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**Aequationes Mathematicae**

# On affine functions with respect to some means

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**Abstract.** The purpose of the present paper is to investigate the functional equation

$$M(f(x), g(y)) = h(N(x, y)),$$

where $f$, $g$ and $h$ are self-mappings of a real interval $I$ and $M,\ N\colon I^2 \to I$ are functions. In particular, we will show that under appropriate assumptions imposed on the functions $M,\ N$ the local boundedness of $f$ implies the continuity of $g$.

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## 1. Introduction: basic definitions and auxiliary lemmas

In this paper $I$ will always denote a non-degenerate interval contained in $\mathbb{R}$.

A function $M\colon I^2 \to I$ is called a *mean* on $I$ if

$$\min\{x, y\} \le M(x, y) \le \max\{x, y\}, \quad x, y \in I.$$

A mean $M$ is called a *strict mean* if

$$\min\{x, y\} < M(x, y) < \max\{x, y\}, \quad x, y \in I,\ x \neq y.$$

Now, let $M$ be a mean on $I$ and let $f\colon I \to I$. We say that $f$ is *affine with respect to M* (or shortly $M$-affine) if

$$f(M(x, y)) = M(f(x), f(y)), \quad x, y \in I.$$

Let us also introduce the following notation. For nonempty sets $X, Y, Z$, $u \in X$, $v \in Y$ and a function $F\colon X \times Y \to Z$, we define functions $F_u\colon Y \to Z$, $F^v\colon X \to Z$ by the formulas:

$$F_u(y) := F(u, y), \quad y \in Y;$$
$$F^v(x) := F(x, v), \quad x \in X.$$

The present paper is devoted to the following functional equation:

$$M(f(x), g(y)) = h(N(x, y)), \quad x, y \in I, \tag{1.1}$$

where $M$ and $N$ are given functions, $M,\ N\colon I^2 \to I$, whereas $f$, $g$ and $h$ are unknown functions, $f,\ g,\ h\colon I \to I$. Our results are analogous to the ones of Ng [1], who has investigated the functional equation

$$f(x) + g(y) = h(T(x, y)),$$

where the unknown functions $f$, $g$ and $h$ act on a connected or a locally connected topological space $X$, and $T\colon X^2 \to X$ is a given function. The methods we apply to (1.1) are modifications of those used by Ng in [1].

In particular, our results cover the case of $L$-affine functions, where $L$ is a logarithmic mean, i.e.

$$L(x, y) := \begin{cases} \frac{x-y}{\ln x - \ln y}, & x \neq y, \\ x, & x = y, \end{cases}$$

which was investigated by Matkowski in [2].

Let us start with a technical lemma.

**Lemma 1.** *Let $M\colon I^2 \to I$ be a function such that for all $u, v \in I$ the mapping $M_u$ is strictly increasing, and the mapping $M^v$ is increasing and continuous. Then, for all $s, S, v_0 \in I$ such that $s \leq S$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $u, \bar{u}, v \in I$ satisfying $M(u, v) = M(\bar{u}, v_0)$, $\bar{u} - u \leq \delta$ and $s \leq u \leq S$, we have $v - v_0 \leq \varepsilon$.*

*Proof.* Suppose (in search of a contradiction) that there exist $s, S, v_0 \in I$ such that $s \leq S$ and

$$\exists_{\varepsilon>0} \forall_{\delta>0} \exists_{u,\bar{u},v\in I} \quad (M(u, v) = M(\bar{u}, v_0), \bar{u} - u \leq \delta, s \leq u \leq S, v - v_0 > \varepsilon).$$

In particular, for every $n \in \mathbb{N}$ there exist $u_n, \bar{u}_n, v_n \in I$ fulfilling the conditions:

$$M(u_n, v_n) = M(\bar{u}_n, v_0), \bar{u}_n - u_n \leq \frac{1}{n}, s \leq u_n \leq S, v_n - v_0 > \varepsilon.$$

Observe that the sequence $(u_n)_{n\in\mathbb{N}}$ has to have a convergent subsequence, thus we may assume that the sequence is convergent to some $u \in I$.

If $u_n + \frac{1}{n} \in I$ for infinitely many $n \in \mathbb{N}$, then we deduce the following inequalities:

$$M(u_n, v_0 + \varepsilon) \leq M(u_n, v_n) = M(\bar{u}_n, v_0) \leq M\left(u_n + \frac{1}{n}, v_0\right).$$

On letting $n$ tend to infinity, we get that $M(u, v_0 + \varepsilon) \leq M(u, v_0)$, which contradicts the fact that $M_u$ is strictly increasing.

If $u_n + \frac{1}{n} \in I$ holds only for finitely many $n \in \mathbb{N}$, then $u = S \in I$ is the right endpoint of the interval $I$. Then we have:

$$M(u_n, v_0 + \varepsilon) \leq M(u_n, v_n) = M(\bar{u}_n, v_0) \leq M(u, v_0)$$

and letting $n$ tend to infinity, we get that $M(u, v_0 + \varepsilon) \leq M(u, v_0)$, which contradicts again the fact that $M_u$ is strictly increasing. $\square$

The proof of the next lemma is analogous and therefore we omit it here.

**Lemma 2.** *Let $M\colon I^2 \to I$ be a function such that for all $u, v \in I$ the mapping $M_u$ is strictly increasing, and the mapping $M^v$ is increasing and continuous. Then, for all $s, S, v_0 \in I$ such that $s \leq S$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $u, \bar{u}, v \in I$ satisfying $M(u, v) = M(\bar{u}, v_0)$, $u - \bar{u} \leq \delta$ and $s \leq \bar{u} \leq S$, we have $v_0 - v \leq \varepsilon$.*

In the proof of our main result we also need the following lemma:

**Lemma 3.** [3] *Let $X$ be a connected and locally connected space, $\theta\colon X \to \mathbb{R}$ a continuous function and $t_1, t_2 \in \theta(X)$, $t_1 < t_2$. Then there exists a connected subset $B \subseteq \theta^{-1}((t_1, t_2))$ such that $\theta(B) = (t_1, t_2)$.*

## 2. Main results

Now let us state and prove our main result:

**Theorem 1.** *Let $M, N\colon I^2 \to I$ be functions such that for all $u, v \in I$ the mapping $M_u$ is strictly increasing, the mapping $M^v$ is strictly increasing and continuous, and the mappings $N_u, N^v$ are continuous. Assume that a triple $(f, g, h)\colon I \to I^3$ is a solution of (1.1) and there exists a subinterval $I_0 \subseteq I$ such that $f$ is nonconstant on $I_0$ and $f(I_0) \subseteq [s, S]$ for some $s, S \in I$. Then $g$ is continuous.*

*Proof.* Choose $x_1, x_2 \in I_0$ such that $f(x_1) \neq f(x_2)$. From (1.1) and the strict monotonicity of $M^{g(y)}$ it follows that $N(x_1, y) \neq N(x_2, y)$ for all $y \in I$.

Fix $y_0 \in I$ arbitrarily. We will prove the continuity of $g$ at $y_0$ using Lemmas 1 and 2 for $v_0 = g(y_0)$. Let $\varepsilon > 0$; there exists a $\delta > 0$ such that the conditions of Lemmas 1 and 2 hold. Denote $l_1 := N(x_1, y_0)$ and $l_2 := N(x_2, y_0)$. We may assume that $l_1 < l_2$. According to Lemma 3 applied to $X = I_0$, $\theta = N^{y_0}$, $t_1 = l_1$, $t_2 = l_2$ there exists an interval $B \subseteq I_0$ such that $N(B, y_0) = (l_1, l_2)$. The function $f$ is bounded on $I_0$ and $B \subseteq I_0$, so $\sup\{f(x) : x \in B\} < +\infty$. Hence, $f(x_0) \geq f(x) - \delta$ for some $x_0 \in B$ and for all $x \in B$.

Let $V := \{y \in I : l_1 < N(x_0, y) < l_2\}$. It is easy to notice that $y_0 \in V$ and $V$ is open, since $N_{x_0}$ is continuous. For arbitrary $y \in V$ we have $N(x_0, y) \in (l_1, l_2) = N(B, y_0)$, so there exists an $x \in B$ such that $N(x_0, y) = N(x, y_0)$. By virtue of (1.1) we get $M(f(x_0), g(y)) = M(f(x), g(y_0))$. Apply Lemma 1 to $v_0 := g(y_0)$, $s$, $S$, $\bar{u} := f(x)$, $u := f(x_0)$, $v := g(y)$ to get that $g(y) - g(y_0) \leq \varepsilon$ for $y \in V$.

Since $N(x_0, y_0) \in (l_1, l_2) = N(B, y_0)$ we can find points $x_3, x_4 \in B$ fulfilling $N(x_3, y_0) < N(x_0, y_0) < N(x_4, y_0)$. Define the set $W$ by

$$W := \{y \in I : N(x_3, y) < N(x_0, y_0) < N(x_4, y)\}.$$

One can check that $W$ is a neighborhood of $y_0$. Moreover, for each $y \in W$ the set $N^y([\min\{x_3, x_4\}, \max\{x_3, x_4\}])$ is an interval which contains the points $N(x_3, y)$ and $N(x_4, y)$. Therefore, the point $N(x_0, y_0)$, which lies between them, belongs to that set. Thus, for every $y \in W$ there exists a point $x$ between $x_3$ and $x_4$, which fulfills $N(x_0, y_0) = N(x, y)$. But such a point $x$ must belong to $B$, since $B$ is connected and $x_3, x_4 \in B$. From (1.1) we get $M(f(x_0), g(y_0)) = M(f(x), g(y))$. From Lemma 2 it follows that $g(y_0) - g(y) < \varepsilon$ for $y \in W$.

Thus, $g$ is continuous at $y_0$. Since $y_0$ is arbitrarily chosen, $g$ is continuous. □

*Remark* 1. Let us note that if $I$ is compact, then the assumption of the boundedness of $f$ by elements from $I$ is obviously fulfilled. Similarly, if $f$ is continuous in some point $x_0$ and $f(x_0) \in intI$, then $f$ is bounded on some neighborhood of $x_0$ by elements from $I$.

*Proof.* If $I = [a, b]$ and $f : I \to I$, then the assumptions of the previous theorem are fulfilled for $s = a,\ S = b,\ I_0 = I$.

If $f$ is continuous in some point $x_0$ and $f(x_0) \in intI$, then there exist $s,\ S \in intI$ such that $s < f(x_0) < S$ and a non-degenerate interval $I_0 \subseteq I$ such that $x_0 \in I_0$ and $f(I_0) \subseteq [s, S]$. □

In the following corollary we will show that under auxiliary assumptions imposed on functions $M, N$ and if $f = g$, then it is enough to assume that $f$ is bounded from one side on some open subset of $I$ by an element from $I$.

**Corollary 1.** *Let $M, N : I^2 \to I$ be functions such that for all $u, v \in I$ the mapping $M_u$ is strictly increasing, the mapping $M^v$ is strictly increasing and continuous and the mappings $N_u, N^v$ are strictly increasing, continuous and $N(x, x) = x$ for every $x \in I$. If a pair $(f, h) : I \to I^2$ is a solution of the equation*

$$h(N(x, y)) = M(f(x), f(y)), \quad x, y \in I \tag{2.1}$$

*and there exist a subinterval $I_0 \subseteq I$ and a constant $S \in I$ (or $s \in I$) such that $f(I_0) \subseteq (-\infty, S]$ (or $f(I_0) \subseteq [s, +\infty)$, respectively) and $f$ is non-constant on $I_0$ and $M^S$ is onto $I$, then $f$ is continuous.*

*Proof.* We show that the function $f$ fulfilling the assumptions of the corollary also fulfills the assumptions of Theorem 1.

Let $f$ be bounded from above on an interval $I_0 = [a, b] \subseteq I$ by a constant $S \in I$ and let $x_0 := \frac{1}{2}(a + b)$. We will prove that there exists a point $x_1 \in (a, x_0)$ such that for every $x \in (x_1, x_0)$ there exists $y \in (a, b)$ such that $N(x, y) = x_0$. First, we show that there exists a pair $(x_1, y_1) \in (a, x_0) \times (a, b)$

such that $N(x_1, y_1) = x_0$. If there were no such pair, then we would have $N(x, (a, b)) \subseteq (a, x_0)$ for each $x \in (a, x_0)$. We infer that for every $\hat{x} \in (x_0, b)$

$$x_0 = N(x_0, x_0) < N(x_0, \hat{x}) = \lim_{x \to x_0^-} N(x, \hat{x}) \le x_0,$$

which is impossible. Let $(x_1, y_1) \in (a, x_0) \times (a, b)$ be a pair for which $N(x_1, y_1) = x_0$. For every $x \in (x_1, x_0)$ we get

$$N(x, x_0) < N(x_0, x_0) = x_0, \quad N(x, y_1) > N(x_1, y_1) = x_0.$$

It means that the interval $N(x, (a, b))$ contains a point greater than $x_0$ and a point smaller than $x_0$. Thus, there exists $y \in (a, b)$ such that $N(x, y) = x_0$.

Now let $x \in (x_1, x_0)$. We may find a $y \in (a, b)$ for which $N(x, y) = x_0$. Use Eq. (2.1) for the pair $(x, y)$ to get

$$h(x_0) = h(N(x, y)) = M(f(x), f(y)) \le M(f(x), S).$$

Since $h(x_0) \in I$ and $M^S$ is a bijection from $I$ onto $I$, there exists exactly one $s \in I$ such that $M^S(s) = h(x_0)$. Observe that $s \le f(x)$ for every $x \in (x_1, x_0)$. Thus, $f$ and $g = f$ satisfy the assumptions of Theorem 1.

The proof when $f(I_0) \subseteq (s, \infty)$ follows in a similar fashion. □

On imposing stronger assumptions upon $N$ we may eliminate the assumption of non-constancy of $f$ on every interval.

**Theorem 2.** *Let $M, N\colon I^2 \to I$ be mappings such that for all $u, v \in I$ the mapping $M_u$ is strictly increasing, the mapping $M^v$ is strictly increasing and continuous and the functions $N_u, N^v$ are continuous. Moreover, let $N$ be a symmetric strict mean. Assume that a triple $(f, g, h)\colon I \to I^3$ is a solution of (1.1) and that there exist a subinterval $I_0 \subset I$ and $s, S \in I$ such that $f(I_0) \subset [s, S]$. Then $g$ is continuous.*

*Proof.* Due to Theorem 1 it is enough to consider the case when $f$ is constant on some non-degenerate subinterval $P \subseteq I$. We prove that in this case $g$ is constant on the whole domain.

Let $c :\equiv f|_P$; we have

$$\begin{aligned} M(c, g(x)) &= M(f(y), g(x)) = h(N(y, x)) = h(N(x, y)) \\ &= M(f(x), g(y)) = M(c, g(y)), \quad x, y \in P. \end{aligned}$$

Since the function $M_c$ is strictly monotone, $g$ is constant on $P$, say $g|_P \equiv c_1$ for some $c_1 \in I$. Thus, for all $x, y \in P$ we get

$$h(N(x, y)) = M(f(x), g(y)) = M(c, c_1),$$

which means that $h|_{N(P \times P)}$ is constant and equal to $c_2 := M(c, c_1)$. On the other hand, $N$ is a mean, thus $N(P \times P) = P$. Thus, we have shown that $h|_P \equiv c_2$.

Now we will verify that $f$ is constant on int$I$. Assume that $P$ is the maximal interval on which $f$ is constant and equals $c$ ( if $Q$ is any interval such that $P \subseteq Q$ and $f|_Q \equiv c$ then $P = Q$ ); we will show that int$P$ = int$I$. Let $\alpha, \beta \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ be chosen in such a way that int$I = (\alpha, \beta)$. Suppose that int$P \neq$ int$I$. Then the left end-point of $P$ is a real number $a > \alpha$ or the right end-point of $P$ is a real number $b < \beta$. Without loss of generality we may assume that $b < \beta$. Choose $d$ so that $[d, b) \subseteq P$. We have $(b, b + \frac{1}{n}) \subseteq I$ for sufficiently large $n \in \mathbb{N}$. For such $n$ we may choose $y_n \in (b, b + \frac{1}{n})$ such that $f(y_n) \neq c$. The mean $N$ is strict, so $d < N(d, b) < b$. Moreover, we have $N(d, u) \in (d, b)$ for all points $u$ taken from some neighbourhood of $b$ because the function $N_d$ is continuous. Thus, $N(d, y_k) \in [d, b)$ for some $k\ in \mathbb{N}$. On the other hand,

$$M(f(y_k), c_1) = M(f(y_k), g(d)) = h(N(y_k, d)) = c_2.$$

Thus, $f(y_k) = c$, because $M(c, c_1) = c_2$ and the function $M^{c_1}$ is strictly increasing. Thus, we get a contradiction.

Obviously, $g|_{\text{int}I} \equiv c_1$ and $h|_{\text{int}I} \equiv c_2$.

To complete the proof it remains to show that if any of the end-points of $I$ belongs to $I$, then in this edge $g$ is also equal to $c_1$. For example, if $\beta \in I$, then from Eq. (2.1) applied to $x \in (\alpha, \beta)$ and $y = \beta$ we get

$$c_2 = h(N(x, \beta)) = M(f(x), g(\beta)) = M(c, g(\beta)),$$

thus, $g(\beta) = c_1$. If $\alpha \in I$, we proceed in a similar way. □

In what follows we apply the just proved theorems to $M-$affine functions. From results of Ng [1] it follows that an $M$-affine function defined on $I$ and bounded from both sides has to be continuous. From the previous results it follows that two-sided boundedness of $f$ may be weakened to one-side boundedness.

**Corollary 2.** *If $M$ is a symmetric, strict mean such that the functions $M_u, M^v$ are continuous and strictly increasing for every $u, v \in I$ and $f$ is an $M-$affine function, then the local boundedness of $f$ from one side by an element from $I$ implies its continuity.*

In particular, setting the logarithmic mean for $M$ we obtain a slightly more general result than those of Matkowski [2, Lemma 2].

**Corollary 3.** *Assume that $I$ is contained in $(0, +\infty)$ and $L$ is the logarithmic mean. If $f : I \to I$ is $L$-affine, bounded at a point from one side by an element from $I$ then $f$ is continuous.*

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