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Existence of Solution for a Katugampola Fractional Differential Equation Using Coincidence Degree Theory

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Abstract. In this paper, the authors study the existence of positive solutions to the fractional boundary value problem at resonance

$$-(D_{a+}^{\alpha,\rho}x)(t) = f(t, x(t), D_{a+}^{\alpha-1,\rho}x(t)), \quad t \in (a, b),$$
$$x(a) = 0, \quad x(b) = \int_a^b x(t)\mathrm{d}A(t),$$

where $1 < \alpha \leq 2$, and $D_{a+}^{\alpha,\rho}$ is a Katugampola fractional derivative, which generalizes the Riemann–Liouville and Hadamard fractional derivatives, and $\int_a^b x(t)\mathrm{d}A(t)$ denotes a Riemann–Stieltjes integral of x with respect to A, where A is a function of bounded variation. Coincidence degree theory is applied to obtain existence results. This appears to be the first work in the literature to deal with a resonant fractional differential equation with a Katugampola fractional derivative. Examples are given to illustrate the application of their results.

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1. Introduction

In this article, we consider the fractional boundary value problem consisting of the equation

$$-(D_{a+}^{\alpha,\rho}x)(t) = f(t, x(t), D_{a+}^{\alpha-1,\rho}x(t)), \quad t \in (a, b) \tag{1}$$

together with the boundary conditions

$$x(a) = 0, \quad x(b) = \int_a^b x(t)\mathrm{d}A(t), \tag{2}$$

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where $1 < \alpha \le 2$ and $D^{\alpha,\rho}_{a+}$ is a Katugampola fractional derivative. This type of fractional derivative generalizes the Riemann–Liouville and Hadamard fractional derivatives (see [19,20]). Here, $\int_a^b x(t)dA(t)$ denotes the Riemann–Stieltjes integral of x with respect to A, where A is a function of bounded variation. We note that the problem (1)–(2) is at resonance in the sense that the corresponding linear homogeneous equation $-(D^{\alpha,\rho}_{a+}x)(t) = 0$, $t \in [a, b]$, has nontrivial solutions with the boundary condition (2).

During the last few years, many researchers have investigated fractional differential equations with various definitions of fractional derivatives and integrals using different techniques; for example, we can see some recent work with Riemann–Liouville derivatives [15,16,29], Caputo derivatives [2,12,13, 31], Hadamard derivatives [5,18], Caputo–Hadamard derivatives [3,10], and ψ-fractional operators [4,22]. However, fractional differential equations with Katugampola derivatives are less studied in the literature, and only recently has attracted researchers to study such problems.

Some recent works on Katugampola fractional differential equations that has motivated us to study the boundary value problem (1)–(2) include the following. In [25], Łupinska and Odzijewicz obtained a Lyapunov-type inequality for the Katugampola fractional problem

$$\begin{cases} D^{\alpha,\rho}_{a+}x(t) + g(t)x(t) = 0, \\ x(a) = x(b) = 0. \end{cases}$$

In [8], Basti et al. used the Guo-Krasnosel'skii and Banach fixed point theorems to study the existence and uniqueness of solutions to the nonlinear Katugampola fractional boundary value problem

$$\begin{cases} D^{\alpha,\rho}_{a+}x(t) + \beta f(t, u(t)) = 0, \quad 1 < \alpha \le 2, \\ x(0) = x(T) = 0, \end{cases}$$

where $\beta \in \mathbb{R}$ and $f : [0, T] \times [0, \infty) \to [h, \infty)$ is a continuous function and h and T are finite positive constants. In another work, Łupiska and Schmeidel [26], obtained a Lyapunov-type inequality and conditions for existence and non-existence of solutions to

$$\begin{cases} D^{\alpha,\rho}_{a+}x(t) + g(t)x(t) = 0, \\ x(a) = D^{\alpha,\rho}_{a+}x(b) = 0. \end{cases}$$

In [6,9,23,24], the authors studied various nonlinear Katugampola fractional differential equations. Moreover, using coincidence degree theory, the existence of solutions to fractional differential equations at resonance with various kinds of fractional derivatives have been studied by a number of authors, for example, see [7,16,17,27,32]. As far as we can determine, there has been no work on Katugampola fractional equations at resonance and this explains our motivation to investigate the problem (1)–(2). We believe that the present work will be an important contribution to the literature on fractional equations and resonance problems.

2. Preliminaries

We begin with some concepts needed to analyze our problem.

Definition 2.1. ([19,25]) Let $\alpha > 0$, $\rho > 0$, $-\infty < a < b \leq \infty$, $p \geq 1$, and $f \in L^p(a,b)$. The operators

$$I_{a+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau)\mathrm{d}\tau$$

and

$$I_{b-}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau)\mathrm{d}\tau,$$

for $t \in (a,b)$, are called the left and right Katugampola integrals of fractional order α, respectively.

Definition 2.2. ([20,25]) Let $\alpha > 0$, $\rho > 0$, $n = [\alpha] + 1$, $0 < a < t < b \leq \infty$, $p \geq 1$, and $f \in L^p(a,b)$. The operators

$$D_{a+}^{\alpha,\rho} f(t) = \left(t^{1-\rho}\frac{d}{\mathrm{d}t}\right)^n I_{a+}^{n-\alpha,\rho} f(t)$$

and

$$D_{b-}^{\alpha,\rho} f(t) = \left(-t^{1-\rho}\frac{d}{\mathrm{d}t}\right)^n I_{b-}^{n-\alpha,\rho} f(t),$$

for $t \in (a,b)$, are called the left and right Katugampola derivatives of fractional order α, respectively.

The Katugampola derivative can be viewed as generalizing two other fractional operators by introducing a new parameter $\rho > 0$ into the definition. In fact, if we take $\rho = 1$, we have the Riemann–Liouville fractional derivative

$$D_{a+}^{\alpha,1} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{\mathrm{d}t}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau.$$

On the other hand, while the Katugampola derivative is only defined for $\rho > 0$, if we formally let $\rho = 0$ in the expression for the Katugampola derivative, it agrees with the Hadamard fractional derivative

$$D_{a+}^{\alpha,0} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{\mathrm{d}t}\right)^n \int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau}.$$

Next, we give some basic lemmas needed in our study.

Lemma 2.3. ([26]) *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\rho > 0$, and $f \in L[a,b]$. Then*

$$I_{a+}^{\alpha,\rho} D_{a+}^{\alpha,\rho} f(t) = f(t) + \sum_{i=0}^{n-1} c_i \left(\frac{t^\rho - a^\rho}{\rho}\right)^{i-n+\alpha},$$

where c_i, $i = 0, 1, \ldots, n-1$, are real constants.

Lemma 2.4. ([25, Proposition 1]) *Let $\alpha > 0$, $\rho > 0$, $a > 0$, and $\lambda > \alpha - 1$. Then*

$$D_{a+}^{\alpha,\rho}\left(\frac{t^\rho - a^\rho}{\rho}\right)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\lambda-\alpha}.$$

Lemma 2.5. ([25, Theorem 1]) *Let $0 < a < b < \infty$, $1 < \alpha \leq 2$, and $h : [a,b] \to \mathbb{R}$ be a continuous function. Then the unique solution of the problem*

$$\begin{cases} (D_{a+}^{\alpha,\rho}x)(t) + h(t) = 0, \\ x(a) = 0, \quad x(b) = 0, \end{cases} \tag{3}$$

is

$$x(t) = \int_a^b G(t,s)h(s)ds, \tag{4}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\begin{cases} \frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}}\left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} - \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}, & a \leq s \leq t \leq b, \\ \frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}}\left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1}, & a \leq t \leq s \leq b. \end{cases} \tag{5}$$

Lemma 2.6. ([25, Theorem 4]) *The Green's function given in (5) has the following properties:*

1) $G(t,s) \geq 0$ *for* $t \in [a,b]$, $s \in [a,b]$,
2) $\max_{t\in[a,b]} G(t,s) \leq \frac{\max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{4\rho}\right)^{\alpha-1}$.

To apply coincidence degree theory (See Theorem 2.7, below), we provide some basic definitions and related properties. Let X and Y be real Banach spaces and $L : dom(L) \subset X \to Y$ be a Fredholm operator of index zero (i.e., $dim\,(Ker(L)) - codim\,(Im(L)) = 0$). Let $P : X \to X$ and $Q : Y \to Y$ be two continuous projectors such that $Im(P) = Ker(L)$, $Ker(Q) = Im(L)$, $X = Ker(L) \oplus Ker(P)$, and $Y = Im(L) \oplus Im(Q)$. Then the inverse operator of $L|_{dom(L)\cap Ker(P)} : dom(L) \cap Ker(P) \to Im(L)$ is known to exist and we denote it by K_p. If we take Ω to be a bounded open subset of X such that $dom(L) \cap \Omega \neq 0$, then the mapping $N : X \to Y$ is said to be *L-compact* if $QN(\overline{\Omega})$ is bounded and the mapping $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. That the equation $Lx = Nx$ is solvable can be seen from [28, Theorem IV.13].

Theorem 2.7. ([28, Theorem 2.4]) *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume the following conditions are satisfied:*

1) $Lx \neq \lambda Nx$ *for every* $(x,\lambda) \in [(dom(L)\backslash Ker(L)) \cap \partial\Omega] \times (0,1)$;
2) $Nx \notin Im(L)$ *for every* $x \in Ker(L) \cap \partial\Omega$;
3) $deg(QN|_{KerL}, KerL \cap \Omega, 0) \neq 0$, *where* $Q : Y \to Y$ *is a projector as above with* $Im(L) = Ker(Q)$.

Then the equation $Lx = Nx$ has at least one solution in $dom(L) \cap \overline{\Omega}$.

In this article, we use the classical Banach space $Y = C[a,b]$ with the norm $\|u\|_\infty = \max_{t\in[a,b]} |u(t)|$ and the Banach space

$$X = \{u : [a,b] \to \mathbb{R} \mid u,\ D_{a+}^{\alpha-1,\rho} u \in C[a,b]\},$$

with the norm $||u||_X = \max\{||u||_\infty, ||D_{a+}^{\alpha,\rho} u||_\infty\}$.

Let us define $L : dom(L) \subset X \to Y$ and $N : X \to Y$ by

$$(Lx)(t) = -(D_{a+}^{\alpha,\rho} x)(t)$$

and

$$(Nx)(t) = f(t, x(t), D_{a+}^{\alpha-1,\rho} x(t))$$

for $t \in [a,b]$, where

$$dom(L) = \left\{ x \in X \mid -D_{a+}^{\alpha,\rho} x \in Y,\ x(a) = 0,\ x(b) = \int_a^b x(t)\mathrm{d}A(t) \right\}.$$

Then the boundary value problem (1)–(2) becomes

$$(Lx)(t) = (Nx)(t), \quad x \in dom(L).$$

To apply Theorem 2.7 in the proofs of the main results in the present paper, we define linear continuous projectors $P : X \to X$ and $Q : Y \to Y$ by

$$(Px)(t) = x(b) \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\rho-1}, \tag{6}$$

and

$$(Qy)(t) = \frac{\alpha}{\frac{(b^\rho - a^\rho)^\alpha}{\rho} - \int_a^b \frac{(t^\rho - a^\rho)^\alpha}{\rho} \mathrm{d}A(t)} \times \left[\int_a^b \frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} y(s)\mathrm{d}s - \int_a^b \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} y(s)\mathrm{d}s\mathrm{d}A(t) \right], \tag{7}$$

and a generalized inverse operator $K_p : Im(L) \to dom(L) \cap Ker(P)$ of L by

$$\begin{aligned}(K_p y)(t) &= \int_a^b G(t,s)y(s)\mathrm{d}s \\ &= \frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} \int_a^b \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-1} y(s)\mathrm{d}s - \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} y(s)\mathrm{d}s,\end{aligned} \tag{8}$$

where $G(t,s)$ is given in (5).

We assume that the following conditions hold throughout the remainder of this paper:

(A1) $\int_a^b \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \mathrm{d}A(t) = \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha-1}$ and $\int_a^b \frac{(t^\rho - a^\rho)^\alpha}{\rho} \mathrm{d}A(t) \neq \frac{(b^\rho - a^\rho)^\alpha}{\rho}$;

(A2) $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodry conditions, that is, $f(\cdot, u, v)$ is measurable for each fixed $(u, v) \in \mathbb{R} \times \mathbb{R}$, $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in [a, b]$, and for each $r > 0$ there exists $\phi_r \in L^\infty[a, b]$ such that $|f(t, u, v)| \leq \phi_r(t)$ for all $|u|$, $|v| \leq r$ and $t \in [a, b]$.

3. Main Results

We set

$$\Delta = \max\left\{ \frac{\max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)} \left(\frac{b^\rho - a^\rho}{4\rho}\right)^{\alpha-1}, \frac{(b^\rho - a^\rho)^\alpha}{\alpha\rho} \right\} \tag{9}$$

and will make use of the following conditions to prove our results.

(A3) There exists μ, σ, $\omega \in C[a, b]$ such that for all u, $v \in \mathbb{R}$ and $t \in [a, b]$,

$$|f(t, u, v)| \leq \mu(t) + \sigma(t)|u| + \omega(t)|v|,$$

with

$$\|\sigma\| + \|\omega\| < \frac{1}{\Psi},$$

where $\|\sigma\| = \|\sigma\|_\infty = \max_{a \leq t \leq b} |\sigma(t)|$, $\|\omega\| = \|\omega\|_\infty = \max_{a \leq t \leq b} |\omega(t)|$, and $\Psi = \Gamma(\alpha)\Delta + 1 + \frac{(b^\rho - a^\rho)^\alpha}{\alpha\rho}$.

(A4) There exists a constant $M > 0$ such that, if $|D_{a+}^{\alpha-1} x(t)| > M$ for all $t \in [a, b]$, then $QNx \neq 0$.

(A5) There exists a constant $B > 0$ such that either

$$cQN\left(c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right) < 0$$

or

$$cQN\left(c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right) > 0$$

for $c \in \mathbb{R}$ with $|c| > B$.

We next prove some lemmas that will facilitate the proof of our main result.

Lemma 3.1. *$L : dom(L) \subset X \to Y$ is a Fredholm operator of index zero.*

Proof. By Lemma 2.3, since $Lx = 0$, we have

$$x(t) = c_1 \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} + c_2 \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-2},$$

and using the first condition in (2) gives $c_2 = 0$. Hence,

$$Ker(L) = \left\{ c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1} : c \in \mathbb{R} \right\}.$$

Also,

$$Im(L)=\left\{y\in Y:\int_a^b \frac{s^{\rho-1}}{(b^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s-\int_a^b\int_a^t \frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s\right\}.$$

Let $x\in dom(L)$ and $Lx=y$. Then by Lemma 2.3,

$$x(t)=c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s.$$

Moreover,

$$x(b)=c_1\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha-1}-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b \frac{s^{\rho-1}}{(b^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s,$$

and

$$\begin{aligned}\int_a^b x(t)\mathrm{d}A(t) &= c_1\int_a^b\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}\mathrm{d}A(t)-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\\ &\quad\int_a^b\int_a^t \frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s\mathrm{d}A(t).\end{aligned}$$

Since $x(b)=\int_a^b x(t)\mathrm{d}A(t)$, we have

$$\int_a^b \frac{s^{\rho-1}}{(b^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s=\int_a^b\int_a^t \frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s\mathrm{d}A(t).$$

On the other hand, if $y\in Y$, then

$$\int_a^b \frac{s^{\rho-1}}{(b^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s=\int_a^b\int_a^t \frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s\mathrm{d}A(t).$$

If

$$\begin{aligned}x(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}\int_a^b \frac{s^{\rho-1}}{(b^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s\\ &\quad-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s,\end{aligned}$$

then $Lx=y$,

$$\begin{aligned}x(b) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\alpha-1}\int_a^b \frac{s^{\rho-1}}{(b^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s\\ &\quad-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b \frac{s^{\rho-1}}{(b^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\int_a^b x(t)\mathrm{d}A(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}\mathrm{d}A(t)\int_a^b \frac{s^{\rho-1}}{(b^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s\\ &\quad-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\int_a^t \frac{s^{\rho-1}}{(t^\rho-s^\rho)^{1-\alpha}}y(s)\mathrm{d}s\mathrm{d}A(t).\end{aligned}$$

Thus, $x \in dom(L)$ implies that $y \in Im(L)$ and $Lx = y$. Hence,

$$Im(L) = \left\{ y \in Y : \int_a^b \frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} y(s)\mathrm{d}s - \int_a^b \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} y(s)\mathrm{d}s = 0 \right\}.$$

Consequently, $dim\ Ker(L) = 1$ and $Im(L)$ is closed.

From (6), we see that P is linear and $(P^2x)(t) = (Px)(t)$, which means that P is a projection operator. Also, $Ker(P) = \{x \in X \mid x(b) = 0\}$ and $Im(P) = Ker(L)$. For any $x \in X$, with $x = (x - Px) + Px$, we have $X = Ker(P) \oplus Ker(L)$. It is easy show that $Ker(L) \cap Ker(P) = \{0\}$, which implies $X = Ker(P) \oplus Ker(L)$. It is not difficult to see that $(Q^2y)(t) = (Qy)(t)$ (see page 12025 in [16] for a similar argument), so Q is a projection operator. Moreover, $Ker(Q) = Im(L)$.

Next, for any $y \in Y$, setting $y_1 = y - Qy$, we have $(Qy_1)(t) = Q(y - Q(y))(t) = Qy(t) - Q^2y(t) = 0$. Hence, $y_1 \in Im(L)$ and $Y = Im(L) + Im(Q)$. Moreover, it is easy to verify that $Im(Q) \cap Im(L) = \{0\}$. Consequently, $Y = Im(L) \oplus Im(Q)$. Since $Im(L)$ is a closed subspace of Y and $dim\,(Ker(L)) = codim\,(Im(L)) = 1$, L is a Fredholm operator of index zero. This proves the lemma. □

Lemma 3.2. *K_p is the inverse of $L|_{dom(L) \cap Ker(P)}$.*

Proof. If $y \in Im(L)$, then

$$\begin{aligned} LK_py &= -D_{a+}^{\alpha,\rho} \left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{b^\rho - a^\rho} \right)^{\alpha-1} \int_a^b \frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} y(s)\mathrm{d}s \right. \\ &\quad \left. - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} y(s)\mathrm{d}s \right) \\ &= y. \end{aligned}$$

For $x \in dom(L) \cap Ker(P)$ and $Lx = y$, we have

$$\begin{aligned} -D_{a+}^{\alpha,\rho} x(t) = y(t), \quad t \in (a,b), \\ x(a) = 0, \quad x(b) = 0. \end{aligned}$$

Furthermore, for $x \in dom(L) \cap Ker(P)$, we have

$$(K_pLx)(t) = \int_a^b G(t,s)(-D_{a+}^{\alpha,\rho} x(s))\mathrm{d}s = \int_a^b G(t,s)y(s)\mathrm{d}s = x(t),$$

that is, $K_p = (L|_{dom(L) \cap Ker(P)})^{-1}$. This completes the proof of the lemma. □

Lemma 3.3. *For $y \in Y$, we have*

$$\|K_py(x)\|_\infty \le \|y\|_\infty \frac{\max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)} \left(\frac{b^\rho - a^\rho}{4\rho} \right)^{\alpha-1},$$

and

$$\|D_{0+}^{\alpha-1} K_py\|_\infty \le \|y\|_\infty \frac{(b^\rho - a^\rho)^\alpha}{\alpha\rho}.$$

Moreover,

$$\|K_py\|_X \le \Delta \|y\|_X.$$

Proof. Consider $K_p y(t)$ given in (8). Applying Lemma 2.4 gives

$$D_{0+}^{\alpha-1,\rho}(K_p y)(t) = \frac{1}{(b^\rho - a^\rho)^{\alpha-1}} \int_a^b \frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} y(s)\mathrm{d}s - \int_a^b y(s)\mathrm{d}s.$$

By Lemma 2.6, we have $G(t,s) > 0$ for $s,t \in (a,b)$,

$$\|K_p y(x)\|_\infty = \| \int_a^b G(t,s) y(s)\mathrm{d}s\|_\infty \le \|y\|_\infty \frac{\max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)} \left(\frac{b^\rho - a^\rho}{4\rho}\right)^{\alpha-1},$$

and

$$\begin{aligned} \|D_{a+}^{\alpha-1,\rho}(K_p y)(t)\|_\infty &\le \|y\|_\infty \left[\frac{1}{(b^\rho - a^\rho)^{\alpha-1}} \int_a^b \frac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} \mathrm{d}s\right] \\ &\le \|y\|_\infty \frac{(b^\rho - a^\rho)^\alpha}{\alpha\rho}. \end{aligned}$$

Thus,

$$\begin{aligned} \|K_p y\|_X &\le \|y\|_X \max\left\{\frac{\max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)} \left(\frac{b^\rho - a^\rho}{4\rho}\right)^{\alpha-1}, \frac{(b^\rho - a^\rho)^\alpha}{\alpha\rho}\right\} \\ &\le \Delta \|y\|_X \end{aligned}$$

where Δ is defined in (9). The proof of the lemma is now complete. □

Lemma 3.4. *$QN : X \to Y$ is continuous and bounded, and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact, where $\Omega \subset X$ is a bounded set.*

Proof. Since f is continuous, $QN(\overline{\Omega})$ and $(I-Q)N(\overline{\Omega})$ are bounded. Hence, there exists a constant $H > 0$, such that $|(I-Q)Nx(t)| \le H$ for $x \in \overline{\Omega}$ and $t \in [a,b]$. Applying the Lebesgue Dominated Convergence Theorem, it is clear that $K_p(I-Q)Ny : Y \to Y$ is completely continuous, so by the Arzelà-Ascoli theorem, $K_p(I-Q)N(\overline{\Omega})$ is compact. This proves the lemma. □

Lemma 3.5. *If conditions (A1)–(A5) are satisfied, then the set*

$$\Omega_1 = \{x \in dom(L) \backslash Ker(L) : Lx = \lambda Nx \text{ for some } \lambda \in [0,1]\},$$

is bounded.

Proof. Let $x(t) \in \Omega_1$; then $Nx \in Im(L) = Ker(Q)$. Therefore, $QNx = 0$. In view of (A4), there exists $t_0 \in [a,b]$ such that $|D_{a+}^{\alpha-1,\rho} x(t_0)| < M$. Since

$$D_{a+}^{\alpha-1,\rho} x(t) = D_{a+}^{\alpha-1,\rho} x(t_0) + \int_{t_0}^t D_{a+}^{\alpha,\rho} x(s)\mathrm{d}s,$$

we have

$$|D_{a+}^{\alpha-1} x(t)| \le M + \int_{t_0}^t |Nx(s)|\mathrm{d}s < M + \|Nx\|_\infty. \tag{10}$$

Since $(I-P)x \in dom(L) \cap Ker(P)$ for all $x \in \Omega_1$, by Lemma 3.3, we have

$$\|(I-P)x\|_X = \|K_p L(I-P)x\|_X = \|K_p Lx\|_X \le \Delta\|Lx\|_X \le \Delta\|Nx\|_\infty,$$

and

$$\|D_{a+}^{\alpha-1,\rho}(I-P)x\|_\infty \le \|D_{a+}^{\alpha-1,\rho}K_pLx\|_\infty \le \left(\frac{(b^\rho-a^\rho)^\alpha}{\alpha\rho}\right)\|Lx\|_\infty$$
$$\le \left(\frac{(b^\rho-a^\rho)^\alpha}{\alpha\rho}\right)\|Nx\|_\infty. \tag{11}$$

Using (10), (11), and Lemma 3.3,

$$\begin{aligned}\Gamma(\alpha)x(b) &= \left|D_{a+}^{\alpha-1,\rho}\left(x(b)\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\right)\right| \\ &= |D_{a+}^{\alpha-1,\rho}Px(t)| \\ &= |D_{a+}^{\alpha-1,\rho}(x(t)-((I-P)x)(t))| \\ &\le |D_{a+}^{\alpha-1,\rho}x(t)| + |D_{a+}^{\alpha-1,\rho}((I-P)x)(t)| \\ &\le M + \|Nx\|_\infty + \left(\frac{(b^\rho-a^\rho)^\alpha}{\alpha\rho}\right)\|Nx\|_\infty \\ &\le M + \left(1+\frac{(b^\rho-a^\rho)^\alpha}{\alpha\rho}\right)\|Nx\|_\infty.\end{aligned}$$

Thus,

$$\begin{aligned}\|x\|_X &\le \|(I-P)x\|_X + \|Px\|_X \\ &\le \Delta\|Nx\|_\infty + |x(b)|\left\|\left(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\right)^{\alpha-1}\right\| \\ &\le \Delta\|Nx\|_\infty + \frac{1}{\Gamma(\alpha)}\left(M+\left(1+\frac{(b^\rho-a^\rho)^\alpha}{\alpha\rho}\right)\|Nx\|_\infty\right).\end{aligned}$$

Hence, for all $x\in\Omega_1$, we have

$$\begin{aligned}\|x\|_X &\le M + \Gamma(\alpha)\Delta\|Nx\|_\infty + \left(1+\frac{(b^\rho-a^\rho)^\alpha}{\alpha\rho}\right)\|Nx\|_\infty \\ &\le M + \left(\Gamma(\alpha)\Delta + 1 + \frac{(b^\rho-a^\rho)^\alpha}{\alpha\rho}\right)\|Nx\|_\infty \\ &\le M + \Psi\|Nx\|_\infty\end{aligned}$$

Applying (A3), we have

$$\begin{aligned}\|x\|_X &\le M + \Psi(\|\mu\| + \|\sigma\|\|x\|_\infty + \|\omega\|\|D_{a+}^{\alpha-1,\rho}x\|_\infty) \\ &\le M + \Psi\|\mu\| + \Psi\|\sigma\|\|x\|_\infty + \Psi\|\omega\|\|D_{a+}^{\alpha-1,\rho}x\|_\infty \\ &\le M + \Psi\|\mu\| + \Psi\|\sigma\|\|x\|_X + \Psi\|\omega\|\|x\|_X \\ &\le M + \Psi\|\mu\| + \Psi(\|\sigma\|+\|\omega\|)\|x\|_X.\end{aligned}$$

Therefore,

$$\|x\|_X \le \frac{M+\Psi\|\mu\|}{1-\Psi(\|\sigma\|+\|\omega\|)}$$

and so Ω_1 is bounded, which is what we wanted to prove. □

Lemma 3.6. *If conditions (A1), (A2), and (A5) are satisfied, then the set*

$$\Omega_2 = \{x : x \in Ker(L), Nx \in Im(L)\},$$

is bounded.

Proof. Let $x \in \Omega_2$ with $x(t) = c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}$ for $c \in \mathbb{R}$; we have $Im(L) = Ker(Q)$, and therefore $QNx(t) = 0$. By (A5), we have $|c| \leq B$. Hence, Ω_2 is bounded. □

Now, we define an isomorphism $J : Ker(L) \to Im(Q)$ by

$$J\left(c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right) = c.$$

Lemma 3.7. *If conditions (A1), (A2), and (A5) hold, then the set*

$$\Omega_3 = \{x : x \in Ker(L),\ \lambda Jx + \beta(1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

with

$$\beta = \begin{cases} -1, & \textit{if } cQN\left(c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right) < 0, \\ 1, & \textit{if } cQN\left(c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right) > 0, \end{cases}$$

is bounded.

Proof. Let $x \in \Omega_3$; we have $x(t) = c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}$ for $c \in \mathbb{R}$, and

$$\lambda c + \beta(1-\lambda)QN\left(c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right) = 0.$$

If $\lambda = 1$, then $c = 0$. If $\lambda = 0$, by condition (A5), we have $|c| \leq B$. Finally, suppose that $\lambda \in (0,1)$. We claim that $|c| \leq B$. If $|c| \geq B$, then $\lambda c^2 = -\beta(1-\lambda)cQN\left(c\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right) < 0$, which contradicts $\lambda c^2 > 0$. Thus, our claim holds, that is, $|c| \leq B$. Thus, Ω_3 is bounded. □

We are now ready to prove the main result in this paper.

Theorem 3.8. *If conditions (A1)–(A5) hold, then problem (1) has at least one solution in X.*

Proof. Let Ω be any bounded open subset of X such that $\overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3} \subset \Omega$. From Lemma 3.4, N is L-Compact. From Lemmas 3.5, 3.6, and 3.7, it is clear that the assumptions 1) and 2) of Theorem 2.7 are satisfied. To complete the proof of the theorem, it remains to show that condition 3) of Theorem 2.7 holds.

Set

$$H(x,\lambda) = \lambda x + \beta(1-\lambda)QNx;$$

then it follows from Lemma 3.7 that $H(x, \lambda) \neq 0$, $x \in Ker(L) \cap \partial\Omega$. Thus, by the homotopy property of degree,

$$\begin{aligned} deg(QN|_{Ker(L)}, \Omega \cap Ker(L), 0) &= deg(H(\cdot, 0), \Omega \cap Ker(L), 0) \\ &= deg(H(\cdot, 1), \Omega \cap Ker(L), 0) \\ &= deg(H(\beta J, \Omega \cap Ker(L), 0) \neq 0. \end{aligned}$$

Hence, by Theorem 2.7, the problem (1)–(2) has at least one solution in $dom(L) \cap \overline{\Omega}$. □

4. Applications

For $\rho \to 1$, $a = 0$, and $b = 1$, problem (1)–(2) becomes a fractional boundary value problem that coincides with the problem studied in [16] for $k = 0$, namely,

$$\begin{cases} -(D_{0+}^{\alpha,1}x)(t) = f(t, x(t), D_{0+}^{\alpha-1,1}x(t)), \quad t \in [0, 1], \\ x(0) = 0, \quad x(1) = \int_0^1 x(t)\mathrm{d}A(t). \end{cases} \tag{12}$$

Example 4.1. Assume that $\alpha = \frac{3}{2}$, $A(t) = \frac{3}{2}t$, and $f(t, u, v) = t + \frac{1}{16}\sin(u) + \frac{1}{8}v$ in the problem (12). Then we obtain $\Gamma(\frac{3}{2}) = 0.886226$, $\Delta = 0.666666667$, $\Psi = 2.257484$, $\|\sigma\| + \|\omega\| = \frac{3}{16} = 0.1875 < \frac{1}{\Psi} = 0.442971024$. Take $M = 9$ and $B = 1$. A straight forward calculation shows that (A1)–(A5) are satisfied. Hence, by Theorem 3.8, problem (12) has at least one nontrivial solution.

Remark 4.1. In example 4.1, we deliberately took values of α, $A(t)$, and $f(t, u, v)$ similar to the those used in [16, Example 1] for the sake of a comparison. It is interesting to note that we obtain a sharper bound of 0.442971024 for $\|\sigma\| + \|\omega\|$ as compared to the estimate 0.501005816 obtained in [16, Example 1].

Next, we give an example of a Katugampola fractional differential equation with $\rho = 2$ in (1)–(2).

Example 4.2. Consider the problem

$$\begin{cases} -D^{\frac{3}{2},2}x(t) = f(t, u(t), D^{\frac{1}{2},2}x(t)), \\ x(1) = 0, \quad x(2) = \int_1^2 x(t)d\left(\frac{1}{\sqrt{6}}(t)\right) \end{cases} \tag{13}$$

where $f(t, u, v) = t + \frac{1}{15}\sin u + \frac{1}{12}v$. Here we have $\alpha = \frac{3}{2}$, $\rho = 2$, $a = 1$, $b = 2$, $A(t) = \frac{1}{\sqrt{6}}t$. It is easy to check that (A1) is satisfied. Also, we see that $\Gamma(\frac{3}{2}) = 0.886226$, $\Delta = 1.732050808$, $\Psi = 4.267039267$, $\|\sigma\| + \|\omega\| = \frac{1}{15} + \frac{1}{12} = 0.15 < \frac{\Gamma(\alpha)}{\Psi} = \frac{1}{4.267039267} = 0.2343545342$, which implies that conditions (A2) and (A3) are satisfied. If we take $M = 25$ and $B = 1$, simple calculations show that (A4) and (A5) are satisfied. Hence, by Theorem 3.8, (13) has at least one nontrivial solution.

As a concluding remark, we point out that by adding additional assumptions on the function f, it would be possible to obtain the uniqueness of solutions.

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