the sequence $\{S^{-n}F^n(x)\}$ be uniformly bounded in a set V. If $\frac{\partial^k \eta^j(0)}{\partial x_{i_1} \ldots \partial x_{i_k}} = 0$ for $j \leq p-1$, $2 \leq k < r$ and $i_1, \ldots, i_k \in \{1, \ldots, N\}$ such that $|s_{i_1} \ldots s_{i_k}| \geq |s_j|$, where $\eta = (\eta^1, \ldots, \eta^N)$ is the formal solution of (1), then the sequence $\{S^{-n}F^n(x)\}$ converges uniformly in V and*

$$\lim_{n\to\infty} (S^{-n}F^n)(x) = \varphi(x), \qquad x \in V$$

where φ is the C^r Schröder function of F.

Proof. Fix $j < p$ and define

$$A_j := \{(i_1, \ldots, i_k) \in \{1, \ldots, N\}^k;\ 1 < k < r,\ |s_{i_1} \ldots s_{i_k}| < |s_j|\}.$$

By (12) we have

$$z_n^j(x) = \sum_{k=2}^{r-1} \sum_{i_1,\ldots,i_k \in A_j} a^j_{i_1,\ldots,i_k} (s_j^{-1} s_{i_1} \ldots s_{i_k})^n y_n^{i_1}(x) \ldots y_n^{i_k}(x).$$

Since $|s_{i_1} \ldots s_{i_k}| < |s_j|$ for $(i_1, \ldots, i_k) \in A_j$ and the sequences $\{y_n^i(x)\}$ are bounded, the sequence $\{z_n^j(x)\}$ converges uniformly in V to zero and, consequently, by (13) and (12), the j-th coordinate of $\{S^{-n}F^n(x)\}$ converges to the j-th coordinate of the Schröder function. For the coordinates $j \geq p$ the assertion follows from Theorem 4. $\square$

In a particular case for $N = 2$ and $r = 3$ Theorem 5 has the following simple form

Corollary 2. *If $F: U \subset \mathbb{R}^2 \to \mathbb{R}^2$ satisfies (i) with $r = 3$, $|s_2|^3 < |s_1| < |s_2|^2$, $S = dF(0)$ is diagonal, $\frac{\partial^2 \eta^1(0)}{\partial x_2 \partial x_2} = 0$ and the sequence $S^{-n}F^n(x)$ is bounded, then formula (2) holds.*

Example 1. Define

$$F(x, y) := \left(\frac{s_1 x(1 + y)}{1 + s_2 y} + g(x, y), s_2 y \right),$$

where g is smooth, $g(0, 0) = 0$, $dg(0, 0) = 0$, $d^{(2)}g(0, 0) = 0$ and g is chosen such that the sequence $\{S^{-n}F^n(x, y)\}$ is bounded. It is easy to verify that the formal solution η of the second order is given by the formula $\eta(x, y) = (x + xy, y)$ and satisfies the assumptions of Corollary 2. In the particular case when $g(x, y) = 0$ it is easy to calculate that

$$S^{-n}F^n(x, y) = \left(\frac{x(1 + y)}{1 + s_2^n y}, y \right).$$

In this case the above sequence converges to $(x + xy, y)$.

Lemma 3. *If* $|s_1| \le \dots \le |s_N| < 1$, *then*

$$\max\{|s_{i_1} \dots s_{i_k}|,\ i_1, \dots, i_k \in \{1, \dots, N\},\ k = 2, \dots, r-1\} = |s_N|^2.$$

If $|s_{N-1}| < |s_N|$ *and* $|s_{i_1} \dots s_{i_k}| = |s_N|^2$, *then* $k = 2$ *and* $i_1 = i_2 = N$.

Proof. If $k \ge 3$, then $|s_{i_1} \dots s_{i_k}| \le |s_N|^k < |s_N|^2$. Let $k = 2$ and suppose that there exist $\bar{i_1}, \bar{i_2}$ such that $|s_{\bar{i_1}} s_{\bar{i_2}}| = |s_N|^2$. Then $|s_N|^2 \le |s_{\bar{i_1}}||s_N|$, hence $|s_N| \le |s_{\bar{i_1}}|$, so $|s_{\bar{i_1}}| = |s_N|$, and consequently also $|s_{\bar{i_2}}| = |s_N|$. Since $|s_{N-1}| < |s_N|$ we get $i_1 = i_2 = N$. □

Lemma 4. *If* $c_n := a_n^1 (q_1)^n + \dots + a_n^k (q_k)^n + b_n q^n$, $|q_i| < |q|$, $i = 1, \dots, k$, $|q| > 1$ *and the sequences* $\{c_n\}$ *and* $\{a_n^i\}$ *are bounded, then* $\lim_{n\to\infty} b_n = 0$.

Proof. The inequality $|c_n| < M$ implies that $|\frac{c_n}{q^n}| \le \frac{M}{q^n} \to 0$, so

$$a_n^1 \left(\frac{q_1}{q}\right)^n + \dots + a_n^k \left(\frac{q_k}{q}\right)^2 + b_n \to 0.$$

By the boundedness of $\{a_n^i\}$ we get the thesis.

Lemma 5. *Let* $r \ge 3$, *hypotheses* (i) *and* (H_r) *be satisfied and the matrix* S *be diagonal. If for every* x *in a neighbourhood of the origin the sequence* $\{S^{-n} F^n(x)\}$ *has a bounded subsequence and* $|s_{N-1}| < |s_N|$ *and* $|s_p| < |s_N|^2$ *then* $a_{NN}^1 = \dots = a_{N,N}^p = 0$, *where* $a_{k,l}^j$ *are the coefficients defined by* (11).

Proof. Let φ be a C^r Schröder function of F. By Lemma 2 there exists a point x_N such that $\varphi^N(x_N) \ne 0$. Let $\{n_\nu\}$ be a sequence such that the subsequence $\{S^{-n_\nu} F^{n_\nu}(x_N)\}$ is bounded. Let $j \le p$. By (13) and (14) we infer that the sequence $\{z_{n_\nu}^j(x_N)\}$ is bounded. The expression (12) may be written in the following form

$$\begin{aligned} &z_{n_\nu}^j(x_N) \\ &\quad = \sum_{k=3}^{r-1} \sum_{i_1,\dots,i_k \in \{1,2,\dots,N\}} a_{i_1,\dots,i_k}^j (s_j^{-1} s_{i_1} \dots s_{i_k})^{n_\nu} y_{n_\nu}^{i_1}(x_N) \dots y_{n_\nu}^{i_k}(x_N) \\ &\qquad + \sum_{i_1,i_2 \in \{1,2,\dots,N\},(i_1,i_2)\ne(N,N)} a_{i_1,i_2}^j (s_j^{-1} s_{i_1} s_{i_2})^{n_\nu} y_{n_\nu}^{i_1}(x_N) y_{n_\nu}^{i_2}(x_N) \\ &\qquad + a_{N,N}^j (s_j^{-1} s_N^2)^n (y_{n_\nu}^N(x_N))^2. \end{aligned} \tag{15}$$

Since $|s_i| \le |s_N| < 1$ we have

$$q_{\hat{i}} := |s^{-j} s_{i_1} \dots s_{i_k}| < |s^{-j} s_N^2| =: q$$

for $k \ge 3$, however, for $k = 2$ the above inequality is true except for the case $i_1 = i_2 = N$. Moreover,

$$q = |s_j^{-1} s_N^2| > 1 \quad \text{for } j \le p$$

and the sequences $\{a^j_{i_1,\ldots,i_k} y^{i_1}_{n_\nu}(x_N)\ldots y^{i_k}_{n_\nu}(x_N)\}$ are bounded. Applying Lemma 4 to the expression (15) we get

$$\lim_{n\to\infty} a^j_{N,N}(y^N_{n_\nu}(x_N))^2 = 0, \qquad j \le p.$$

On the other hand, by Remark 3, $\lim_{n\to\infty} y^N_{n_\nu}(x_N) = \varphi^N(x_N) \neq 0$, so $a^j_{N,N} = 0$, for $j \le p$. □

From the argumentation of the above proof and Remark 3 we get the following

Corollary 3. *Let $r \geq 3$ and (i) be satisfied, matrix S be diagonal and (1) have a formal solution of r-order. If $|s_{N-1}| < |s_N|$, $|s_p| < |s_N|^2$ for such a p that $a^p_{N,N} \neq 0$, then in every neighbourhood of zero there exists x_0 such that*

$$\lim_{n\to\infty} |(S^{-n}F^n)^N(x_0)| = \infty.$$

The following result shows that for $r \geq 3$ formula (6) in Theorem 1 cannot be replaced by (2).

Theorem 6. *Let $r \geq 3$ and hypotheses (i), (H_r) be satisfied, S be diagonal, $|s_N|^r < |s_1|$, $|s_{N-1}| < |s_N|$ and $\eta = (\eta^1,\ldots,\eta^N)$ be the formal solution of (1) of order 2. If $\frac{\partial^2\eta^p}{\partial x_N \partial x_N}(0) \neq 0$ for a p such that $|s_p| < |s_N|^2$, then in every neighbourhood of the origin there exists $x_0 \neq 0$ such that the sequence $\{S^{-n}F^n(x_0)\}$ is unbounded. More precisely, $\lim_{n\to\infty} |(S^{-n}F^n)^N(x_0)| = \infty$.*

Proof. The assertion is a consequence of Lemma 5, Corollary 3 and the equalities $d^{(2)}\varphi(0) = d^{(2)}\eta(0)$, $\frac{\partial^2\eta^p}{\partial x_N \partial x_N}(0) = a^p_{N,N}$, where φ is the C^r Schröder function of F. □

This theorem gives the answer to the main question of the paper. More precisely we have

Corollary 4. *Let $r \geq 3$ and F satisfy the assumptions of* Theorem 3*. Let $dF(0)$ be a diagonal matrix and $|s_{N-1}| < |s_N|$. If φ is a Schröder function of F and $\frac{\partial^2\varphi^p(0)}{\partial x_N \partial x_N} \neq 0$ for a p such that $|s_p| < |s_N|^2$, then formula (2) does not hold.*

We give an example of a function F satisfying the assumptions of Theorem 6 and Corollary 4.

Example 2. Let $r \geq 3$, $|s_1| \leq \cdots \leq |s_{N-1}| < |s_N| < 1$, $|s_N|^r < |s_1| < |s_N|^2$ and $\varphi \in C^r$ be a given function such that $\varphi(0) = 0$, $d\varphi(0) = E$ and $\frac{\partial^2\varphi^1(0)}{\partial x_N \partial x_N} \neq 0$. The mapping φ is invertible in a neighbourhood of the origin and we may define the following function

$$F(x) := \varphi^{-1}(S\varphi(x)),$$

where $S := [s_1 \dots, s_N]$ is a diagonal matrix. We have $dF(0) = [s_1 \dots, s_N]$, thus F satisfies the assumptions of Corollary 4. Consequently for this function formula (2) does not hold.

Note that every function $\varphi = (\varphi^1, \dots, \varphi^N)$, where

$$\varphi^j(x_1, \dots, x_N) = x_j + \sum_{i_1, i_2 \in \{1, \dots, N\}} a^j_{i_1, i_2} x_{i_1} x_{i_2}$$

and $a^1_{N,N} \neq 0$ satisfies the above mentioned conditions.

Example 3. We consider the case where $r = 3$, $N = 2$ and $p = 1$. Let $|s_2|^3 < |s_1| < |s_2|^2 < 1$. Define

$$F(x, y) = (s_1 x + a(s_1 - s_2^2) y^2, s_2 y), \qquad a \neq 0.$$

We have $dF(0, 0) = [s_1, s_2]$. Solving system (4) we get that the formal solution of order 2 of Eq. (1) is expressed by the formula

$$\eta(x, y) = (x + ay^2, y).$$

Obviously $\frac{\partial^2 \eta(0)}{\partial y \partial y} = (a, 0)$. Hence, by Corollary 3, $\lim_{n\to\infty} (S^{-n} F^n)^1(x) = \infty$.

Remark 4. *If* $F(x_1, \dots, x_N) = (f_1(x_1), \dots, f_N(x_N))$, $f \in C^r(-a, a)$, $f_i(0) = 0$, $f_i'(0) = s_i$, $0 < |s_i| < 1$, $i = 1, \dots, N$, *then Eq. (1) has a unique* C^r *solution* φ *which is given by (2).*

Proof. We have

$$S^{-n} F^n(x) = \begin{bmatrix} \frac{f_1^n(x_1)}{s_1^N} \\ \vdots \\ \frac{f_n^n(x_N)}{s_N^n} \end{bmatrix}.$$

It follows, by Proposition 1 in the one dimensional case, that there exists a $\delta > 0$ such that for every $1 \le i \le N$ there exists the limit $\lim_{n\to\infty} \frac{F_i^n(u)}{s_i^n} = \varphi^i(u)$ for $|u| < \delta$ and that φ^i are of class C^r. Hence we get (2). $\square$

Remark. Note that if the sequence $\{S^{-n} F^n(x)\}$ is convergent to $\varphi(x)$, then

$$F^n(x) \approx S^n \varphi(x),$$

which means that the sequence $\{F^n(x)\}$ has the same asymptotic behavior as the sequence $S^n \varphi(x)$. The matrices sequences S^n are well described in the literature (see, for e.g., Elaydi [4]). As an application of this fact we obtain the assertion that the solutions of the system of nonlinear difference equations $\hat{x}_{n+1} = F(\hat{x}_n)$ in $\mathbb{R}^N$ space can be approximated by the solutions of a linear system $\hat{y}_{n+1} = S\hat{y}_n$ with the initial condition $\hat{y}_0 = \varphi(\hat{x}_0)$.

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