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Complete Nonsingular Holomorphic Foliations on Stein Manifolds

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Dedicated to Josip Globevnik.

Abstract. Let X be a Stein manifold of complex dimension $n>1$ endowed with a Riemannian metric $\mathfrak{g}$. We show that for every integer k with $\left[\frac{n}{2}\right] \le k \le n-1$ there is a nonsingular holomorphic foliation of dimension k on X all of whose leaves are closed and $\mathfrak{g}$-complete. The same is true if $1 \le k < \left[\frac{n}{2}\right]$ provided that there is a complex vector bundle epimorphism $TX \to X \times \mathbb{C}^{n-k}$. We also show that if $\mathcal{F}$ is a proper holomorphic foliation on $\mathbb{C}^n$ $(n>1)$ then for any Riemannian metric $\mathfrak{g}$ on $\mathbb{C}^n$ there is a holomorphic automorphism Φ of $\mathbb{C}^n$ such that the image foliation $\Phi_*\mathcal{F}$ is $\mathfrak{g}$-complete. The analogous result is obtained on every Stein manifold with Varolin's density property.

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1. Introduction

Let X be a complex manifold endowed with a Riemannian metric $\mathfrak{g}$. A locally closed subvariety Y of X is said to be $\mathfrak{g}$-complete if every smooth divergent path $\gamma : [0,1) \to Y$ (i.e., one leaving every compact subset of Y) has infinite $\mathfrak{g}$-length: $\int_0^1 \mathfrak{g}(\gamma(t), \dot\gamma(t))\,dt = +\infty$. If $\mathfrak{g}$ is a complete metric on X then clearly every closed connected complex subvariety of X of positive dimension is $\mathfrak{g}$-complete. A holomorphic foliation on X is said to be $\mathfrak{g}$-complete if every leaf is $\mathfrak{g}$-complete. For the theory of holomorphic foliations, see Scárdua [26].

The results of this paper concern the problem, posed by Yang in 1977 [30,31], whether there exist bounded complex submanifolds of a Euclidean space $\mathbb{C}^n$ for $n>1$ which are complete in the Euclidean metric. We begin with a brief overview of the main developments on this subject, referring the interested reader to the more complete recent survey by Alarcón [3]. We also

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point out that Yang's problem is related to the Calabi–Yau problem in the theory of minimal surfaces; see [7, Chapter 7] for a recent survey of the latter subject.

The first affirmative result on Yang's problem was obtained by Jones [23], who constructed a holomorphic immersion of the unit disc into $\mathbb{C}^2$ and an embedding into $\mathbb{C}^3$ with bounded image and complete induced metric. His method is based on the BMO duality theorem. Much later, Martín, Umehara, and Yamada [25] used a geometric method to construct complete bounded complex curves in $\mathbb{C}^2$ with arbitrary finite genus and finitely many ends. This was followed by Alarcón and López [10] who showed that any topological type is possible. The methods used in these constructions do not provide any control of the complex structure on the curve, except in the simply connected case when any such curve is biholomorphic to the disc. In [4], the authors used the Riemann–Hilbert boundary value problem and gluing methods to prove that every bordered Riemann surface admits a complete proper holomorphic immersion into the ball of $\mathbb{C}^2$ and a complete proper holomorphic embedding into the ball of $\mathbb{C}^3$. A related result for Riemann surfaces with finite genus and countably many ends was obtained by the authors in [6, Theorem 1.8]. These are still the only results on Yang's problem with a complete control of the conformal structure on the underlying Riemann surface.

All results mentioned so far pertain to complex curves. This is how things stood until the seminal work of Globevnik [21] in 2015. In his landmark construction, Globevnik showed that for every pair of integers $1 \leq k < n$ there exists a holomorphic foliation of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ by closed complete complex subvarieties of complex dimension k, most of which are smooth (without singularities). The leaves of foliations in his construction are connected components of the level sets of holomorphic maps $f : \mathbb{B}^n \to \mathbb{C}^{n-k}$. Completeness of the leaves is ensured by choosing the map f to grow sufficiently fast on components of a suitable labyrinth $\Gamma \subset \mathbb{B}^n$ having the property that any divergent path in $\mathbb{B}^n$ avoiding all but finitely many components of Γ has infinite Euclidean length (see Lemma 2.1). The construction of such labyrinths was one of the main new results of Globevnik's paper. In the sequel [22], Globevnik extended his construction to any pseudoconvex domain in $\mathbb{C}^n$. Further improvements and generalizations of his results were made by Alarcón [1,2].

This is a suitable point to state our first main result. It generalizes Globevnik's theorem to an arbitrary Riemannian Stein manifold, and the foliations that we find are nonsingular.

Theorem 1.1. *Let X be a Stein manifold of complex dimension $n > 1$ endowed with a Riemannian metric $\mathfrak{g}$. For every integer k with $\left[\frac{n}{2}\right] \leq k \leq n-1$ there exists a nonsingular holomorphic foliation of dimension k on X all of whose leaves are closed and $\mathfrak{g}$-complete. The same holds if $1 \leq k < \left[\frac{n}{2}\right]$ provided that there is a complex vector bundle epimorphism $TX \to X \times \mathbb{C}^{n-k}$. In particular, if X is parallelizable then it admits a nonsingular holomorphic foliation of any dimension $k \in \{1, \ldots, n-1\}$ with closed $\mathfrak{g}$-complete leaves.*

Foliations in Theorem 1.1 are given by holomorphic submersions $X \to \mathbb{C}^{n-k}$. The proof (see Sect. 2) combines the methods of Globevnik [21,22] with those of Forstnerič [16]; the latter provide the h-principle for holomorphic submersions from any Stein manifold X to Euclidean spaces of dimension $< \dim X$.

The construction methods used in the proof of Theorem 1.1 (like those in [1,2,21,22]) do not provide any control of the topology or the complex structure of the leaves. Soon after Globevnik's work, Alarcón, Globevnik, and López [8,9] combined the use of labyrinths with the approximation theory for holomorphic automorphisms of complex Euclidean spaces to construct complete properly embedded complex hypersurfaces in the ball $\mathbb{B}^n$ of $\mathbb{C}^n$ $(n > 1)$ with partial control on their topology, and with prescribed topology if $n = 2$. The main idea is to inductively deform a given closed complex submanifold $Y \subset \mathbb{C}^n$ by holomorphic automorphisms of $\mathbb{C}^n$ such that the images avoid more and more pieces of a given labyrinth $\Gamma \subset \mathbb{B}^n$, and they converge to a properly embedded complete complex submanifold of $\mathbb{B}^n$ which is biholomorphic to a Runge domain in Y. This construction is possible if Γ is polynomially convex and the connected components of Γ are holomorphically contractible. The labyrinths used in the aforementioned papers consist of balls in suitably placed affine real hyperplanes in $\mathbb{C}^n$. A bit later, the authors proved in [5] that the ball $\mathbb{B}^n$ for $n > 1$ admits a nonsingular holomorphic foliation by complete holomorphic discs. Using labyrinths in pseudoconvex shells in $\mathbb{C}^n$, constructed by Charpentier and Kosiński in [13], one obtains the analogous result with the ball replaced by an arbitrary Kobayashi hyperbolic pseudoconvex Runge domain in $\mathbb{C}^n$ with $n > 1$ (see [5, Remark 1]). If the domain fails to be hyperbolic then some leaves of the foliation may be complex lines. See the survey [3] for more information.

To extend this technique to more general Stein manifolds, we must assume that the manifold has many holomorphic automorphisms. The suitable class are Stein manifolds with the density property; see Definition 1.6. Another technical issue is to find suitable labyrinths in X with holomorphically contractible components. We do this in Sect. 3 in the course of proof of Theorem 1.7 (see Lemma 3.2, which is an important new tool).

We shall consider foliations satisfying the following condition.

Definition 1.2. Let X be a connected Stein manifold of dimension > 1. A (possibly singular) holomorphic foliation $\mathcal{F}$ on X is *proper* if every leaf $\mathcal{F}_x$ $(x \in X)$ is closed and satisfies $\dim \mathcal{F}_x \geq 1$, and for every compact subset $K \subset X$ the set

$$\mathcal{F}(K) := \overline{\bigcup_{x \in K} \mathcal{F}_x} \tag{1.1}$$

is such that $X \setminus \mathcal{F}(K)$ is nonempty and not relatively compact in X.

A biholomorphic map clearly takes a proper foliation to another such foliation.

Proposition 1.3. *Every holomorphic foliation $\mathcal{F}$ on a connected Stein manifold X given by a nonconstant holomorphic map $f = (f_1, \ldots, f_q) : X \to \mathbb{C}^q$ with $1 \le q < \dim X$ is proper.*

Proof. Every irreducible component of a nonempty fibre $f^{-1}(z)$ ($z \in \mathbb{C}^q$) is a closed complex subvariety of X of some dimension $d \in \{1, \ldots, \dim X - 1\}$. In particular, the foliation does not have any zero dimensional leaves. Given a compact set $K \subset X$ we clearly have that

$$\mathcal{F}(K) \subset f^{-1}(f(K)) \subset f_j^{-1}(f_j(K)) \text{ for every } j = 1, \ldots, q. \tag{1.2}$$

Since f is nonconstant, the function $f_j : X \to \mathbb{C}$ is nonconstant for some j, hence an open map. Thus, $f_j(X)$ is an open subset of $\mathbb{C}$ containing the compact subset $f_j(K)$. Choose a nonempty open subset $U \subset f_j(X) \setminus f_j(K)$. The set $X \setminus f_j^{-1}(f_j(K))$ then contains $f_j^{-1}(U)$, hence is nonempty and not relatively compact. By (1.2) the same is true for $X \setminus \mathcal{F}(K)$. □

The proof of Proposition 1.3 also gives the following criterium for properness.

Corollary 1.4. *Assume that X is a connected Stein manifold of dimension > 1 and $\mathcal{F}$ is a holomorphic foliation on X with all leaves closed and of positive dimension. If there is a nonconstant holomorphic function on X which is constant on every leaf of $\mathcal{F}$, then $\mathcal{F}$ is a proper foliation.*

An example of a nonproper (singular) holomorphic foliation is given by punctured complex lines through the origin in $\mathbb{C}^2$, with the origin a leaf of dimension zero. Taking a Cartesian product with $\mathbb{C}$ gives a nonproper holomorphic foliation of $\mathbb{C}^3$ with a closed leaf of dimension 1 and nonclosed leaves of dimension 2. Proposition 1.3 and Corollary 1.4 show that there are no simple examples of nonproper holomorphic foliations having closed leaves.

Our second main result is the following. It is proved in Sect. 3.

Theorem 1.5. *Let $\mathfrak{g}$ be a Riemannian metric on $\mathbb{C}^n$, $n > 1$. For every proper holomorphic foliation $\mathcal{F}$ on $\mathbb{C}^n$ (see Definition 1.2) there is a holomorphic automorphism $\Phi \in \mathrm{Aut}(\mathbb{C}^n)$ such that the image foliation $\Phi_*\mathcal{F}$ with leaves $\Phi(\mathcal{F}_z)$ ($z \in \mathbb{C}^n$) is $\mathfrak{g}$-complete.*

This theorem is nontrivial if the metric $\mathfrak{g}$ is not complete on $\mathbb{C}^n$, and its main interest is when $\mathfrak{g}$ decays fast at infinity. Note that the foliation $\Phi_*\mathcal{F}$ has exactly the same leaves as the original foliation $\mathcal{F}$ up to biholomorphisms, but they are now sufficiently twisted in $\mathbb{C}^n$ to become $\mathfrak{g}$-complete.

Using labyrinths provided by Lemma 3.2, we shall also prove the analogue of Theorem 1.5 for every Stein manifold with the density property, a notion introduced by Varolin [29].

Recall that a holomorphic vector field on a complex manifold X is called complete if its flow exists for all complex values of time, so it forms a complex one-parameter group of holomorphic automorphisms of X.

Definition 1.6. (See Varolin [29] or Definition 4.10.1 in [18]) A complex manifold X has the *density property* if every holomorphic vector field on X can be approximated uniformly on compacts by sums and commutators of complete holomorphic vector fields on X.

The fact that the Euclidean space $\mathbb{C}^n$ for $n > 1$ has the density property was discovered by Andersén and Lempert [11], thereby giving birth to this theory. It is known that most complex Lie groups and complex homogeneous manifolds have the density property. Surveys of this subject can be found in [18, Chapter 4], [19], and [24]. An important point is that, on any Stein manifold X with the density property, one can approximate isotopies of biholomorphic maps between Stein Runge domains in X by isotopies of holomorphic automorphisms of X; see Forstnerič and Rosay [20] for the case $X = \mathbb{C}^n$ and [18, Theorem 4.10.5] for the general case. This result plays an essential role in the proofs of Theorems 1.5, 1.7, and 1.8 .

The following result generalizes Theorem 1.5; it is proved in Sect. 3. As above, it is nontrivial only if the given metric $\mathfrak{g}$ on X is not complete.

Theorem 1.7. *Let X be a Stein manifold with the density property, endowed with a Riemannian metric $\mathfrak{g}$. For every proper holomorphic foliation $\mathcal{F}$ on X there is a holomorphic automorphism $\Phi \in \mathrm{Aut}(X)$ such that the image foliation $\Phi_*\mathcal{F}$ is $\mathfrak{g}$-complete.*

The analogous construction applies on any pseudoconvex Runge domain in a Stein manifold with the density property. This gives the following result, generalizing the aforementioned results in [5,8,9] for domains in $\mathbb{C}^n$ with $n > 1$. See also Theorem 3.3.

Theorem 1.8. *Let X be a Stein manifold of dimension > 1 with the density property, and let Ω be a pseudoconvex Runge domain in X endowed with a Riemannian metric $\mathfrak{g}$. For every proper holomorphic foliation $\mathcal{F}_0$ on X there is a $\mathfrak{g}$-complete holomorphic foliation $\mathcal{F}$ on Ω such that every leaf of $\mathcal{F}$ is biholomorphic to a pseudoconvex Runge domain in a leaf of $\mathcal{F}_0$.*

Theorems 1.5, 1.7, and 1.8 suggest that the main obstruction to finding complete holomorphic foliations with specific types of leaves on a given Stein Riemannian manifold with the density property lies in the topological and the complex structure of the manifold, and not in the choice of the Riemannian metric.

2. Proof of Theorem 1.1

We begin by sketching the proof of Globevnik's main theorem in [21]. The essential ingredient are labyrinths Γ as in the following lemma from [21]. A simpler construction can be found in [9], where the connected components of Γ are balls in affine real hyperplanes.

Lemma 2.1. *Given numbers $0 < r < s$ and $M > 0$, there is a compact set $\Gamma \subset s\mathbb{B}^n \setminus r\overline{\mathbb{B}}^n$ with connected complement satisfying the following two conditions:*

- (a) *the compact set $r\overline{\mathbb{B}}^n \cup \Gamma$ is polynomially convex, and*
- (b) *every piecewise smooth path $\gamma : [0,1] \to \mathbb{C}^n \setminus \Gamma$ with $\gamma(0) \in r\overline{\mathbb{B}}^n$ and $\gamma(1) \in \mathbb{C}^n \setminus s\mathbb{B}^n$ has Euclidean length $\geq M$.*

Remark 2.2. Given a Riemannian metric $\mathfrak{g}$ on $\mathbb{C}^n$ and a constant $M > 0$, there is a labyrinth Γ as in the lemma such that any path crossing the shell $s\mathbb{B}^n \setminus r\overline{\mathbb{B}}^n$ and avoiding Γ has $\mathfrak{g}$-length $\geq M$. The reason is that any two Riemannian metrics are comparable on a compact set. When condition (b) holds for a metric $\mathfrak{g}$, we say that Γ enlarges the $\mathfrak{g}$-distance by M.

Granted the lemma, the proof in [21] proceeds as follows. The main case is to find a complete foliation by hypersurfaces given as level sets of a holomorphic function $f : \mathbb{B}^n \to \mathbb{C}$. (By Sard's theorem, most level sets of f are nonsingular.) Pick a sequence $0 < r_1 < r_2 < \cdots < 1$ with $\lim_{i\to\infty} r_i = 1$. In each spherical shell $S_i = \{z \in \mathbb{C}^n : r_i < |z| < r_{i+1}\}$ we choose a labyrinth Γ_i satisfying Lemma 2.1 with a constant $M_i > 0$, chosen such that

$$\sum_{i=1}^{\infty} M_i = +\infty. \tag{2.1}$$

Since the compact set $K_i = r_i\overline{\mathbb{B}}^n \cup \Gamma_i$ is polynomially convex for every $i \in \mathbb{N}$, a standard construction using the Oka–Weil theorem gives a holomorphic function f on $\mathbb{B}^n$ satisfying

$$\lim_{i\to\infty} \min_{z\in\Gamma_i} |f(z)| = +\infty. \tag{2.2}$$

It follows that any divergent path $\gamma : [0, 1) \to \mathbb{B}^n$ on which f is bounded avoids Γ_i for all sufficiently big $i \in \mathbb{N}$, and hence γ has infinite Euclidean length by Lemma 2.1 (b) and (2.1). Thus, every nonempty level set $\{f = c\}$ is a complete closed (possibly singular) complex hypersurface in $\mathbb{B}^n$. Foliations of lower dimensions are obtained by adding generically chosen additional component functions, noting that the leaves are automatically complete.

The construction in the proof of Theorem 1.1 is similar to the one of Globevnik, except that we also use the results of Forstnerič [16] on the existence of holomorphic submersions from Stein manifolds to Euclidean spaces. His main result (see [16, Theorem 2.5]) is that a holomorphic submersion $f : X \to \mathbb{C}^q$ always exists if $1 \leq q \leq \left[\frac{n+1}{2}\right]$ where $n = \dim X$, so we obtain a nonsingular holomorphic foliation of X of any dimension $k = n - q$ with $\left[\frac{n}{2}\right] \leq k \leq n-1$. If on the other hand $\left[\frac{n+1}{2}\right] < q \leq n-1$ then a holomorphic submersion $X^n \to \mathbb{C}^q$ exists if and only if the holomorphic tangent bundle of X admits a surjective complex vector bundle map $\theta : TX \to X \times \mathbb{C}^q$ onto the trivial bundle of rank q. In fact, the h-principle holds: every surjective complex vector bundle map θ as above is homotopic through maps of the same kind to the tangent map of a holomorphic submersion $f : X \to \mathbb{C}^q$. In [16, Theorem 2.5] it is also shown that the analogous results hold with interpolation on closed complex subvarieties and approximation on compact $\mathscr{O}(X)$-convex subsets of X, in analogy to the constructions of holomorphic functions in the Oka–Cartan theory and of holomorphic maps in Oka theory. Noncritical holomorphic functions also exist on reduced Stein spaces with singularities, see [17].

Proof of Theorem 1.1. We embed the Stein manifold X as a closed complex submanifold in a Euclidean space $\mathbb{C}^N$; see [18, Theorems 2.4.1 and 9.3.1] for a survey of the classical embedding theorems. For every $i \in \mathbb{N}$ the set $X_i = X \cap i\overline{\mathbb{B}}^N$ is compact and $\mathscr{O}(X)$-convex, i.e., holomorphically convex in X. Pick a number $c_i > 0$ such that

$$\mathfrak{g}(x,v) \geq c_i|v| \ \text{ holds for every } x \in X_{i+1} \text{ and } v \in T_xX. \tag{2.3}$$

Here, $|v|$ denotes the Euclidean length of a tangent vector $v \in T_xX \subset T_x\mathbb{C}^N \cong \mathbb{C}^N$. In every spherical shell $S_i = \{z \in \mathbb{C}^N : i < |z| < i+1\}$ $(i \in \mathbb{N})$ we choose a labyrinth Γ_i satisfying Lemma 2.1 with a constant $M_i > 0$, chosen such that

$$\sum_{i=1}^{\infty} c_i M_i = +\infty. \tag{2.4}$$

We explain the basic case with $k = n-1$. Using the Oka–Weil type approximation theorem for noncritical holomorphic functions (see [16, Theorem 2.1]) inductively, we find a holomorphic function $f : X \to \mathbb{C}$ without critical points such that condition (2.2) holds for all $i \in \mathbb{N}$ with Γ_i replaced by $\Gamma_i' := \Gamma_i \cap X$. To be precise, choose constants $C_i > 0$ $(i \in \mathbb{N})$ with $\lim_{i\to\infty} C_i = +\infty$. In the induction step, we are given a noncritical holomorphic function f_i on a neighbourhood of X_i in X. Next, we define f_i on a neighbourhood of Γ_i' in X to be an arbitrary noncritical holomorphic function satisfying $\min_{\Gamma_i'} |f_i| > C_i$. Since $X_i \cup \Gamma_i'$ is a compact $\mathscr{O}(X)$-convex set, we can apply [16, Theorem 2.1] to find a noncritical holomorphic function f_{i+1} on a neighbourhood of X_{i+1} in X which approximates f_i on $X_i \cup \Gamma_i'$. Assuming that the approximation is close enough at every step, the sequence f_i converges uniformly on compacts in X to a noncritical holomorphic function $f : X \to \mathbb{C}$ satisfying $\min_{\Gamma_i'} |f| > C_i$ for every $i \in \mathbb{N}$. Pick a divergent path $\gamma : [0,1) \to X$ contained in a level set of f and choose $i_0 \in \mathbb{N}$ such that $\gamma(0) \in i_0\mathbb{B}^N$. Note that γ also diverges in $\mathbb{C}^N$ since X is a closed noncompact submanifold. We see as before that there is an integer $i_1 \geq i_0$ such that γ avoids the labyrinth Γ_i', and hence also the labyrinth $\Gamma_i \subset \mathbb{C}^N$ for all $i \geq i_1$. From (2.3) and (2.4) it follows that

$$\int_0^1 \mathfrak{g}(\gamma(t), \dot\gamma(t))dt \ \geq \ \sum_{i=i_1}^{\infty} c_i M_i = +\infty.$$

Hence, every nonempty level set $f^{-1}(c)$ is a closed complete complex hypersurface in X, which is smooth since f has no critical points.

The same argument gives nonsingular $\mathfrak{g}$-complete holomorphic foliations of X of any dimension $k \geq \left[\frac{n}{2}\right]$. In this case, one inductively uses the Oka-Weil type approximation theorem for holomorphic submersions (see [16, Theorem 2.5]) in the above proof; we leave out the obvious details. The same holds for $1 \leq k < \left[\frac{n}{2}\right]$ if we assume the existence of a surjective complex vector bundle map $TX \to X \times \mathbb{C}^q$ with $q = n-k > \left[\frac{n+1}{2}\right]$. □

Using the full power of [16, Theorem 2.5] we obtain the following stronger statement.

Theorem 2.3. *Let X be a closed complex submanifold of dimension $n > 1$ in $\mathbb{C}^N$ endowed with a Riemannian metric $\mathfrak{g}$. For every integer k with*

$N - \left[\frac{n+1}{2}\right] \le k \le N-1$ there is a nonsingular holomorphic foliation $\mathcal{F}$ of dimension k on $\mathbb{C}^N$, given by a holomorphic submersion $f : \mathbb{C}^N \to \mathbb{C}^{N-k}$, such that every leaf $\mathcal{F}_z$ is transverse to X and the induced foliation on X is $\mathfrak{g}$-complete.

Proof. Theorem 1.1 furnishes a holomorphic submersion $f_0 : X \to \mathbb{C}^q$ with $\mathfrak{g}$-complete leaves, where $q = N - k \le \left[\frac{n+1}{2}\right]$. Clearly, f_0 extends to a holomorphic submersion $U \to \mathbb{C}^q$ from an open neighbourhood U of X in $\mathbb{C}^N$. By [16, Theorem 2.5] there is a holomorphic submersion $f : \mathbb{C}^N \to \mathbb{C}^q$ which agrees with f_0 to the second order along X. The foliation on $\mathbb{C}^N$ determined by f then clearly satisfies Theorem 2.3. Furthermore, given a Riemannian metric $\mathfrak{h}$ on $\mathbb{C}^N$, one can choose $\mathcal{F}$ to be $\mathfrak{h}$-complete, transverse to X, and such that its trace on X is $\mathfrak{g}$-complete. We leave further details to the reader. □

Remark 2.4. The proof of Theorem 1.1 actually gives for any $1 \le q \le \left[\frac{n+1}{2}\right]$ (where $n = \dim X$) a holomorphic submersion $f : X \to \mathbb{C}^q$ which is unbounded on every divergent path of finite $\mathfrak{g}$-length in X. The same holds also for $q \in \{\left[\frac{n+1}{2}\right] + 1, \ldots, n-1\}$ if there is a surjective complex vector bundle map $TX \to X \times \mathbb{C}^q$. This is in the spirit of Globevnik's main theorem in [21] when X is the ball in $\mathbb{C}^n$ and f is a single holomorphic function, possibly with critical points. Furthermore, one can strengthen the statement in the spirit of those obtained by Charpentier and Kosiński [14] for holomorphic functions on pseudoconvex domains in $\mathbb{C}^n$. In particular, under the above conditions, a holomorphic submersion $f : X \to \mathbb{C}^q$ can be chosen such that the image of any divergent path of finite $\mathfrak{g}$-length in X is everywhere dense in $\mathbb{C}^q$. It follows that the foliation of X defined by f is nonsingular and $\mathfrak{g}$-complete.

3. Proofs of Theorems 1.5, 1.7, and 1.8

We shall need the following result generalizing [5, Lemma 2].

Lemma 3.1. *Let B be a compact polynomially convex set in $\mathbb{C}^n$ $(n > 1)$, and let $\Gamma = \bigcup_{j=1}^m \Gamma_j \subset \mathbb{C}^n \setminus B$ be a union of finitely many pairwise disjoint compact convex sets Γ_j such that the set $B \cup \Gamma$ is polynomially convex. If E is a closed subset of $\mathbb{C}^n$ with unbounded complement, then for any $\epsilon > 0$ there exists an automorphism $\Theta \in \mathrm{Aut}(\mathbb{C}^n)$ such that*

(a) $\Theta(E) \cap \Gamma = \varnothing$, *and*
(b) $|\Theta(z) - z| < \epsilon$ *for all* $z \in B$.

Proof. Pick a compact neighbourhood K_0 of B and compact convex neighbourhoods K_j of Γ_j for $j = 1, \ldots, m$ such that the sets $K_0, \ldots, K_m$ are pairwise disjoint and $K = \bigcup_{j=0}^m K_j$ is polynomially convex. (The existence of such sets follows from standard results on polynomial convexity; see Stout [28].) Let $\Psi_0 = \mathrm{Id} \in \mathrm{Aut}(\mathbb{C}^n)$ be the identity and set $K_0' = K_0$. For every $j = 1, \ldots, m$ we choose an automorphism $\Psi_j \in \mathrm{Aut}(\mathbb{C}^n)$ such that the compact sets $K_j' := \Psi_j(K_j)$ are pairwise disjoint, we have that

$$K_j' \cap (E \cup K_0') = \varnothing \quad \text{for } j = 1, \ldots, m, \tag{3.1}$$

and the union $\bigcup_{j=0}^m K_j'$ is polynomially convex. Such Ψ_j are obtained by squeezing K_j $(j = 1, \ldots, m)$ by a dilation into a small neighbourhood of an interior point, then translating the images into small pairwise disjoint balls around some points in $\mathbb{C}^n \setminus (E \cup K_0')$ (this set is open and nonempty by the assumption on E), and applying [20, Theorem 1.1] to approximate the final map by an automorphism of $\mathbb{C}^n$. By [20, Theorem 2.3] (see also [18, Corollary 4.12.4]), given $\delta > 0$ there is an automorphism $\Psi \in \mathrm{Aut}(\mathbb{C}^n)$ satisfying

$$|\Psi(z) - \Psi_j(z)| < \delta \quad \text{for all } z \in K_j, j = 0, 1, \ldots, m. \tag{3.2}$$

Let $\Theta = \Psi^{-1}$. If $\delta > 0$ is small enough then condition (b) holds, and we have that $\Psi(\Gamma_j) \subset \mathring{K}_j'$ and hence $\Gamma_j \subset \Theta(\mathring{K}_j')$ for every $j = 1, \ldots, m$, which by (3.1) also yields (a). □

Proof of Theorem 1.5. Let $\mathbb{B}$ denote the open unit ball of $\mathbb{C}^n$. Set $r_1 = 1$, $B_1 = r_1\overline{\mathbb{B}} = \overline{\mathbb{B}}$, $\mathcal{F}_1 = \mathcal{F}$, and $E_1 = \mathcal{F}_1(B_1)$; see (1.1). Choose a sequence $M_j > 0$ such that

$$\sum_{j=1}^{\infty} M_j = +\infty. \tag{3.3}$$

Pick a closed ball $B_1' \subset \mathbb{C}^n$ centred at 0 and containing B_1 in its interior. By Lemma 2.1 and Remark 2.2 there is a labyrinth $\Gamma_1 \subset \mathring{B}_1' \setminus B_1$ which enlarges the $\mathfrak{g}$-distance by $M_1 > 0$ such that $B_1 \cup \Gamma_1$ is polynomially convex. Fix a number $0 < \epsilon_1 < 1$. Lemma 3.1 furnishes an automorphism $\phi_1 \in \mathrm{Aut}(\mathbb{C}^n)$ such that

(a$_1$) $\phi_1(E_1) \cap \Gamma_1 = \varnothing$, and
(b$_1$) $|\phi_1(z) - z| < \epsilon_1$ for all $z \in B_1$.

In the second step, we choose a number $r_2 > 2$ such that $\phi_1(\overline{\mathbb{B}}) \subset (r_2 - 1)\mathbb{B}$ and set

$$B_2 = r_2\overline{\mathbb{B}}, \quad \mathcal{F}_2 = (\phi_1)_*\mathcal{F}_1, \quad E_2 = \mathcal{F}_2(B_2).$$

Pick a slightly bigger ball $B_2' \supset B_2$ containing B_2 in its interior. By Lemma 2.1 there is a labyrinth $\Gamma_2 \subset \mathring{B}_2' \setminus B_2$ which enlarges the $\mathfrak{g}$-distance by $M_2 > 0$. Given $\epsilon_2 > 0$, Lemma 3.1 furnishes an automorphism $\phi_2 \in \mathrm{Aut}(\mathbb{C}^n)$ such that

(a$_2$) $\phi_2(E_2) \cap \Gamma_2 = \varnothing$, and
(b$_2$) $|\phi_2(z) - z| < \epsilon_2$ for all $z \in B_2$.

Continuing inductively, we obtain an increasing sequence of numbers $r_j > 0$, balls

$$B_j = r_j\overline{\mathbb{B}} \subset B_j' \subset B_{j+1} = r_{j+1}\overline{\mathbb{B}}, \tag{3.4}$$

labyrinths $\Gamma_j \subset \mathring{B}_j' \setminus B_j$, constants $\epsilon_j > 0$, automorphisms $\phi_j \in \mathrm{Aut}(\mathbb{C}^n)$, and foliations $\mathcal{F}_j$ of $\mathbb{C}^n$ such that, setting

$$E_j = \mathcal{F}_j(B_j), \Phi_j = \phi_j \circ \cdots \circ \phi_1 \in \mathrm{Aut}(\mathbb{C}^n), \text{ and } \mathcal{F}_{j+1} = (\Phi_j)_*\mathcal{F}, \tag{3.5}$$

the following conditions hold for every for $j = 1, 2, \ldots$:

(i$_j$) $\phi_j(E_j) \cap \Gamma_j = \varnothing$.
(ii$_j$) $|\phi_j(z) - z| < \epsilon_j$ for $z \in B_j$.

(iii$_j$) $0 < \epsilon_{j+1} < \frac{1}{2}\min\{\epsilon_j, \mathrm{dist}(\phi_j(E_j), \Gamma_j)\}$. (Note that the set $\phi_j(E_j)$ is closed and Γ_j is compact, so these sets are at positive distance by (ii$_j$).)

(iv$_j$) $\Phi_j(j\overline{\mathbb{B}}) \cup B'_j \subset (r_{j+1} - 1)\mathbb{B}$.

(v$_j$) The labyrinth Γ_j enlarges the $\mathfrak{g}$-distance by M_j and $B_j \cup \Gamma_j$ is polynomially convex.

Assuming that we have obtained these quantities for indices $\leq j$, the induction step goes as follows. Let $\mathcal{F}_{j+1}$ and E_{j+1} be given by (3.5). Pick a number ϵ_{j+1} satisfying (iii$_j$) and a number $r_{j+1} > r_j$ satisfying (iv$_j$), and set $B_{j+1} = r_{j+1}\overline{\mathbb{B}}$. Choose a ball $B'_{j+1} \supsetneq B_{j+1}$. Lemma 2.1 gives a labyrinth $\Gamma_{j+1} \subset \mathring{B}'_{j+1} \setminus B_{j+1}$ satisfying (v$_{j+1}$). Then, Lemma 3.1 gives an automorphism $\phi_{j+1} \in \mathrm{Aut}(\mathbb{C}^n)$ satisfying (i$_{j+1}$) and (ii$_{j+1}$), closing the induction.

We claim that the sequence $\Phi_j \in \mathrm{Aut}(\mathbb{C}^n)$ converges uniformly on compacts in $\mathbb{C}^n$ to an automorphism $\Phi \in \mathrm{Aut}(\mathbb{C}^n)$. Indeed, since

$$|\phi_j(z) - z| < \epsilon_j < 1 < \mathrm{dist}(B_j, \mathbb{C}^n \setminus B_{j+1}) \quad \text{for every } j = 1, 2, \ldots$$

and $\sum_{j=1}^{\infty} \epsilon_j < \infty$ (see (ii$_j$)–(iv$_j$)), the sequence Φ_j converges uniformly on compacts in the domain $\Omega = \bigcup_{j=1}^{\infty} \Phi_j^{-1}(B_j) \subset \mathbb{C}^n$ to a biholomorphic map $\Phi : \Omega \to \mathbb{C}^n$ onto $\mathbb{C}^n$ (see [18, Corollary 4.4.2]). Furthermore, condition (iv$_j$) ensures that $j\overline{\mathbb{B}} \subset \Omega$ for every j, so $\Omega = \mathbb{C}^n$ and hence $\Phi \in \mathrm{Aut}(\mathbb{C}^n)$. Hence, the sequence of foliations $\mathcal{F}_{j+1} = (\Phi_j)_*\mathcal{F}$ converges uniformly on compacts in $\mathbb{C}^n$ to a limit foliation $\mathcal{G} = \Phi_*(\mathcal{F})$.

It remains to show that the foliation $\mathcal{G}$ is $\mathfrak{g}$-complete. We must show that for any divergent path $\gamma : [0, 1) \to \mathbb{C}^n$ contained in a leaf of $\mathcal{F}$, the path $\tilde{\gamma} = \Phi \circ \gamma : [0, 1) \to \mathbb{C}^n$ has infinite $\mathfrak{g}$-length. (Since the foliation $\mathcal{F}$ is assumed to have closed leaves of positive dimension, every divergent path in a leaf of $\mathcal{F}$ is also divergent in $\mathbb{C}^n$. Note that $\tilde{\gamma}$ is a divergent path in $\mathbb{C}^n$ contained in a leaf of $\mathcal{G}$.) For every $k \in \mathbb{N}$ let $\gamma_k = \Phi_k \circ \gamma : [0, 1) \to \mathbb{C}^n$; this is a divergent path contained in a leaf of the foliation $\mathcal{F}_{k+1} = (\Phi_k)_*\mathcal{F}$. Pick $j \in \mathbb{N}$ such that $\gamma(0) \in j\mathbb{B}$. By (iv$_j$) we have that $\gamma_j(0) = \Phi_j(\gamma(0)) \in (r_{j+1} - 1)\mathbb{B} \subset B_{j+1}$, and hence $\gamma_j([0, 1)) \subset E_{j+1}$ (see (3.5)). Conditions (i$_j$)–(iii$_j$) imply that for every $k > j$ we have that $\gamma_k(0) \in B_{j+1} = r_{j+1}\overline{\mathbb{B}}$ and the trace of γ_k avoids the labyrinths Γ_i for $i = j + 1, \ldots, k$. Since γ_k is a divergent path in $\mathbb{C}^n$, its $\mathfrak{g}$-length is at least $\sum_{i=j+1}^{k} M_i$ by (i$_i$). As $k \to \infty$, it follows that γ_k converges to a divergent path $\tilde{\gamma} = \Phi \circ \gamma : [0, 1) \to \mathbb{C}^n$ which has infinite $\mathfrak{g}$-length in view of (3.3). $\square$

Proof of Theorem 1.7. The proof will follow that of Theorem 1.5 if we find labyrinths Γ in the given Stein manifold X which increase the length of divergent paths avoiding Γ by a given amount and whose pieces are holomorphically contractible in X. (Note that the labyrinths used in the proof of Theorem 1.1, which are obtained by intersecting labyrinths in $\mathbb{C}^N$ whose pieces are balls in affine real hyperplanes with the embedded submanifold $X \subset \mathbb{C}^N$, need not satisfy this property.) To this end, we shall prove the following lemma whose first part generalizes Lemma 2.1, as well as [13, Lemma 2.4] by Charpentier and Kosiński, while the second part is an analogue of Lemma 3.1 adjusted to this situation. As before, $\mathbb{B}$ denotes the unit ball of $\mathbb{C}^N$.

Lemma 3.2. *Let X be a closed complex submanifold of $\mathbb{C}^N$, and let $\mathfrak{g}$ be a Riemannian metric on X. Given numbers $0 < r < s$ and $M > 0$, there is a compact set $\Gamma \subset X \cap (s\mathbb{B} \backslash r\overline{\mathbb{B}})$ satisfying the following conditions.*

(a) *The compact set $\Gamma \cup (X \cap (r\overline{\mathbb{B}}))$ is $\mathscr{O}(X)$-convex.*
(b) *$\Gamma = \bigcup_{i=1}^m \Gamma_i$ is the union of finitely many pairwise disjoint compact sets Γ_i such that every Γ_i is convex in a certain local holomorphic chart on X.*
(c) *Every piecewise smooth path $\gamma : [0,1] \to X \setminus \Gamma$ with $\gamma(0) \in X \cap r\overline{\mathbb{B}}$ and $\gamma(1) \in X \backslash s\mathbb{B}$ has $\mathfrak{g}$-length at least M.*

Given such Γ and assuming in addition that X has the density property, then for every closed subset $E \subsetneq X$ whose complement $X \setminus E$ is not relatively compact in X and for any $\epsilon > 0$ there exists an automorphism $\Theta \in \mathrm{Aut}(X)$ such that

(A) *$\Theta(E) \cap \Gamma = \varnothing$, and*
(B) *$|\Theta(z) - z| < \epsilon$ for all $z \in X \cap r\overline{\mathbb{B}}$.*

Proof. We construct the labyrinth Γ in two stages.

In the first stage, we apply Lemma 2.1 to find a labyrinth $\Gamma^0 = \bigcup_{j=1}^k \Gamma_j^0$ in the spherical shell $S_{r,s} = s\mathbb{B} \setminus r\overline{\mathbb{B}} \subset \mathbb{C}^N$ which increases the $\mathfrak{g}$-length in X by the given amount $M > 0$ and the set $\Gamma^0 \cup r\overline{\mathbb{B}}$ is polynomially convex. Set $B_0 = X \cap r\overline{\mathbb{B}}$. It follows that each of the compact sets $\Gamma_j^X = \Gamma_j^0 \cap X$ $(j = 1, \ldots, k)$, $\Gamma^X = \bigcup_{j=1}^k \Gamma_j^X$, and $\Gamma^X \cup B_0$ are $\mathscr{O}(X)$-convex. The construction of such labyrinths in [9] shown that the connected components Γ_j^0 of Γ^0 (which are closed balls in affine real hyperplanes in $\mathbb{C}^N$) may be chosen with arbitrarily small diameter (this is an immediate consequence of Pythagoras theorem, see [1, Lemma 2.3]); in particular we can choose them small enough such that $\Gamma_j^X = \Gamma_j^0 \cap X$ is contained in a holomorphic coordinate chart $U_j \subset X$ which is Runge in X for every $j = 1, \ldots, k$. (Most of the sets Γ_j^X are empty, and we discard them from the above collection and adjust the indexes accordingly.) More precisely, for any $\Gamma_j^X \neq \varnothing$ we pick a point $p_j \in \Gamma_j^X$ and let $\Sigma_j = T_{p_j} X \cong \mathbb{C}^n$ with $n = \dim X$ be the tangent plane of X at p_j, with the orthogonal $\mathbb{C}$-linear projection $\pi_j : \mathbb{C}^N \to \Sigma_j$. Then, we may assume that there is a Runge neighbourhood $U_j \subset X$ of Γ_j^X such that the restricted projection

$$\pi_j|_{U_j} : U_j \to \pi_j(U_j) = V_j \subset \Sigma_j \cong \mathbb{C}^n$$

is a biholomorphic map onto a ball $V_j \subset \Sigma_j$ around the point p_j. It follows that a compact set $K \subset U_j$ is $\mathscr{O}(X)$-convex if and only if $\pi_j(K)$ is polynomially convex in $\Sigma_j \cong \mathbb{C}^n$.

In the second stage, we choose for every $j = 1, \ldots, k$ a compact $\mathscr{O}(X)$-convex neighbourhood $B_j \subset U_j$ of Γ_j^X such that the sets $B_0 = X \cap r\overline{\mathbb{B}}, B_1, \ldots, B_k$ are pairwise disjoint and $\bigcup_{j=0}^k B_j$ is $\mathscr{O}(X)$-convex. (Note that every compact $\mathscr{O}(X)$-set has a basis of compact $\mathscr{O}(X)$-convex neighbourhoods.) We now apply the construction of labyrinths with holomorphically contractible pieces in pseudoconvex domains in [13, Theorem 1.1] by Charpentier and Kosiński, to find for each $j = 1, \dots, k$ a labyrinth $\Gamma_j \subset \mathring{B}_j \backslash \Gamma_j^X$ which is $\mathscr{O}(B_j)$-convex (and hence $\mathscr{O}(X)$-convex), its connected components are holomorphically contractible sets in X (in fact, under the projection $\pi_j : U_j \to V_j \subset \Sigma_j$ they correspond to closed balls in affine real hyperplanes of $\Sigma_j \cong \mathbb{C}^n$), and the $\mathfrak{g}$-distance in $X \setminus \Gamma_j$ from Γ_j^X to bB_j is at least M. In other words, any path in U_j from Γ_j^X to bB_j which avoids Γ_j has $\mathfrak{g}$-length at least M.

We claim that the labyrinth $\Gamma = \bigcup_{j=1}^k \Gamma_j \subset X \cap S_{r,s}$ satisfies conditions (a)–(c) in the lemma. The set Γ_j is $\mathscr{O}(B_j)$ convex for every $j = 1, \dots, k$. Since $\bigcup_{j=0}^k B_j$ is $\mathscr{O}(X)$-convex, it follows that $\Gamma \cup B_0 = \Gamma \cup (X \cap r\overline{\mathbb{B}})$ is $\mathscr{O}(X)$-convex, so (a) holds. Condition (b) holds by the construction. Let γ be a path as in part (c) avoiding Γ. If γ avoids the initial labyrinth $\Gamma^X = \Gamma^0 \cap X$ then its $\mathfrak{g}$-length is at least M by the choice of Γ^0. If on the other hand γ intersects Γ_j^X for some $j \in \{1, \dots, k\}$, then γ connects a point in bB_j to Γ_j^X avoiding the labyrinth Γ_j, hence its $\mathfrak{g}$-length is at least M by the choice of Γ_j. This proves (c).

The second part of the lemma is obtained by following the proof of Lemma 3.1. The only point deserving an explanation is the construction of the automorphisms Ψ_j ($j = 1, \dots, k$) and Ψ in the proof of Lemma 3.1. Recall that the labyrinth $\Gamma_j = \bigcup_{i=1}^{k_j} \Gamma_{j,i}$ has holomorpically contractible $\mathscr{O}(X)$-convex connected components $\Gamma_{j,i}$. More precisely, there is a 1-parameter family of biholomorphic contractions on a pseudoconvex Runge neighbourhood of $\Gamma_{j,i}$ in X which shrinks this set within itself almost to a point. We can then move the images of these small sets to $X \setminus (B_0 \cup E)$ by an isotopy of biholomorphic maps through pseudoconvex Runge domains in X, ensuring that the traces of these isotopies for $j = 1, \dots, k$ are pairwise disjoint, contained in $X \setminus B_0$, and their unions (together with B_0) are Runge in X for every value of the parameter. (The set B_0 remains fixed during this process.) Assuming that the Stein manifold X has the density property, we can apply the approximation theorem for such isotopies of injective holomorphic maps (see [18, Theorem 4.10.5]) to obtain an automorphism $\Psi \in \mathrm{Aut}(X)$ as in the proof of Lemma 3.1. The remainder of the proof is exactly as in the case $X = \mathbb{C}^n$. □

With Lemma 3.2 in hand, we obtain Theorem 1.7 by following the proof of Theorem 1.5, and we leave the obvious details to the reader. □

Proof of Theorem 1.8. Let $\mathcal{F}_0$ be a proper holomorphic foliation on a Stein manifold X with the density property, and let Ω be a pseudoconvex Runge domain in X. Using Lemma 3.2 inductively, we find a normal exhaustion $\Omega_1 \Subset \Omega_2 \Subset \cdots \subset \bigcup_{i=1}^\infty \Omega_i = \Omega$ by open relatively compact pseudoconvex Runge domains and for each $i = 1, 2, \dots$ a labyrinth $\Gamma_i \subset \Omega_{i+1} \backslash \overline{\Omega}_i$ having the properties (a)–(c) in Lemma 3.2 for a given constant $M_i > 0$ chosen such that $\sum_i M_i = +\infty$. Furthermore, we can ensure that the compact set $\Gamma_{i+1} \cup \overline{\Omega}_i$ is $\mathscr{O}(\Omega)$-convex (and hence also $\mathscr{O}(X)$-convex) for every $i \in \mathbb{N}$. Since the components of the labyrinth $\Gamma = \bigcup_{i=1}^\infty \Gamma_i$ are holomorphically contractible, we can apply the proof of Theorem 1.5 to inductively twist the foliation $\mathcal{F}_0$ by a sequence of automorphisms $\Phi_i \in \mathrm{Aut}(X)$ ($i = 1, 2, \dots$), chosen so that they converge uniformly on compacts in Ω, and the leaves of the foliations $\mathcal{F}_i = (\Phi_i)_* \mathcal{F}_{i-1}$ avoid more and more components of Γ as $i \to \infty$. The limit holomorphic foliation $\mathcal{F} = \lim_{i\to\infty} \mathcal{F}_i$ on Ω is such that every leaf avoids all but finitely many labyrinths Γ_i, and hence it is $\mathfrak{g}$-complete. Furthermore, the construction ensures that every leaf is a pseudoconvex Runge domain in a leaf of the initial foliation $\mathcal{F}_0$ on X. The details can be found (for the case $X = \mathbb{C}^n$) in [9] and [5]. □

In Theorems 1.5, 1.7, and 1.8 we focused on constructing complete nonsingular holomorphic foliations. However, the proofs of these results also apply to individual closed complex submanifolds and yield the following result.

Theorem 3.3. *Let X be a Stein manifold with the density property, and let Y be a closed complex submanifold of X. The following assertions hold.*

(i) *Given a Riemannian metric $\mathfrak{g}$ on X there is a holomorphic automorphism $\Phi \in \mathrm{Aut}(X)$ such that the submanifold $\Phi(Y)$ is $\mathfrak{g}$-complete.*
(ii) *Let Ω be a pseudoconvex Runge domain in X such that $\Omega \cap Y \neq \varnothing$. Given a Riemannian metric $\mathfrak{g}$ on Ω and a connected compact subset $K \subset \Omega \cap Y$, there is a $\mathfrak{g}$-complete closed complex submanifold of Ω which is biholomorphic to a pseudoconvex Runge domain in Y containing K.*

By combining assertion (i) in Theorem 3.3 with the embedding theorems for Stein manifolds (see [12, 15, 27]) we obtain the following corollary.

Corollary 3.4. *Every Stein manifold Y of dimension $n \geq 1$ admits a proper $\mathfrak{g}$-complete holomorphic embedding in $(\mathbb{C}^N, \mathfrak{g})$ for any $N \geq \max\left\{3, \left[\frac{3n}{2}\right] + 1\right\}$, and a proper $\mathfrak{g}$-complete holomorphic embedding in $(X, \mathfrak{g})$ for any Riemannian Stein manifold X with the density property of dimension $\dim X \geq 2n+1$.*

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Declarations

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