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The Duality Theory of Fractional Calculus and a New Fractional Calculus of Variations Involving Left Operators Only

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Abstract. Through duality, it is possible to transform left fractional operators into right fractional operators and vice versa. In contrast to existing literature, we establish integration by parts formulas that exclusively involve either left or right operators. The emergence of these novel fractional integration by parts formulas inspires the introduction of a new calculus of variations, where only one type of fractional derivative (left or right) is present. This applies to both the problem formulation and the corresponding necessary optimality conditions. As a practical application, we present a new Lagrangian that relies solely on left-hand side fractional derivatives. The fractional variational principle derived from this Lagrangian leads us to the equation of motion for a dissipative/damped system.

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1. Introduction

Fractional differentiation means "differentiation of arbitrary order". Its origin goes back more than 325 years, when in 1695 L'Hopital asked Leibniz the meaning of $\frac{d^n x}{dt^n}$ for $n = 1/2$. After that, many famous mathematicians, like Fourier, Abel, Liouville, and Riemann, among others, contributed to the development of Fractional Calculus [16].

In 1931, Bauer proved that it is impossible to use a variational principle to derive a linear dissipative equation of motion with constant coefficients [5]. Bauer's result expresses the well-known belief that there is no direct method of applying variational principles to nonconservative systems, which are characterized by friction or other dissipative processes. It turns out that fractional derivatives provide an elegant solution to the problem. Indeed, the proof of Bauer relies, implicitly, on the assumption that derivatives are of integer order.

The Fractional Calculus of Variations (FCoV) was born in 1996–1997 with the work of Riewe [14,15], precisely with the aim to obtain a Lagrangian for a simple dissipative system with a damping force proportional to the velocity. Riewe proposed to represent the dissipative effects with a Lagrangian dependent on left and right fractional Riemann–Liouville derivatives of order 1/2, showing that in such case one can obtain the equations of motion of a dissipative linear system with constant coefficients by a (fractional) variational principle. Riewe's idea is very simple and natural: if the Lagrangian contains a term proportional to $\left(\frac{d^n x}{dt^n}\right)^2$, then the respective Euler–Lagrange equation has a corresponding term proportional to $\frac{d^{2n}x}{dt^{2n}}$. Therefore, a damping force of the form $c\frac{dx}{dt}$ should follow from a Lagrangian containing a term proportional to $\left(\frac{d^{1/2}x}{dt^{1/2}}\right)^2$. The FCoV has therefore significant importance in physics and engineering, as a means to circumvent Bauer's result [9]. This explains the fact why FCoV is under strong development. For those interested on the subject, we refer to the books [1,2,10,11] and the survey papers [3,12].

The Lagrangian proposed by Riewe marked the beginning of several discussions in the literature, in particular among physicists, due to the presence of right fractional derivatives and other related issues. For a good and recent account about the fractional Lagrangian proposed by Riewe and other proposals by different mathematicians, we refer the reader to [8]. Here we remark that, to the best of our knowledge, all such fractional Lagrangians involve always a right derivative, which causes some physical concerns of non-causality. From a strictly mathematical point of view, however, the right operators appear naturally in the FCoV due to the central role of integration by parts in the proof of the Euler–Lagrange necessary optimality conditions. Indeed, under fractional integration by parts, a left fractional operator is transformed into a right fractional operator and vice versa (see Sect. 2), so even if the Lagrangian only includes left operators, then the right operators will appear in the Euler–Lagrange equation. To circumvent the undesirable phenomenon of appearance of right-fractional operators either on the Lagrangian or the Euler–Lagrange equation, here we develop the theory of duality for fractional calculus (see Sect. 3.1), as formulated by Caputo and Torres in 2015 [7], with early contributions discussed in [16].

The duality theory gives a way to express left fractional operators in terms of right fractional operators and the other way around. In simple terms, the duality theory shows that the right fractional operators of a function are the dual of the left operators of the dual function or, equivalently, the left fractional derivative/integral of a function is the dual of the right fractional derivative/integral of the dual function. Here, we further develop and use such duality theory to derive new fractional integration by parts formulas involving only left (or right) fractional operators (see Sect. 3.2). With the help of the new fractional integration by parts formulas, as well as results relating fractional derivatives with classical ones (see Sect. 3.3), we provide a new perspective to the FCoV by proposing a new FCoV where both Lagrangians and respective Euler–Lagrange equations involve only one type of operators, e.g., both variational problem and respective necessary optimality conditions with left-hand-side operators only (see Sect. 3.4). As an application of the obtained results, we go back to the original motivation behind the seminal papers of Riewe and provide a new Lagrangian that circumvents the Bauer theorem, for which the respective Euler–Lagrange equation coincide with the equation of motion of a dissipative system (see Sect. 4). In contrast with the example of Riewe, and others available in the literature, our quadratic Lagrangian for the linear friction problem is a real valued Lagrangian involving let-hand side fractional derivatives only. This provides a better physical meaning to the proposed Lagrangian, in contrast with the ones available in the literature. We end up with a conclusion, summarizing the main contributions of the paper (see Sect. 5).

2. Preliminaries

We recall some well-known definitions and results from fractional calculus, fixing also our notations. Here, we only give the notions and results that will be useful in the sequel. For more details, we refer to the book [11], where all such definitions and results are found, and references therein.

Definition 2.1. (Fractional integrals of order α) Let $\varphi \in L^1([a,b],\mathbb{R})$. The integrals

$$\left(\mathcal{I}^{\alpha}_{a+}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \varphi(\tau)(t-\tau)^{\alpha-1}d\tau, \quad t>a, \tag{2.1}$$

and

$$\left(\mathcal{I}^{\alpha}_{b-}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \varphi(\tau)(\tau-t)^{\alpha-1}d\tau, \quad t<b, \tag{2.2}$$

where $\alpha>0$ and $\Gamma(\cdot)$ is the Gamma function, are called, respectively, the left and the right fractional integrals of order α. Additionally, we define

$$\mathcal{I}^0_{a+}\varphi = \mathcal{I}^0_{b-}\varphi = \varphi,$$

that is, for $\alpha = 0$ (2.1) and (2.2) are the identity operator.

Proposition 2.2. (Integration by parts for fractional integrals) *Let $\alpha > 0$ and $1/p + 1/q \leq 1+\alpha$, $p \geq 1$, $q \geq 1$, with $p \neq 1$ and $q \neq 1$ in the case $1/p+1/q = 1+\alpha$. If $\varphi \in L^p([a,b],\mathbb{R})$ and $\psi \in L^q([a,b],\mathbb{R})$, then the following equality holds:*

$$\int_a^b \varphi(\tau)\left(\mathcal{I}^{\alpha}_{a+}\psi\right)(\tau)d\tau = \int_a^b \left(\mathcal{I}^{\alpha}_{b-}\varphi\right)(\tau)\psi(\tau)d\tau. \tag{2.3}$$

Proposition 2.3. *If $f \in L^p([a,b],\mathbb{R})$, $1 \leq p \leq \infty$, $\alpha>0$ and $\beta>0$, then*

$$\left(\mathcal{I}^{\alpha}_{a+}\left(\mathcal{I}^{\beta}_{a+}f\right)\right)(t) = \left(\mathcal{I}^{\alpha+\beta}_{a+}f\right)(t) \tag{2.4}$$

almost everywhere on $[a,b]$.

Definition 2.4. (Left/right Riemann–Liouville fractional derivatives of order α) Let $0 < \alpha < 1$ and $\varphi \in W^{1,1}([a,b],\mathbb{R})$. The left Riemann–Liouville derivative of order α is defined by

$$\left(\mathcal{D}^{\alpha}_{a+}\varphi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \varphi(\tau)(t-\tau)^{-\alpha}d\tau, \tag{2.5}$$

while the right Riemann–Liouville derivative of order α is defined by

$$\left(\mathcal{D}^{\alpha}_{b-}\varphi\right)(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b \varphi(\tau)(\tau-t)^{-\alpha}d\tau. \tag{2.6}$$

For $\alpha = 0$, we have $\mathcal{D}^{0}_{a+}\varphi = \mathcal{D}^{0}_{b-}\varphi = \varphi$; while for $\alpha = 1$, one has $\left(\mathcal{D}^{1}_{a+}\varphi\right)(t) = \frac{d}{dt}\varphi(t)$ and $\left(\mathcal{D}^{1}_{b-}\varphi\right)(t) = -\frac{d}{dt}\varphi(t)$.

Remark 2.5. Let $\alpha \in [0,1]$. For functions $\varphi \in W^{1,1}([a,b],\mathbb{R})$, each of the following expressions hold:

$$\left(\mathcal{D}^{\alpha}_{a+}\varphi\right)(t) = \frac{d}{dt}\left[\left(\mathcal{I}^{1-\alpha}_{a+}\varphi\right)(t)\right] \tag{2.7}$$

and

$$\left(\mathcal{D}^{\alpha}_{b-}\varphi\right)(t) = -\frac{d}{dt}\left[\left(\mathcal{I}^{1-\alpha}_{b-}\varphi\right)(t)\right]. \tag{2.8}$$

The following properties can be found, for example, in Propositions 2.4 and 2.5 of [11]. They are valid under appropriate space of functions $\mathcal{I}^{\alpha}_{a+}\left(L^p(a,b)\right)$ and $\mathcal{I}^{\alpha}_{b-}\left(L^p(a,b)\right)$, $1 \le p \le \infty$, defined, respectively, by

$$\mathcal{I}^{\alpha}_{a+}\left(L^p(a,b)\right) := \left\{\varphi : \varphi(t) = \left(\mathcal{I}^{\alpha}_{a+}\psi\right)(t), \psi \in L^p([a,b],\mathbb{R})\right\}$$

and

$$\mathcal{I}^{\alpha}_{b-}\left(L^p(a,b)\right) := \left\{\varphi : \varphi(t) = \left(\mathcal{I}^{\alpha}_{b-}\psi\right)(t), \psi \in L^p([a,b],\mathbb{R})\right\}.$$

Proposition 2.6. *If $\alpha > 0$, then*

$$\left(\mathcal{D}^{\alpha}_{a+}\left(\mathcal{I}^{\alpha}_{a+}\varphi\right)\right)(t) = \varphi(t) \tag{2.9}$$

for any $\varphi \in L^1([a,b],\mathbb{R})$, while

$$\left(\mathcal{I}^{\alpha}_{a+}\left(\mathcal{D}^{\alpha}_{a+}\varphi\right)\right)(t) = \varphi(t) \tag{2.10}$$

is satisfied for $\varphi \in \mathcal{I}^{\alpha}_{a+}\left(L^1(a,b)\right)$.

Proposition 2.7. (Integration by parts for fractional derivatives) *Let $0 < \alpha < 1$. If $\varphi \in \mathcal{I}^{\alpha}_{b-}\left(L^p(a,b)\right)$ and $\psi \in \mathcal{I}^{\alpha}_{a+}\left(L^q(a,b)\right)$ with $1/p + 1/q \le 1 + \alpha$, then the following equality holds:*

$$\int_a^b \varphi(\tau)\left(\mathcal{D}^{\alpha}_{a+}\psi\right)(\tau)d\tau = \int_a^b \left(\mathcal{D}^{\alpha}_{b-}\varphi\right)(\tau)\psi(\tau)d\tau. \tag{2.11}$$

3. Main Results

In what follows the α's are always real numbers between $0 < \alpha < 1$.

3.1. A Duality Theory for the FCoV

Here, we extend the duality of [7] to the context of the calculus of variations.

Definition 3.1. (Dual function and the dual operator $*$) Given $f : [a, b] \to \mathbb{R}$, we define the dual function $f^* : [a, b] \to \mathbb{R}$ (and the dual operator $*$) by

$$f^*(t) = f(b - t + a) \tag{3.1}$$

for all $t \in [a, b]$.

The following properties are straightforward but important.

Proposition 3.2. *For any function $f : [a, b] \to \mathbb{R}$, let $f^{**} := (f^*)^*$. Then,*

$$f^{**} = f. \tag{3.2}$$

If $c_1, c_2 \in \mathbb{R}$ and $f_1, f_2 : [a, b] \to \mathbb{R}$, then

$$(f_1 \cdot f_2)^* = f_1^* \cdot f_2^*; \tag{3.3}$$

$$(c_1 f_1 + c_2 f_2)^* = c_1 f_1^* + c_2 f_2^*. \tag{3.4}$$

Proof. The proof is a trivial exercise. □

Remark 3.3. An immediate consequence of the linearity given by (3.4) is that the sign change of the dual is the dual of the sign change. This property will be useful for us to establish a relation between left and right fractional derivatives. Precisely, for any function $f : [a, b] \to \mathbb{R}$ we have

$$(-f)^* = -f^*. \tag{3.5}$$

Proposition 3.4. *Under integration, the following relation holds:*

$$\int_a^b f^*(t) \cdot g(t)dt = \int_a^b f(t) \cdot g^*(t)dt. \tag{3.6}$$

Proof. From Definition 3.1 of duality, one has

$$\int_a^b f^*(t) \cdot g(t)dt = \int_a^b f(b - t + a) \cdot g(t)dt$$

and doing the change of variables $s = b - t + a$ we have

$$\int_a^b f^*(t) \cdot g(t)dt = -\int_b^a f(s) \cdot g(b - s + a)ds = \int_a^b f(s) \cdot g^*(s)ds.$$

The proof is complete. □

The following important lemma asserts that the left fractional integral of the dual is the dual of the right fractional integral (of the same order). See equation (2.19) in the classical book of Samko–Kilbas–Marichev [16] and also [4].

Lemma 3.5. *Let $\varphi \in L^1([a, b], \mathbb{R})$. The following relation holds:*

$$\mathcal{I}^\alpha_{a+} f^* = \left(\mathcal{I}^\alpha_{b-} f\right)^*. \tag{3.7}$$

As a corollary of Lemma 3.5, it follows that the right fractional integral of the dual is the dual of the left fractional integral.

Corollary 3.6. *If $f \in L^1([a,b],\mathbb{R})$, then*

$$\mathcal{I}^{\alpha}_{b-} f^* = \left(\mathcal{I}^{\alpha}_{a+} f\right)^* . \tag{3.8}$$

An equivalent way to look to the duality of the fractional integral operators consists to say that the left fractional integral is the dual of the right fractional integral of the dual; and the right fractional integral is the dual of the left fractional integral of the dual.

Corollary 3.7. *If $\varphi \in L^1([a,b],\mathbb{R})$, then*

$$\mathcal{I}^{\alpha}_{a+} f = \left(\mathcal{I}^{\alpha}_{b-} f^*\right)^* \tag{3.9}$$

and

$$\mathcal{I}^{\alpha}_{b-} f = \left(\mathcal{I}^{\alpha}_{a+} f^*\right)^* . \tag{3.10}$$

Roughly speaking, now we show that, up to a minus signal, the duality relations proved for the fractional integrals also hold true for fractional derivatives.

Lemma 3.8 asserts that the left fractional derivative of the dual is the sign change of the dual of the right fractional derivative (of the same order).

Lemma 3.8. *Let $f \in W^{1,1}([a,b],\mathbb{R})$. The following relation holds:*

$$\mathcal{D}^{\alpha}_{a+} f^* = -\left(\mathcal{D}^{\alpha}_{b-} f\right)^* . \tag{3.11}$$

Proof. From relations (2.7) and (3.7), we can write that

$$\left(\mathcal{D}^{\alpha}_{a+} f^*\right)(t) = \frac{d}{dt}\left[\left(\mathcal{I}^{1-\alpha}_{a+} f^*\right)(t)\right] = \frac{d}{dt}\left[\left(\mathcal{I}^{1-\alpha}_{b-} f\right)^*(t)\right]. \tag{3.12}$$

Using now (3.1) and relation (2.8) between the right fractional derivative of order α and the right fractional integral of order $1-\alpha$, we obtain from (3.12) that

$$\begin{aligned}\left(\mathcal{D}^{\alpha}_{a+} f^*\right)(t) &= \frac{d}{dt}\left[\left(\mathcal{I}^{1-\alpha}_{b-} f\right)(b-t+a)\right]\\ &= -\left(\mathcal{D}^{\alpha}_{b-} f\right)(b-t+a) = -\left(\mathcal{D}^{\alpha}_{b-} f\right)^*(t),\end{aligned}$$

which completes the proof. □

As a corollary, we can also say that the right fractional derivative of the dual is the sign change of the dual of the left fractional derivative.

Corollary 3.9. *If $f \in W^{1,1}([a,b],\mathbb{R})$, then*

$$\mathcal{D}^{\alpha}_{b-} f^* = -\left(\mathcal{D}^{\alpha}_{a+} f\right)^* . \tag{3.13}$$

Proof. Let $g = f^*$, which by Proposition 3.2 is equivalent to $f = g^*$. Using Lemma 3.8 for g, we can write from (3.11) that

$$\left(\mathcal{D}^{\alpha}_{b-} g\right)^*(t) = -\left(\mathcal{D}^{\alpha}_{a+} g^*\right)(t)$$

for all $t \in [a,b]$, that is,

$$\left(\mathcal{D}^{\alpha}_{b-} f^*\right)^*(t) = -\left(\mathcal{D}^{\alpha}_{a+} f\right)(t), \quad t \in [a,b]. \tag{3.14}$$

Applying the dual operator $*$ both sides of (3.14) gives, by Proposition 3.2 and (3.5), that

$$\left(\mathcal{D}^{\alpha}_{b-} f^*\right)(t) = -\left(\mathcal{D}^{\alpha}_{a+} f\right)^*(t)$$

and the result is proved. □

Recalling Remark 3.3, together with the linearity of the fractional operators, allow us to look to Lemma 3.8 as saying that the left fractional derivative of the dual is the dual of the right fractional derivative of the sign change or, in other words, the right fractional derivative is the sign change of the dual of the left fractional derivative of the dual.

Corollary 3.10. *Let $f \in W^{1,1}([a,b],\mathbb{R})$. The following relations hold:*

$$\mathcal{D}^{\alpha}_{a+} f^* = \left(\mathcal{D}^{\alpha}_{b-}(-f)\right)^* \tag{3.15}$$

and

$$\mathcal{D}^{\alpha}_{b-} f = -\left(\mathcal{D}^{\alpha}_{a+} f^*\right)^*. \tag{3.16}$$

Proof. Using (3.11) and (3.5) we can write that

$$\mathcal{D}^{\alpha}_{a+} f^* = -\left(\mathcal{D}^{\alpha}_{b-} f\right)^* = \left(-\mathcal{D}^{\alpha}_{b-} f\right)^*$$

and equality (3.15) follows by the linearity of operator $\mathcal{D}^{\alpha}_{b-}$. Applying the dual operator $*$ to both sides of (3.15), already proved, gives

$$\mathcal{D}^{\alpha}_{b-}(-f) = \left(\mathcal{D}^{\alpha}_{a+} f^*\right)^*,$$

which is equivalent, by the linearity of $\mathcal{D}^{\alpha}_{b-}$, to

$$-\mathcal{D}^{\alpha}_{b-} f = \left(\mathcal{D}^{\alpha}_{a+} f^*\right)^*.$$

The result is proved. □

Similarly, Remark 3.3 together with the linearity of the fractional derivatives give us a new look to the duality of Corollary 3.9: the right fractional derivative of the dual is the dual of the left fractional derivative of the sign change or, equivalently, the left fractional derivative is the sign change of the dual of the right fractional derivative of the dual.

Corollary 3.11. *If $f \in W^{1,1}([a,b],\mathbb{R})$, then*

$$\mathcal{D}^{\alpha}_{b-} f^* = \left(\mathcal{D}^{\alpha}_{a+}(-f)\right)^* \tag{3.17}$$

and

$$\mathcal{D}^{\alpha}_{a+} f = -\left(\mathcal{D}^{\alpha}_{b-} f^*\right)^*. \tag{3.18}$$

Proof. We prove Corollary 3.11 as a corollary of Corollary 3.10. Let $g = f^*$, that is, $f = g^*$. From relation (3.16) applied to g we know that

$$\mathcal{D}^{\alpha}_{b-} g = -\left(\mathcal{D}^{\alpha}_{a+} g^*\right)^* \Leftrightarrow \mathcal{D}^{\alpha}_{b-} f^* = -\left(\mathcal{D}^{\alpha}_{a+} f\right)^*,$$

which, by (3.5) and the linearity of $\mathcal{D}^{\alpha}_{a+}$, is equivalent to

$$\mathcal{D}^{\alpha}_{b-} f^* = \left(-\mathcal{D}^{\alpha}_{a+} f\right)^* \Leftrightarrow \mathcal{D}^{\alpha}_{b-} f^* = \left(\mathcal{D}^{\alpha}_{a+}(-f)\right)^*.$$

We have just proved (3.17). From (3.15) applied to g, we also know that

$$\mathcal{D}^{\alpha}_{a+} g^* = \left(\mathcal{D}^{\alpha}_{b-}(-g)\right)^* \Leftrightarrow \mathcal{D}^{\alpha}_{a+} f = \left(\mathcal{D}^{\alpha}_{b-}(-f^*)\right)^*,$$

which by the linearity of the operator $\mathcal{D}^{\alpha}_{b-}$ and (3.5) is equivalent to

$$\mathcal{D}^{\alpha}_{a+} f = \left(-\mathcal{D}^{\alpha}_{b-} f^*\right)^* \Leftrightarrow \mathcal{D}^{\alpha}_{a+} f = -\left(\mathcal{D}^{\alpha}_{b-} f^*\right)^*.$$

The result is proved. □

3.2. New Fractional Formulas of Integration by Parts

Now we are in a good position to prove some important results: formulas of integration by parts, for both fractional integrals and fractional derivatives, that involve only left operators or only right operators. This contrasts with the results available in the literature (cf. Propositions 2.2 and 2.7) for which a left operator on the left-hand side is converted into a right operator on the right-hand side—see formulas (2.3) and (2.11).

As we shall see in Sect. 3.4, our new results allow us to develop a new fractional calculus of variations (FCoV) more suitable for mechanics.

Theorem 3.12. (Integration by parts involving left fractional integrals only) *Let $1/p + 1/q \leq 1 + \alpha$, $p \geq 1$, $q \geq 1$, with $p \neq 1$ and $q \neq 1$ in the case $1/p+1/q = 1+\alpha$. If $f \in L^p([a,b],\mathbb{R})$ and $g \in L^q([a,b],\mathbb{R})$, then the following equality holds:*

$$\int_a^b f(\tau) \cdot \left(\mathcal{I}^\alpha_{a+} g\right)^* (\tau) d\tau = \int_a^b \left(\mathcal{I}^\alpha_{a+} f\right)(\tau) \cdot g^*(\tau) d\tau. \tag{3.19}$$

Proof. From (3.8) and (2.3), we can write that

$$\begin{aligned} \int_a^b f(\tau) \left(\mathcal{I}^\alpha_{a+} g\right)^* (\tau) d\tau &= \int_a^b f(\tau) \cdot \left(\mathcal{I}^\alpha_{b-} g^*\right)(\tau) d\tau \\ &= \int_a^b \left(\mathcal{I}^\alpha_{a+} f\right)(\tau) \cdot g^*(\tau) d\tau, \end{aligned}$$

which proves the intended relation. □

Similarly to Theorem 3.12, one can prove an integration by parts formula involving right fractional integrals only.

Theorem 3.13. (Integration by parts involving right fractional integrals only) *Let $1/p + 1/q \leq 1 + \alpha$, $p \geq 1$, $q \geq 1$, with $p \neq 1$ and $q \neq 1$ in the case $1/p + 1/q = 1 + \alpha$. If $f \in L^p([a,b],\mathbb{R})$ and $g^* \in L^q([a,b],\mathbb{R})$, then the following equality holds:*

$$\int_a^b f(\tau) \cdot \left(\mathcal{I}^\alpha_{b-} g\right)^* (\tau) d\tau = \int_a^b \left(\mathcal{I}^\alpha_{b-} f\right)(\tau) \cdot g^*(\tau) d\tau. \tag{3.20}$$

Proof. From (3.7) and (2.3), it follows that

$$\begin{aligned} \int_a^b f(\tau) \left(\mathcal{I}^\alpha_{b-} g\right)^* (\tau) d\tau &= \int_a^b f(\tau) \cdot \left(\mathcal{I}^\alpha_{a+} g^*\right)(\tau) d\tau \\ &= \int_a^b \left(\mathcal{I}^\alpha_{b-} f\right)(\tau) \cdot g^*(\tau) d\tau \end{aligned}$$

and the result is proved. □

We now give a proper formula of integration by parts for the FCoV that involves left fractional derivatives only.

Theorem 3.14. (Integration by parts involving left fractional derivatives only) *If* $f \in \mathcal{I}^{\alpha}_{a+}(L^p(a,b))$ *and* $g^* \in \mathcal{I}^{\alpha}_{b-}(L^q(a,b))$ *with* $1/p + 1/q \leq 1 + \alpha$*, then the following equality holds:*

$$\int_a^b f(t) \cdot \left(\mathcal{D}^{\alpha}_{a+} g\right)^* (t) dt = - \int_a^b \left(\mathcal{D}^{\alpha}_{a+} f\right)(t) \cdot g^*(t) dt. \tag{3.21}$$

Proof. From (3.13) and (2.11), we can write that

$$\begin{aligned}\int_a^b f(t) \left(\mathcal{D}^{\alpha}_{a+} g\right)^* (t) dt &= - \int_a^b f(t) \cdot \left(\mathcal{D}^{\alpha}_{b-} g^*\right)(t) dt \\ &= - \int_a^b \left(\mathcal{D}^{\alpha}_{a+} f\right)(t) \cdot g^*(t) dt,\end{aligned}$$

which proves the intended relation. □

In classical mechanics, it makes sense to consider causal operators. For this reason, we shall develop our new FCoV based on left operators only. For completeness, however, we also provide here a similar relation to (3.21) involving right fractional derivatives only.

Theorem 3.15. (Integration by parts involving right fractional derivatives only) *If* $f \in \mathcal{I}^{\alpha}_{b-}(L^p(a,b))$ *and* $g^* \in \mathcal{I}^{\alpha}_{a+}(L^q(a,b))$ *with* $1/p + 1/q \leq 1 + \alpha$*, then the following equality holds:*

$$\int_a^b f(t) \cdot \left(\mathcal{D}^{\alpha}_{b-} g\right)^* (t) dt = - \int_a^b \left(\mathcal{D}^{\alpha}_{b-} f\right)(t) \cdot g^*(t) dt. \tag{3.22}$$

Proof. Using Lemma 3.8 and Proposition 2.7, we obtain that

$$\begin{aligned}\int_a^b f(t) \cdot \left(\mathcal{D}^{\alpha}_{b-} g\right)^* (t) dt &\overset{(3.11)}{=} - \int_a^b f(t) \cdot \left(\mathcal{D}^{\alpha}_{a+} g^*\right)(t) dt \\ &\overset{(2.11)}{=} - \int_a^b \left(\mathcal{D}^{\alpha}_{b-} f\right)(t) \cdot g^*(t) dt.\end{aligned}$$

The result is proved. □

3.3. From Fractional to Classical Derivatives

Before formulating our fundamental problem of the fractional calculus of variations and proving its Euler–Lagrange necessary optimality condition, we prove here several auxiliary results that relate expressions involving fractional derivatives into expressions involving classical derivatives only. The results are here formulated and proved for left derivatives but similar formulas for right derivatives can also be obtained.

Proposition 3.16. *Let* $\alpha_1, \alpha_2 \in [0,1]$ *with* $\alpha_1 + \alpha_2 = 1$*. If* $h \in C^1([a,b];\mathbb{R})$*, then*

$$\left(\mathcal{D}^{\alpha_1}_{a+}\left(\mathcal{D}^{\alpha_2}_{a+} h\right)\right)(t) = h'(t) \tag{3.23}$$

for all $t \in [a,b]$.

Proof. Let $\psi = \mathcal{D}^{\alpha_2}_{a+}h$. Then,

$$\left(\mathcal{D}^{\alpha_1}_{a+}\psi\right)(t) \overset{(2.7)}{=} \frac{d}{dt}\left[\left(\mathcal{I}^{1-\alpha_1}_{a+}\psi\right)(t)\right] = \frac{d}{dt}\left[\left(\mathcal{I}^{\alpha_2}_{a+}\mathcal{D}^{\alpha_2}_{a+}h\right)(t)\right] \overset{(2.10)}{=} \frac{d}{dt}\left[h(t)\right] = h'(t).$$

The proof is complete. □

Theorem 3.17. *Let $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 + \alpha_2 = 1$. If $f \in C^1([a,b];\mathbb{R})$, then*

$$\int_a^b \left(\mathcal{D}^{\alpha_1}_{a+}f\right)(t) \cdot \left(\mathcal{D}^{\alpha_2}_{a+}g\right)^*(t)dt = -\int_a^b f'(t)\cdot g^*(t)dt. \tag{3.24}$$

Proof. Using Theorem 3.14 and Proposition 3.16, we obtain:

$$\int_a^b \left(\mathcal{D}^{\alpha_1}_{a+}f\right)(t) \cdot \left(\mathcal{D}^{\alpha_2}_{a+}g\right)^*(t)dt \overset{(3.21)}{=} -\int_a^b \left(\mathcal{D}^{\alpha_2}_{a+}\left(\mathcal{D}^{\alpha_1}_{a+}f\right)\right)(t)\cdot g^*(t)dt \overset{(3.23)}{=} -\int_a^b f'(t)\cdot g^*(t)dt. \tag{3.25}$$

The result is proved. □

Remark 3.18. As a consequence of Proposition 3.4, the left-hand side of (3.24) can be written as

$$\int_a^b \left(\mathcal{D}^{\alpha_1}_{a+}f\right)^*(t) \cdot \left(\mathcal{D}^{\alpha_2}_{a+}g\right)(t)dt$$

while the right-hand side of (3.24) can be written as

$$-\int_a^b \left(f'\right)^*(t)\cdot g(t)dt.$$

Corollary 3.19. *Let $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 + \alpha_2 = 1$. If $f \in C^1([a,b];\mathbb{R})$ and $h \in L^1([a,b],\mathbb{R})$, then*

$$\int_a^b \left(\mathcal{D}^{\alpha_1}_{a+}f\right)(t) \cdot \left(\mathcal{D}^{\alpha_2}_{b-}h\right)(t)dt = \int_a^b f'(t)\cdot h(t)dt. \tag{3.26}$$

Proof. Using (3.24) with $g = h^* \Leftrightarrow h = g^*$, we obtain that

$$\int_a^b \left(\mathcal{D}^{\alpha_1}_{a+}f\right)(t) \cdot \left(\mathcal{D}^{\alpha_2}_{a+}h^*\right)^*(t)dt = -\int_a^b f'(t)\cdot h(t)dt. \tag{3.27}$$

From (3.15) and Proposition 3.2, we know that

$$\left(\mathcal{D}^{\alpha_2}_{a+}h^*\right)^* = \mathcal{D}^{\alpha_2}_{b-}(-h) = -\mathcal{D}^{\alpha_2}_{b-}h$$

and (3.27) can be rewritten as

$$-\int_a^b \left(\mathcal{D}^{\alpha_1}_{a+}f\right)(t) \cdot \left(\mathcal{D}^{\alpha_2}_{b-}h\right)(t)dt = -\int_a^b f'(t)\cdot h(t)dt,$$

which proves the result. □

Proposition 3.20. *Let $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 + \alpha_2 = 1$. If $f^* \in C^1([a,b],\mathbb{R})$, then*

$$\mathcal{D}_{a+}^{\alpha_1}\left(\mathcal{D}_{b-}^{\alpha_2} f\right)^* = -\left(f^*\right)'. \tag{3.28}$$

Proof. Let $\psi = \left(\mathcal{D}_{b-}^{\alpha_2} f\right)^*$. Then,

$$\begin{aligned}
\left(\mathcal{D}_{a+}^{\alpha_1}\left(\mathcal{D}_{b-}^{\alpha_2} f\right)^*\right)(t) &= \left(\mathcal{D}_{a+}^{\alpha_1}\psi\right)(t) \\
&\overset{(2.7)}{=} \frac{d}{dt}\left[\left(\mathcal{I}_{a+}^{\alpha_2}\psi\right)(t)\right] \\
&= \frac{d}{dt}\left[\left(\mathcal{I}_{a+}^{\alpha_2}\left(\mathcal{D}_{b-}^{\alpha_2} f\right)^*\right)(t)\right] \\
&\overset{(3.11)}{=} \frac{d}{dt}\left[\left(\mathcal{I}_{a+}^{\alpha_2}\left(-\mathcal{D}_{a+}^{\alpha_2} f^*\right)\right)(t)\right] \\
&\overset{(2.10)}{=} -\frac{d}{dt}\left[f^*(t)\right] = -\left(f^*\right)'(t),
\end{aligned}$$

and the proof is complete. □

3.4. A New FCoV Involving Left Fractional Operators Only

Here, we propose a different approach to the FCoV so that both variational problems and their necessary optimality conditions involve left fractional derivatives only. This contrasts completely with the results found in the literature.

Let $0 < \alpha < 1$ and let us use the standard notation of mechanics for the derivative with respect to time: $\dot{x}(t) = x'(t)$. Consider the following problem: find a function $x \in C^1([a,b];\mathbb{R})$ that gives an extremum (minimum or maximum) to the integral functional $\mathcal{J}$,

$$\mathcal{J}[x] = \int_a^b L\left(t, x(t), \dot{x}(t), \left(\mathcal{D}_{a+}^{\alpha_1} x\right)(t), \left(\mathcal{D}_{a+}^{\alpha_2} x\right)^*(t)\right) dt, \tag{3.29}$$

when subject to given boundary conditions

$$x(a) = x_a, \quad x(b) = x_b. \tag{3.30}$$

We assume that the Lagrangian $L \in C^2([a,b] \times \mathbb{R}^4;\mathbb{R})$ and that $\partial_4 L$ and $\partial_5 L$ (the partial derivatives of $L(\cdot,\cdot,\cdot,\cdot,\cdot)$ with respect to its 4th and 5th arguments, respectively) have continuous Riemann–Liouville fractional derivatives.

Definition 3.21. A function $x \in C^1([a,b];\mathbb{R})$ that satisfies the given boundary conditions (3.30) is said to be an admissible trajectory for problem (3.29)–(3.30).

Trivially, for the particular case when the Lagrangian L only depends on t, x, and $\dot{x}$, then problem (3.29)–(3.30) reduces to the classical fundamental problem of the calculus of variations [13]. We now prove an extension of the classical Euler–Lagrange equation: when $L = L(t,x,\dot{x})$ in (3.29), then our necessary optimality condition (3.31) reduces to the classical Euler–Lagrange equation:

$$\partial_2 L - \frac{d}{dt}\left(\partial_3 L\right) = 0.$$

Theorem 3.22. (The fractional Euler–Lagrange equation) *If x is an extremizer (minimizer or maximizer) to problem* (3.29)–(3.30)*, then x satisfies the Euler–Lagrange equation*

$$\partial_2 L - \frac{d}{dt}\left(\partial_3 L\right) - \left(\mathcal{D}_{a+}^{\alpha_1}\left(\partial_4 L\right)^*\right)^* - \left(\mathcal{D}_{a+}^{\alpha_2}\left(\partial_5 L\right)\right)^* = 0. \tag{3.31}$$

Proof. Suppose that x is a solution of (3.29)–(3.30). Note that $\hat{x}_\epsilon(t) = x(t) + \epsilon h(t)$ is admissible for our problem for any $h \in C^1([a,b];\mathbb{R})$ with $h(a) = h(b) = 0$ and for any $\epsilon \in \mathbb{R}$. Clearly, for $\epsilon = 0$ one gets the solution of the problem: $\hat{x}_0 = x$. Let us define the real function $J : \mathbb{R} \to \mathbb{R}$ by

$$J(\epsilon) = \mathcal{J}[\hat{x}_\epsilon] = \int_a^b L\left(t, \hat{x}_\epsilon(t), \dot{\hat{x}}_\epsilon(t), \left(\mathcal{D}_{a+}^{\alpha_1}\hat{x}_\epsilon\right)(t), \left(\mathcal{D}_{a+}^{\alpha_2}\hat{x}_\epsilon\right)^*(t)\right) dt.$$

Since $\epsilon = 0$ is an extremizer for function J, the classical Fermat's theorem asserts that a necessary optimality condition for our problem is given by $J'(0) = 0$. Using the linearity of the involved operators, a direct computation tells us that

$$\begin{aligned} 0 = J'(0) &= \left.\frac{d}{d\epsilon}J(\epsilon)\right|_{\epsilon=0} \\ &= \int_a^b \left[\partial_2 L \cdot h(t) + \partial_3 L \cdot \dot{h}(t) + \partial_4 L \cdot \left(\mathcal{D}_{a+}^{\alpha_1}h\right)(t) + \partial_5 L \cdot \left(\mathcal{D}_{a+}^{\alpha_2}h\right)^*(t)\right] dt, \end{aligned} \tag{3.32}$$

where we omit the arguments $\left(t, x(t), \dot{x}(t), \left(\mathcal{D}_{a+}^{\alpha_1}x\right)(t), \left(\mathcal{D}_{a+}^{\alpha_2}x\right)^*(t)\right)$ of $\partial_i L$, for $i = 2, \ldots, 5$. Using the classical formula of integration by parts, and having in mind that $h(a) = h(b) = 0$, it follows that

$$\int_a^b \partial_3 L \cdot \dot{h}(t)\, dt = -\int_a^b \frac{d}{dt}\left(\partial_3 L\right) \cdot h(t)\, dt. \tag{3.33}$$

On the other hand, our Proposition 3.2 and Theorem 3.14 allow us to write that

$$\begin{aligned} \int_a^b \partial_4 L \cdot \left(\mathcal{D}_{a+}^{\alpha_1}h\right)(t)\, dt &\overset{(3.2)}{=} \int_a^b \left(\left(\partial_4 L\right)^*\right)^* \cdot \left(\mathcal{D}_{a+}^{\alpha_1}h\right)(t)\, dt \\ &\overset{(3.21)}{=} -\int_a^b \left(\mathcal{D}_{a+}^{\alpha_1}\left(\partial_4 L\right)^*\right)^* \cdot h(t)\, dt, \end{aligned} \tag{3.34}$$

while from Theorem 3.14 and Proposition 3.4, we obtain that

$$\begin{aligned} \int_a^b \partial_5 L \cdot \left(\mathcal{D}_{a+}^{\alpha_2}h\right)^*(t)\, dt &\overset{(3.21)}{=} -\int_a^b \mathcal{D}_{a+}^{\alpha_2}\left(\partial_5 L\right) \cdot h^*(t)\, dt \\ &\overset{(3.6)}{=} -\int_a^b \left(\mathcal{D}_{a+}^{\alpha_2}\left(\partial_5 L\right)\right)^* \cdot h(t)\, dt. \end{aligned} \tag{3.35}$$

Substituting (3.33), (3.34) and (3.35) into (3.32), we conclude that

$$\int_a^b \left[\partial_2 L - \frac{d}{dt}\left(\partial_3 L\right) - \left(\mathcal{D}_{a+}^{\alpha_1}\left(\partial_4 L\right)^*\right)^* - \left(\mathcal{D}_{a+}^{\alpha_2}\left(\partial_5 L\right)\right)^*\right] \cdot h(t)\, dt = 0. \tag{3.36}$$

We obtain the intended necessary optimality condition (3.31) applying the fundamental lemma of the calculus of variations to (3.36). □

As an application of our new proposed calculus of variations, in Sect. 4, we give the first example of the literature of a fractional mechanical Lagrangian L involving left fractional derivatives only, for which the respective Euler–Lagrange equation coincides with the motion of a dissipative/damped system.

4. A Nonconservative/Dissipative Equation of Motion

Let $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 + \alpha_2 = 1$ and the Lagrangian L be given by

$$L\left(t, x, \dot{x}, \mathcal{D}_{a+}^{\alpha_1} x, \left(\mathcal{D}_{a+}^{\alpha_2} x\right)^*\right) = \frac{1}{2} m\dot{x}^2 - U(x) + \frac{1}{2} c \left(\mathcal{D}_{a+}^{\alpha_1} x\right) \cdot \left(\mathcal{D}_{a+}^{\alpha_2} x\right)^* . \quad (4.1)$$

In this case, one has

$$\begin{aligned} \partial_2 L &= -U'(x), \\ \partial_3 L &= m\dot{x}, \\ \partial_4 L &= \frac{1}{2} c \left(\mathcal{D}_{a+}^{\alpha_2} x\right)^* \Rightarrow (\partial_4 L)^* = \frac{1}{2} c \left(\mathcal{D}_{a+}^{\alpha_2} x\right), \\ \partial_5 L &= \frac{1}{2} c \left(\mathcal{D}_{a+}^{\alpha_1} x\right), \end{aligned}$$

and thus the Euler–Lagrange equation (3.31) takes the form

$$-U'(x) - \frac{d}{dt}(m\dot{x}) - \frac{1}{2} c \left(\mathcal{D}_{a+}^{\alpha_1}\left(\mathcal{D}_{a+}^{\alpha_2} x\right)\right)^* - \frac{1}{2} c \left(\mathcal{D}_{a+}^{\alpha_2}\left(\mathcal{D}_{a+}^{\alpha_1} x\right)\right)^* = 0. \quad (4.2)$$

It follows from Proposition 3.16 that (4.2) is equivalent to

$$U'(x) + c(\dot{x})^* + m\ddot{x} = 0, \quad (4.3)$$

that is, we have obtained an equation of motion (4.3) with a nonconservative Rayleigh term [6].

5. Conclusion

In 1996, Riewe has proved that a calculus of variations can be formulated to include derivatives of fractional (non-integer) order [14]. By doing that, Riewe has shown possible to define Lagrangians, involving both left and right fractional derivatives, that lead directly to equations of motion with nonconservative forces, such as friction, circumventing Bauer's corollary that "The equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle" [5]. Here, we continued the development of the fractional-derivative calculus of variations by providing a completely new perspective to the subject. Our main contributions are: (i) new formulas of integration by parts that involve left fractional operators only; (ii) a new fractional calculus of variations where both the Lagrangian and respective Euler–Lagrange equation involve left fractional derivatives only; (iii) a new example of a Lagrangian for which the respective

Euler–Lagrange equation coincide with the equation of motion of a dissipative system. Such contributions are radically different from other results found in the vast literature on the subject. Indeed, to the best of our knowledge, (i) all available formulas of fractional integration by parts involve both left and right fractional operators; (ii) all available forms of the fractional calculus of variations for which the Lagrangian involves left fractional derivatives result in Euler–Lagrange equations involving a right fractional derivative; (iii) all available examples of fractional Lagrangians giving rise to equations of motion of a dissipative system involve both left and right fractional derivatives.

In this paper, we have restricted ourselves to ideas and to the central result of any calculus of variations: the celebrated Euler–Lagrange equation, which is a first-order necessary optimality condition. Of course our results can be extended in many different ways and directions. We trust that the new perspective introduced here marks the beginning of a fruitful road for fractional mechanics, the fractional calculus of variations and fractional optimal control.

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Declarations

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