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Mann-Type Inertial Projection and Contraction Method for Solving Split Pseudomonotone Variational Inequality Problem with Multiple Output Sets

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Abstract. In this paper, we study the concept of split variational inequality problem with multiple output sets when the cost operators are pseudomonotone and non-Lipschitz. We introduce a new Mann-type inertial projection and contraction method with self-adaptive step sizes for approximating the solution of the problem in the framework of Hilbert spaces. Under some mild conditions on the control parameters and without prior knowledge of the operator norms, we prove a strong convergence theorem for the proposed algorithm. We point out that while the cost operators are non-Lipschitz, our proposed method does not require any linesearch method but uses a more efficient self-adaptive step size technique that generates a non-monotonic sequence of step sizes. Finally, we apply our result to study certain classes of optimization problems and we present several numerical experiments to illustrate the applicability of the proposed method. Several of the existing results in the literature could be viewed as special cases of our result in this study.

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Keywords. Split inverse problems, variationl inequalities, non-Lipschitz operators, projection & contraction method, inertial technique.

1. Introduction

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and induced norm $||\cdot||$. Let C be a nonempty, closed and convex subset of H, and let $A : H \to H$ be a mapping. The variational inequality problem (VIP) is formulated as finding a point $p \in C$ such that

$$\langle x - p, Ap \rangle \geq 0, \quad \forall\, x \in C. \tag{1.1}$$

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We denote the solution set of the VIP (1.1) by $VI(C, A)$. Variational inequality theory was first introduced independently by Fichera [13] and Stampacchia [34]. The VIP is a fundamental problem in optimization theory, which unifies several important concepts in applied mathematics, such as the necessary network equilibrium problems, optimality conditions, systems of nonlinear equations and complementarity problems (e.g. see [4,5,20]). In the recent years, the VIP has attracted the attention of researchers due to its numerous applications in diverse fields, such as in optimization theory, economics, structural analysis, operations research, sciences and engineering (see [10,17,36] and the references therein). Several authors have proposed and studied different iterative methods for approximating the solution of the VIP (see [2,7,16,25,26] and references therein).

The *split inverse problem* (SIP) is another area of research which has recently received great research attention (see [42] and the references therein) due to its several applications in different fields, for instance, in signal processing, phase retrieval, medical image reconstruction, data compression, intensity-modulated radiation therapy, etc. (e.g. see [8,9,18,22,29]). The SIP model is formulated as follows:

$$\text{Find } \hat{x} \in H_1 \quad \text{that solves } \mathrm{IP}_1 \tag{1.2}$$

such that

$$\hat{y} := T\hat{x} \in H_2 \quad \text{solves } \mathrm{IP}_2, \tag{1.3}$$

where H_1 and H_2 are real Hilbert spaces, IP_1 denotes an inverse problem formulated in H_1 and IP_2 denotes an inverse problem formulated in H_2, and $T : H_1 \to H_2$ is a bounded linear operator.

In 1994, Censor and Elfving in [9] introduced the first instance of the SIP called the *split feasibility problem* (SFP) for modelling inverse problems that arise from medical image reconstruction. The SFP finds application in the control theory, approximation theory, signal processing, geophysics, communications, biomedical engineering, etc. [8,23,31,32]. Let C and Q be nonempty, closed and convex subsets of Hilbert spaces H_1 and H_2, respectively, and let $T : H_1 \to H_2$ be a bounded linear operator. The SFP is defined as follows:

$$\text{Find } \hat{x} \in C \text{ such that } \hat{y} = T\hat{x} \in Q. \tag{1.4}$$

Several iterative algorithms for solving the SFP (1.4) have been constructed and investigated by researchers (see, e.g. [8,23,24] and the references therein). An important generalization of the SFP is the *split variational inequality problem* (SVIP) introduced by Censor et al. [10]. The SVIP is formulated as follows:

$$\text{Find } \hat{x} \in C \text{ that solves } \langle A_1\hat{x}, x - \hat{x}\rangle \geq 0, \quad \forall x \in C \tag{1.5}$$

such that

$$\hat{y} = T\hat{x} \in H_2 \text{ solves } \langle A_2\hat{y}, y - \hat{y}\rangle \geq 0, \quad \forall y \in Q, \tag{1.6}$$

where $A_1 : H_1 \to H_1, A_2 : H_2 \to H_2$ are single-valued operators. Several authors have studied and proposed different iterative methods for approximating the solution of SVIP (see [19,21,37] and the references therein).

In 2020, Reich and Tuyen [28] introduced and studied the concept of *split feasibility problem with multiple output sets* in Hilbert spaces (SFPMOS), which is formulated as follows: Find a point $u^\dagger$ such that

$$u^\dagger \in \Gamma := C \cap \left(\bigcap_{i=1}^{N} T_i^{-1}(Q_i) \right) \neq \emptyset. \tag{1.7}$$

where $T_i : H \to H_i,\ i = 1, 2, ..., N$, are bounded linear operators, C and Q_i are nonempty, closed and convex subsets of Hilbert spaces H and $H_i, i = 1, 2, \ldots, N$, respectively.

Moreover, Reich and Tuyen [30] proposed the following two algorithms for approximating the solution of SFPMOS (1.7) in Hilbert spaces:

$$x_{n+1} = P_C \left[x_n - \gamma_n \sum_{i=1}^{N} T_i^*(I - P_{Q_i})T_i x_n \right], \tag{1.8}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_C \left[x_n - \gamma_n \sum_{i=1}^{N} T_i^*(I - P_{Q_i})T_i x_n) \right], \tag{1.9}$$

where $f : C \to C$ is a strict contraction, $\{\gamma_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$. The authors obtained weak and strong convergence result for Algorithm (1.8) and Algorithm (1.9), respectively.

In this paper, we study the split variational inequality problem with multiple output sets. Let $H, H_i, i = 1, 2, ..., N$, be real Hilbert spaces and let C, C_i be nonempty, closed and convex subsets of real Hilbert spaces H and $H_i, i = 1, 2, ..., N$, respectively. Let $T_i : H \to H_i, i = 1, 2, ..., N$, be bounded linear operators and let $A : H \to H, A_i : H_i \to H_i, i = 1, 2, ..., N$, be single-valued operators. The *split variational inequality problem with multiple output sets* (SVIPMOS) is formulated as finding a point $x^* \in C$ such that

$$x^* \in \Omega := VI(C, A) \cap (\bigcap_{i=1}^{N} T_i^{-1} VI(C_i, A_i)) \neq \emptyset. \tag{1.10}$$

It is clear that the SVIPMOS (1.10) generalizes the SFPMOS (1.7).

In the last couple of years, developing iterative methods with a high rate of convergence for solving optimization problems has become of great interest to researchers. One of the approaches employed by researchers to achieve this objective is the inertial technique. This technique originates from an implicit time discretization method (the heavy ball method) of second-order dynamical systems. In recent years, several authors have constructed highly efficient iterative methods by employing the inertial technique, see, e.g., [1, 3, 11, 14, 38, 40].

In this paper, we propose and analyze a new Mann-type inertial projection and contraction algorithm with self-adaptive step sizes for approximating the solution SVIPMOS (1.10) when the cost operators are pseudomonotone and non-Lipschitz. While the cost operators are non-Lipschitz, our proposed method does not involve any line search method but uses a more efficient self-adaptive step size technique which generates a non-monotonic sequence

of step sizes. Furthermore, we prove that the sequence generated by our proposed method converges to the minimum-norm solution of the problem in Hilbert spaces. Finally, we apply our result to study certain classes of optimization problems and we present several numerical experiments to demonstrate the applicability of our proposed algorithm.
The outline of the paper is as follows: In Sect. 2, we give some definitions and results required for the convergence analysis. In Sect. 3, we present the proposed algorithm and in Sect. 4 we analyze the convergence of our proposed method. In Sect. 5 we apply our result to study certain classes of optimization problems, and in Sect. 6 we carry out several numerical experiments with graphical illustrations. Finally, we give some concluding remarks in Sect. 7.

2. Preliminaries

Definition 2.1. [2,16] An operator $A : H \to H$ is said to be

(i) *α-strongly monotone*, if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|x - y\|^2, \ \forall \ x, y \in H;$$

(ii) *monotone*, if

$$\langle x - y, Ax - Ay\rangle \geq 0, \quad \forall \ x, y \in H;$$

(iii) *pseudomonotone*, if

$$\langle Ay, x - y\rangle \geq 0 \implies \langle Ax, x - y\rangle \geq 0, \ \forall x, y \in H,$$

(iv) *L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$||Ax - Ay|| \leq L||x - y||, \quad \forall \ x, y \in H;$$

(v) *uniformly continuous*, if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that

$$\|Ax - Ay\| < \epsilon \quad \text{whenever} \quad \|x - y\| < \delta, \quad \forall x, y \in H;$$

Remark 2.2. We note that the following implications hold: $(i) \implies (ii) \implies (iii)$ but the converses are not generally true. We also point out that uniform continuity is a weaker notion than Lipschitz continuity.

It is well known that if D is a convex subset of H, then $A : D \to H$ is uniformly continuous if and only if, for every $\epsilon > 0$, there exists a constant $K < +\infty$ such that

$$\|Ax - Ay\| \leq K\|x - y\| + \epsilon \quad \forall x, y \in D. \tag{2.1}$$

Lemma 2.3. [27,39] *Let H be a real Hilbert space. Then the following results hold for all $x, y \in H$ and $\delta \in (0, 1)$:*

(i) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle$;
(ii) $||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2$;
(iii) $||\delta x + (1 - \delta)y||^2 = \delta||x||^2 + (1 - \delta)||y||^2 - \delta(1 - \delta)||x - y||^2$.

Lemma 2.4. ([33]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers,* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *with* $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\{b_n\}$ *be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n \quad \textit{for all } n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0,$ *then* $\lim_{n\to\infty} a_n = 0.$

Lemma 2.5. [35] *Suppose* $\{\lambda_n\}$ *and* $\{\theta_n\}$ *are two nonnegative real sequences such that*

$$\lambda_{n+1} \leq \lambda_n + \phi_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty}\phi_n < \infty$*, then* $\lim_{n\to\infty}\lambda_n$ *exists.*

Lemma 2.6. [12] *Consider the* VIP (1.1) *with* C *being a nonempty, closed, convex subset of a real Hilbert space* H *and* $A : C \to H$ *being pseudomonotone and continuous. Then* p *is a solution of* VIP *(1.1) if and only if*

$$\langle Ax, x-p\rangle \geq 0, \ \forall x \in C$$

3. Main Results

In this section, we present our proposed algorithm for solving the SVIPMOS (1.10). We analyze the convergence of the proposed method under the following conditions:
Let C, C_i be nonempty, closed and convex subsets of real Hilbert spaces $H, H_i, i = 1,2,...,N$, respectively, and let $T_i : H \to H_i, i = 1,2,...,N$, be bounded linear operators with adjoints T_i^*. Let $A : H \to H, A_i : H_i \to H_i, i = 1,2,...,$
N, be uniformly continuous pseudomonotone operators satisfying the following property:

$$\text{whenever } \{T_i x_n\} \subset C_i, \ T_i x_n \rightharpoonup T_i z, \ \text{then } \|A_i T_i z\| \leq \liminf_{n\to\infty}\|A_i T_i x_n\|, \ i = 0,1,2\ldots,N, C_0 = C, A_0 = A, T_0 = I^H. \quad (3.1)$$

Moreover, we assume that the solution set $\Omega \neq \emptyset$ and the control parameters satisfy the following conditions:

Assumption A. (A1) $\{\alpha_n\} \subset (0,1), \lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = +\infty,$ $\lim_{n\to\infty}\frac{\epsilon_n}{\alpha_n} = 0, \{\xi_n\} \subset [a,b] \subset (0, 1-\alpha_n), \theta > 0;$

(A2) $0 < c_i < c_i' < 1, 0 < \phi_i < \phi_i' < 1, 0 < k_i < k_i' < 2, \{c_{n,i}\}, \{\phi_{n,i}\}, \{k_{n,i}\} \subset \mathbb{R}_+, \lim_{n\to\infty} c_{n,i} = \lim_{n\to\infty}\phi_{n,i} = \lim_{n\to\infty} k_{n,i} = 0, \lambda_{1,i} > 0, \ \forall \ i = 0,1,2,\ldots,N;$

(A3) $\{\rho_{n,i}\} \subset \mathbb{R}_+, \sum_{n=1}^{\infty}\rho_{n,i} < +\infty, 0 < a_i \leq \delta_{n,i} \leq b_i < 1, \sum_{i=0}^{N}\delta_{n,i} = 1$ for each $n \geq 1$.

Now, the algorithm is presented as follows:

Remark 3.2. By conditions (C1) and (C2), it follows from (3.2) that

$$\lim_{n\to\infty}\theta_n||x_n - x_{n-1}|| = 0 \quad \text{and} \quad \lim_{n\to\infty}\frac{\theta_n}{\alpha_n}||x_n - x_{n-1}|| = 0.$$

Algorithm 3.1.

Step 0. Select initial points $x_0, x_1 \in H$. Let $C_0 = C,\ T_0 = I^H,\ A_0 = A$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta,\ \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.2}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$y_{n,i} = P_{C_i}(T_i w_n - \lambda_{n,i} A_i T_i w_n)$$

$$\lambda_{n+1,i} = \begin{cases} \min\{\frac{(c_{n,i}+c_i)\|T_i w_n - y_{n,i}\|}{\|A_i T_i w_n - A_i y_{n,i}\|},\ \lambda_{n,i} + \rho_{n,i}\}, & \text{if } \ A_i T_i w_n - A_i y_{n,i} \neq 0, \\ \lambda_{n,i} + \rho_{n,i}, & \text{otherwise.} \end{cases}$$

$$z_{n,i} = T_i w_n - \beta_{n,i} r_{n,i},$$

where

$$r_{n,i} = T_i w_n - y_{n,i} - \lambda_{n,i}(A_i T_i w_n - A_i y_{n,i})$$

and

$$\beta_{n,i} = \begin{cases} (k_i + k_{n,i})\frac{\langle T_i w_n - y_{n,i}, r_{n,i}\rangle}{\|r_{n,i}\|^2}, & \text{if } \ r_{n,i} \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Step 4. Compute

$$b_n = \sum_{i=0}^{N} \delta_{n,i}\big(w_n + \eta_{n,i} T_i^*(z_{n,i} - T_i w_n)\big),$$

where

$$\eta_{n,i} = \begin{cases} \frac{(\phi_{n,i}+\phi_i)\|T_i w_n - z_{n,i}\|^2}{\|T_i^*(T_i w_n - z_{n,i})\|^2}, & \text{if } \ \|T_i^*(T_i w_n - z_{n,i})\| \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Step 5. Compute

$$x_{n+1} = (1 - \alpha_n - \xi_n) w_n + \xi_n b_n.$$

Set $n := n + 1$ and return to **Step 1**.

Remark 3.3. Observe that while the cost operators $A_i, i = 0, 1, 2, \ldots, N$ are non-Lipschitz, our method does not require any linesearch technique, which could be computationally too expensive too implement. Rather, we employ self-adaptive step sizes that only require simple computations of known information per iteration.

4. Convergence Analysis

First, we prove some lemmas needed for our strong convergence theorem.

Lemma 4.1. *Suppose* $\{\lambda_{n,i}\}$ *is the sequence generated by Algorithm 3.1 such that Assumption A holds. Then* $\{\lambda_{n,i}\}$ *is well defined for each* $i = 0, 1, 2, \ldots, N$ *and* $\lim_{n\to\infty} \lambda_{n,i} = \lambda_{1,i} \in [\min\{\frac{c_i}{M_i}, \lambda_{1,i}\}, \lambda_{1,i} + \Phi_i]$, *where* $\Phi_i = \sum_{n=1}^{\infty} \rho_{n,i}$.

Proof. Since A_i is uniformly continuous for each $i = 0, 1, 2, \ldots, N$, then by (2.1) we have that for any given $\epsilon_i > 0$, there exists $K_i < +\infty$ such that $\|A_iT_iw_n - A_iy_{n,i}\| \le K_i\|T_iw_n - y_{n,i}\| + \epsilon_i$. Hence, for the case $A_iT_iw_n - A_iy_{n,i} \neq 0$ for all $n \ge 1$ we have

$$\frac{(c_{n,i} + c_i)\|T_iw_n - y_{n,i}\|}{\|A_iT_iw_n - A_iy_{n,i}\|} \ge \frac{(c_{n,i} + c_i)\|T_iw_n - y_{n,i}\|}{K_i\|T_iw_n - y_{n,i}\| + \epsilon_i}$$
$$= \frac{(c_{n,i} + c_i)\|T_iw_n - y_{n,i}\|}{(K_i + \mu_i)\|T_iw_n - y_{n,i}\|} = \frac{(c_{n,i} + c_i)}{M_i} \ge \frac{c_i}{M_i},$$

where $\epsilon_i = \mu_i\|T_iw_n - y_{n,i}\|$ for some $\mu_i \in (0, 1)$ and $M_i = K_i + \mu_i$. Thus, by the definition of λ_{n+1}, the sequence $\{\lambda_{n,i}\}$ has lower bound $\min\{\frac{c_i}{M_i}, \lambda_{1,i}\}$ and has upper bound $\lambda_{1,i} + \Phi_i$. By Lemma 2.5, the limit $\lim_{n\to\infty} \lambda_{n,i}$ exists and we denote by $\lambda_i = \lim_{n\to\infty} \lambda_{n,i}$. It is clear that $\lambda_i \in \big[\min\{\frac{c_i}{M_i}, \lambda_{1,i}\}, \lambda_{1,i}+\Phi_i\big]$ for each $i = 0, 1, 2 \ldots, N$. □

Lemma 4.2. *If $\|T_i^*(T_iw_n - z_{n,i})\| \neq 0$, then the sequence $\{\eta_{n,i}\}$ defined by (3.3) has a positive lower bounded for each $i = 0, 1, 2, \ldots, N$.*

Proof. If $\|T_i^*(T_iw_n - z_{n,i})\| \neq 0$, we have for each $i = 0, 1, 2, \ldots, N$

$$\eta_{n,i} = \frac{(\phi_{n,i} + \phi_i)\|T_iw_n - z_{n,i}\|^2}{\|T_i^*(T_iw_n - z_{n,i})\|^2}.$$

Since T_i is a bounded linear operator and $\lim_{n\to\infty} \phi_{n,i} = 0$ for each $i = 0, 1, 2, \ldots,$
N, we have

$$\frac{(\phi_{n,i} + \phi_i)\|T_iw_n - z_{n,i}\|^2}{\|T_i^*(T_iw_n - z_{n,i})\|^2} \ge \frac{(\phi_{n,i} + \phi_i)\|T_iw_n - z_{n,i}\|^2}{\|T_i\|^2\|T_iw_n - z_{n,i}\|\|^2} \ge \frac{\phi_i}{\|T_i\|^2},$$

which implies that $\frac{\phi_i}{\|T_i\|^2}$ is a lower bound of $\{\eta_{n,i}\}$ for each $i = 0, 1, 2, \ldots, N$. □

Lemma 4.3. *Suppose Assumption A of Algorithm 3.1 holds. Then, there exists a positive integer N such that*

$$k_i + k_{n,i} \in (0, 2), \phi_i + \phi_{n,i} \in (0, 1), \quad \textit{and} \quad \frac{\lambda_{n,i}(c_{n,i} + ci)}{\lambda_{n+1,i}} \in (0, 1), \quad \forall n \ge N.$$

Proof. Since $0 < k_i < k_i' < 2$ and $\lim_{n\to\infty} k_{n,i} = 0$ for each $i = 0, 1, 2, \ldots, N$, there exists a positive integer $N_{1,i}$ such that

$$0 < k_i + k_{n,i} \le k_i' < 2,\ \forall n \ge N_{1,i}.$$

By similar argument, there exists a positive integer $N_{2,i}$ for each $i = 0, 1, 2, \ldots,$ N, such that

$$0 < \phi_i + \phi_{n,i} \le \phi_i' < 1,\ \forall n \ge N_{2,i}.$$

In addition, since $0 < c_i < c_i' < 1, \lim_{n\to\infty} c_{n,i} = 0$ and $\lim_{n\to\infty} \lambda_{n,i} = \lambda_i$ for each $i = 0, 1, 2, \ldots, N$, we have

$$\lim_{n\to\infty}\Big(1 - \frac{\lambda_{n,i}(c_{n,i} + c_i)}{\lambda_{n+1,i}}\Big) = 1 - c_i > 1 - c_i' > 0.$$

Therefore, for each $i = 0, 1, 2, \ldots, N$, there exists a positive integer $N_{3,i}$ such that

$$1 - \frac{\lambda_{n,i}(c_{n,i} + c_i)}{\lambda_{n+1,i}} > 0, \ \forall n \geq N_{3,i}.$$

Now, by setting $N = \max\{N_{1,i},\ N_{2,i},\ N_{3,i} : i = 0, 1, 2, \ldots, N\}$, the required result follows. □

Lemma 4.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that Assumption A holds. Then $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega$. This implies that $T_i p \in VI(C_i, A_i),\ i = 0, 1, 2, \ldots, N$. Then, by applying the triangular inequality, it follows from the definition of w_n that

$$\begin{aligned} \|w_n - p\| &= \|x_n + \theta_n(x_n - x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|. \end{aligned} \tag{4.1}$$

By Remark (3.2), there exists $M_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1, \ \forall\ n \geq 1.$$

Thus, it follows from (4.1) that

$$\|w_n - p\| \leq \|x_n - p\| + \alpha_n M_1, \ \forall n \geq 1. \tag{4.2}$$

Since $y_{n,i} = P_{C_i}(T_i w_n - \lambda_{n,i} A_i T_i w_n)$ and $T_i p \in VI(C_i, A_i),\ i = 0, 1, 2 \ldots, N$, by the property of the projection map it follows that

$$\langle y_{n,i} - T_i w_n + \lambda_{n,i} A_i T_i w_n, y_{n,i} - T_i p\rangle \leq 0. \tag{4.3}$$

Moreover, since $y_{n,i} \in C_i,\ i = 0, 1, 2, \ldots, N$, we have

$$\langle A_i T_i p, y_{n,i} - T_i p\rangle \geq 0,$$

which follows from the pseudomonotonicity of A_i that $\langle A_i y_{n,i}, y_{n,i} - T_i p\rangle \geq 0$. Since $\lambda_{n,i} > 0,\ i = 0, 1, 2, \ldots, N$, we have

$$\langle \lambda_{n,i} A_i y_{n,i}, y_{n,i} - T_i p\rangle \geq 0. \tag{4.4}$$

From (4.3) and (4.4) we obtain

$$\langle T_i w_n - y_{n,i} - \lambda_{n,i}(A_i T_i w_n - A_i y_{n,i}), y_{n,i} - T_i p\rangle \geq 0. \tag{4.5}$$

Now, applying the definition of $r_{n,i}$ and (4.5) we get

$$\begin{aligned} \langle T_i w_n - T_i p, r_{n,i}\rangle &= \langle T_i w_n - y_{n,i}, r_{n,i}\rangle + \langle y_{n,i} - T_i p, r_{n,i}\rangle \\ &= \langle T_i w_n - y_{n,i}, r_{n,i}\rangle \\ &\quad + \langle T_i w_n - y_{n,i} - \lambda_{n,i}(A_i T_i w_n - A_i y_{n,i}), y_{n,i} - T_i p\rangle \\ &\geq \langle T_i w_n - y_{n,i}, r_{n,i}\rangle. \end{aligned} \tag{4.6}$$

Since $z_{n,i} = T_i w_n - \beta_{n,i} r_{n,i}$, it follows that

$$\|\beta_{n,i} r_{n,i}\|^2 = \|z_{n,i} - T_i w_n\|^2. \tag{4.7}$$

By Lemma 4.3, there exists a positive integer N such that $0 < k_i + k_{n,i} < 2\ \forall n \geq N$. From the definition of $\beta_{n,i}$, if $r_{n,i} \neq 0\ i = 0, 1, 2, \ldots, N$, we have

$$\beta_{n,i}\|r_{n,i}\|^2 = (k_i + k_{n,i})\langle T_i w_n - y_{n,i}, r_{n,i}\rangle. \tag{4.8}$$

Now, by applying Lemma 2.3, (4.6), (4.7) and (4.8) we get

$$\begin{aligned}
\|z_{n,i} - T_i p\|^2 &= \|T_i w_n - \beta_{n,i} r_{n,i} - T_i p\|^2 \\
&= \|T_i w_n - T_i p\|^2 + \beta_{n,i}^2\|r_{n,i}\|^2 - 2\beta_{n,i}\langle T_i w_n - T_i p, r_{n,i}\rangle \\
&\leq \|T_i w_n - T_i p\|^2 + \beta_{n,i}^2\|r_{n,i}\|^2 - 2\beta_{n,i}\langle T_i w_n - y_{n,i}, r_{n,i}\rangle \\
&= \|T_i w_n - T_i p\|^2 + \beta_{n,i}^2\|r_{n,i}\|^2 - \frac{2}{k_i + k_{n,i}}\beta_{n,i}^2\|r_{n,i}\|^2 \\
&= \|T_i w_n - T_i p\|^2 + \left(1 - \frac{2}{k_i + k_{n,i}}\right)\|z_{n,i} - T_i w_n\|^2 \\
&\leq \|T_i w_n - T_i p\|^2.
\end{aligned} \tag{4.9}$$

Observe that if $r_{n,i} = 0,\ i = 0, 1, 2, \ldots, N$, (4.9) still holds.
Next, since the function $\|\cdot\|^2$ is convex, we have

$$\begin{aligned}
\|b_n - p\|^2 &= \|\sum_{i=0}^{N}\delta_{n,i}\big(w_n + \eta_{n,i}T_i^*(z_{n,i} - T_i w_n)\big) - p\|^2 \\
&\leq \sum_{i=0}^{N}\delta_{n,i}\|w_n + \eta_{n,i}T_i^*(z_{n,i} - T_i w_n) - p\|^2.
\end{aligned} \tag{4.10}$$

By Lemma 4.3, there exists a positive integer N such that $0 < \phi_{n,i} + \phi_i < 1,\ i = 0, 1, 2, \ldots, N$ for all $n \geq N$. Now, from (4.10) and by applying Lemma 2.3 and (4.9) we have

$$\begin{aligned}
&\|w_n + \eta_{n,i}T_i^*(z_{n,i} - T_i w_n) - p\|^2 \\
&\quad = \|w_n - p\|^2 + \eta_{n,i}^2\|T_i^*(z_{n,i} - T_i w_n)\|^2 + 2\eta_{n,i}\langle w_n - p, T_i^*(z_{n,i} - T_i w_n)\rangle \\
&\quad = \|w_n - p\|^2 + \eta_{n,i}^2\|T_i^*(z_{n,i} - T_i w_n)\|^2 + 2\eta_{n,i}\langle T_i w_n - T_i p, z_{n,i} - T_i w_n\rangle \\
&\quad = \|w_n - p\|^2 + \eta_{n,i}^2\|T_i^*(z_{n,i} - T_i w_n)\|^2 + \eta_{n,i}[\|z_{n,i} - T_i p\|^2 - \|T_i w_n - T_i p\|^2 \\
&\qquad - \|z_{n,i} - T_i w_n\|^2] \\
&\quad \leq \|w_n - p\|^2 + \eta_{n,i}^2\|T_i^*(z_{n,i} - T_i w_n)\|^2 - \eta_{n,i}\|z_{n,i} - T_i w_n\|^2 \\
&\quad = \|w_n - p\|^2 - \eta_{n,i}[\|z_{n,i} - T_i w_n\|^2 - \eta_{n,i}\|T_i^*(z_{n,i} - T_i w_n)\|^2].
\end{aligned} \tag{4.11}$$

If $\|T_i^*(z_{n,i} - T_i w_n)\| \neq 0$, then using the definition of $\eta_{n,i}$ we have

$$\|z_{n,i} - T_i w_n\|^2 - \eta_{n,i}\|T_i^*(z_{n,i} - T_i w_n)\|^2 = [1 - (\phi_{n,i} + \phi_i)]\|T_i w_n - z_{n,i}\|^2 \geq 0. \tag{4.12}$$

Thus, by applying (4.12) in (4.11) and substituting in (4.10) we have

$$\begin{aligned}
\|b_n - p\|^2 &\leq \|w_n - p\|^2 - \sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1 - (\phi_{n,i} + \phi_i)]\|T_i w_n - z_{n,i}\|^2 \\
&\leq \|w_n - p\|^2.
\end{aligned} \tag{4.13}$$

Observe that if $\|T_i^*(z_{n,i} - T_i w_n)\| = 0$, (4.13) still holds from (4.11).
By the definition of x_{n+1}, we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n-\xi_n)(w_n-p) + \xi_n(b_n-p) - \alpha_n p\| \\ &\leq \|(1-\alpha_n-\xi_n)(w_n-p) + \xi_n(b_n-p)\| + \alpha_n\|p\|. \end{aligned} \tag{4.14}$$

Applying Lemma 2.3(ii) together with (4.13) we have

$$\begin{aligned}&\|(1-\alpha_n-\xi_n)(w_n-p)+\xi_n(b_n-p)\|^2 \\ &\quad = (1-\alpha_n-\xi_n)^2\|w_n-p\|^2 + 2(1-\alpha_n-\xi_n)\xi_n\langle w_n-p, b_n-p\rangle \\ &\qquad + \xi_n^2\|b_n-p\|^2 \\ &\quad \leq (1-\alpha_n-\xi_n)^2\|w_n-p\|^2 + 2(1-\alpha_n-\xi_n)\xi_n\|w_n-p\|\|b_n-p\| \\ &\qquad + \xi_n^2\|b_n-p\|^2 \\ &\quad \leq (1-\alpha_n-\xi_n)^2\|w_n-p\|^2 + (1-\alpha_n-\xi_n)\xi_n\big[\|w_n-p\|^2 + \|b_n-p\|^2\big] \\ &\qquad + \xi_n^2\|b_n-p\|^2 \\ &\quad = (1-\alpha_n-\xi_n)(1-\alpha_n)\|w_n-p\|^2 + \xi_n(1-\alpha_n)\|b_n-p\|^2 \\ &\quad \leq (1-\alpha_n-\xi_n)(1-\alpha_n)\|w_n-p\|^2 + \xi_n(1-\alpha_n)\|w_n-p\|^2 \\ &\quad = (1-\alpha_n)^2\|w_n-p\|^2,\end{aligned}$$

which implies that

$$\|(1-\alpha_n-\xi_n)(w_n-p)+\xi_n(b_n-p)\| \leq (1-\alpha_n)\|w_n-p\|. \tag{4.15}$$

Now, applying (4.2) and (4.15) in (4.14), we have for all $n \geq N$

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\alpha_n)\|w_n-p\| + \alpha_n\|p\| \\ &\leq (1-\alpha_n)\big[\|x_n-p\| + \alpha_n M_1\big] + \alpha_n\|p\| \\ &\leq (1-\alpha_n)\|x_n-p\| + \alpha_n\big[\|p\| + M_1\big] \\ &\leq \max\big\{\|x_n-p\|, \|p\|+M_1\big\} \\ &\vdots \\ &\leq \max\big\{\|x_N-p\|, \|p\|+M_1\big\}.\end{aligned}$$

which implies that $\{x_n\}$ is bounded. Hence, $\{w_n\}, \{y_{n,i}\}, \{z_{n,i}\}, \{y_{n,i}\}, \{r_{n,i}\}$ and $\{b_n\}$ are all bounded. □

Lemma 4.5. *Suppose $\{w_n\}$ and $\{b_n\}$ are two sequences generated by Algorithm 3.1 with subsequences $\{w_{n_k}\}$ and $\{b_{n_k}\}$, respectively, such that $\lim_{k\to\infty}\|w_{n_k} - b_{n_k}\| = 0$. If $w_{n_k} \rightharpoonup z \in H$, then $z \in \Omega$.*

Proof. From (4.13), we have

$$\|b_{n_k}-p\|^2 \leq \|w_{n_k}-p\|^2 - \sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_i w_{n_k} - z_{n_k,i}\|^2. \tag{4.16}$$

From this, we obtain

$$\begin{aligned}&\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_iw_{n_k}-z_{n_k,i}\|^2\\&\quad\le\|w_{n_k}-p\|^2-\|b_{n_k}-p\|^2\\&\quad\le\|w_{n_k}-b_{n_k}\|^2+2\|w_{n_k}-b_{n_k}\|\|b_{n_k}-p\|.\end{aligned}\tag{4.17}$$

Since by the hypothesis of the lemma $\lim_{k\to\infty}\|w_{n_k}-b_{n_k}\|=0$, it follows from (4.17) that

$$\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_iw_{n_k}-z_{n_k,i}\|^2\to 0,\quad k\to\infty,$$

which implies that

$$\delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_iw_{n_k}-z_{n_k,i}\|^2\to 0,\quad k\to\infty,\ \forall i=0,1,2,\ldots,N.$$

By the definition of $\eta_{n,i}$, we have

$$\begin{aligned}&\delta_{n_k,i}(\phi_{n_k,i}+\phi_i)[1-(\phi_{n_k,i}+\phi_i)]\frac{\|T_iw_{n_k}-z_{n_k,i}\|^4}{\|T_i^*(T_iw_{n_k}-z_{n_k,i})\|^2}\to 0,\\&\quad k\to\infty,\ \forall i=0,1,2,\ldots,N,\end{aligned}$$

which implies that

$$\frac{\|T_iw_{n_k}-z_{n_k,i}\|^2}{\|T_i^*(T_iw_{n_k}-z_{n_k,i})\|}\to 0,\quad k\to\infty,\ \forall i=0,1,2,\ldots,N,$$

Since $\{\|T_i^*(T_iw_{n_k}-z_{n_k,i})\|\}$ is bounded, it follows that

$$\|T_iw_{n_k}-z_{n_k,i}\|\to 0,\quad k\to\infty,\ \forall i=0,1,2,\ldots,N.\tag{4.18}$$

Thus, we have

$$\begin{aligned}&\|T_i^*(T_iw_{n_k}-z_{n_k,i})\|\le\|T_i^*\|\|(T_iw_{n_k}-z_{n_k,i})\|=\|T_i\|\|(T_iw_{n_k}-z_{n_k,i})\|\to 0,\\&\quad k\to\infty,\ \forall i=0,1,2,\ldots,N.\end{aligned}\tag{4.19}$$

By the definition of $\lambda_{n+1,i}$, it follows that

$$\begin{aligned}&\langle T_iw_{n_k}-y_{n_k,i},r_{n_k,i}\rangle\\&\quad=\langle T_iw_{n_k}-y_{n_k,i},T_iw_{n_k}-y_{n_k,i}-\lambda_{n_k,i}(A_iT_iw_{n_k}-A_iy_{n_k,i})\rangle\\&\quad=\|T_iw_{n_k}-y_{n_k,i}\|^2-\langle T_iw_{n_k}-y_{n_k,i},\lambda_{n_k,i}(A_iT_iw_{n_k}-A_iy_{n_k,i})\rangle\\&\quad\ge\|T_iw_{n_k}-y_{n_k,i}\|^2-\lambda_{n_k,i}\|T_iw_{n_k}-y_{n_k,i}\|\|A_iT_iw_{n_k}-A_iy_{n_k,i}\|\\&\quad\ge\|T_iw_{n_k}-y_{n_k,i}\|^2-\frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i}+c_i)\|T_iw_{n_k}-y_{n_k,i}\|^2\\&\quad=\left(1-\frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i}+c_i)\right)\|T_iw_{n_k}-y_{n_k,i}\|^2.\end{aligned}\tag{4.20}$$

From Lemma 4.1 we know that $\lim_{k\to\infty}\lambda_{n_k,i}=\lambda_i,\ i=0,1,2,\ldots,N$ and by Lemma 4.3, there exists a positive integer N such that $1-\frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i}+c_i)>$

$0,\ \forall n \geq N,\ i = 0,1,2,\ldots,N$. If $r_{n,i} \neq 0$, then by applying the continuity of A_i, the definitions of $\beta_{n,i}, r_{n,i}$ and $z_{n,i}$ $i = 0,1,2,\ldots,N$, from (4.20) we have

$$\begin{aligned}
&\|T_i w_{n_k} - y_{n_k,i}\|^2 \\
&\quad \leq \frac{1}{\Big(1 - \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)} \langle T_i w_{n_k} - y_{n_k,i}, r_{n_k,i}\rangle \\
&\quad = \frac{1}{(k_i + k_{n_k,i})\Big(1 - \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)} \beta_{n_k,i}\|r_{n_k,i}\|^2 \\
&\quad = \frac{1}{(k_i + k_{n_k,i})\Big(1 - \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)} \beta_{n_k,i}\|r_{n_k,i}\| \|T_i w_{n_k} - y_{n_k,i} \\
&\qquad -\lambda_{n_k,i}(A_i T_i w_{n_k} - A_i y_{n_k,i}) \\
&\quad \leq \frac{1}{(k_i + k_{n_k,i})\Big(1 - \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)} \beta_{n_k,i}\|r_{n_k,i}\| \Big(\|T_i w_{n_k} - y_{n_k,i}\| \\
&\qquad +\lambda_{n_k,i}\|A_i T_i w_{n_k} - A_i y_{n_k,i}\|\Big) \\
&\quad \leq \frac{1}{(k_i + k_{n_k,i})\Big(1 - \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)} \beta_{n_k,i}\|r_{n_k,i}\| \\
&\qquad \Big(1 + \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)\|T_i w_{n_k} - y_{n_k,i}\| \\
&\quad = \frac{\Big(1 + \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)}{(k_i + k_{n_k,i})\Big(1 - \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)} \|T_i w_{n_k} - z_{n_k,i}\| \|T_i w_{n_k} - y_{n_k,i}\|.
\end{aligned} \tag{4.21}$$

Thus, we have

$$\|T_i w_{n_k} - y_{n_k,i}\| \leq \frac{\Big(1 + \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)}{(k_i + k_{n_k,i})\Big(1 - \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}}(c_{n_k,i} + c_i)\Big)} \|T_i w_{n_k} - z_{n_k,i}\|. \tag{4.22}$$

Since $\lim_{k\to\infty} c_{n_k,i} = k_{n_k,i} = 0$ and by Lemma 4.1 $\lim_{k\to\infty} \frac{\lambda_{n_k,i}}{\lambda_{n_k+1,i}} = 1,\ i = 0,1,2,\ldots,N$, then from (4.22) and by applying (4.18) we have

$$\|T_i w_{n_k} - y_{n_k,i}\| \to 0, \quad k \to \infty,\ \forall i = 0,1,2,\ldots,N. \tag{4.23}$$

If $r_{n,i} = 0$, from (4.20) we know that (4.23) still holds.
Since $y_{n,i} = P_{C_i}(T_i w_n - \lambda_{n,i} A_i T_i w_n)$, by the property of the projection map we have

$$\begin{aligned}
&\langle T_i w_{n_k} - \lambda_{n_k,i} A_i T_i w_{n_k} - y_{n_k,i}, T_i x - y_{n_k,i}\rangle \leq 0, \quad \forall\ T_i x \in C_i, \\
&\quad i = 0,1,2,\ldots,N,
\end{aligned}$$

which implies that

$$\frac{1}{\lambda_{n_k,i}}\langle T_iw_{n_k}-y_{n_k,i},T_ix-y_{n_k,i}\rangle \le \langle A_iT_iw_{n_k},T_ix-y_{n_k,i}\rangle,\\ \forall\ T_ix\in C_i,\quad i=0,1,2,\ldots,N.$$

From the last inequality, we get

$$\frac{1}{\lambda_{n_k,i}}\langle T_iw_{n_k}-y_{n_k,i},T_ix-y_{n_k,i}\rangle + \langle A_iT_iw_{n_k},y_{n_k,i}-T_iw_{n_k}\rangle\\ \le \langle A_iT_iw_{n_k},T_ix-T_iw_{n_k}\rangle,\ \forall\ T_ix\in C_i,\ i=0,1,2,\ldots,N. \quad (4.24)$$

By applying (4.23) and the fact that $\lim_{k\to\infty}\lambda_{n_k,i}=\lambda_i>0$, from (4.24) we obtain

$$\liminf_{k\to\infty}\langle A_iT_iw_{n_k},T_ix-T_iw_{n_k}\rangle \ge 0,\quad \forall\ T_ix\in C_i,\ i=0,1,2,\ldots,N. \quad (4.25)$$

Observe that

$$\langle A_iy_{n_k,i},T_ix-y_{n_k,i}\rangle = \langle A_iy_{n_k,i}-A_iT_iw_{n_k},T_ix-T_iw_{n_k}\rangle\\ +\langle A_iT_iw_{n_k},T_ix-T_iw_{n_k}\rangle + \langle A_iy_{n_k,i},T_iw_{n_k}-y_{n_k,i}\rangle. \quad (4.26)$$

By the continuity of A_i, from (4.23) we have

$$\|A_iT_iw_{n_k}-A_iy_{n_k,i}\| \to 0,\quad k\to\infty,\ \forall i=0,1,2,\ldots,N. \quad (4.27)$$

By applying (4.23) and (4.27), we obtain from (4.25) and (4.26) that

$$\liminf_{k\to\infty}\langle A_iy_{n_k,i},T_ix-y_{n_k,i}\rangle \ge 0,\quad \forall\ T_ix\in C_i,\ i=0,1,2,\ldots,N. \quad (4.28)$$

Next, let $\{\Theta_{k,i}\}$ be a decreasing sequence of positive numbers such that $\Theta_{k,i}\to 0$ as $k\to\infty,\ i=0,1,2,\ldots,N$. For each k, let N_k denote the smallest positive integer such that

$$\langle A_iy_{n_j,i},T_ix-y_{n_j,i}\rangle + \Theta_{k,i} \ge 0,\quad \forall\ j\ge N_k,\ T_ix\in C_i,\ i=0,1,2,\ldots,N, \quad (4.29)$$

where the existence of N_k follows from (4.28). Since $\{\Theta_{k,i}\}$ is decreasing, then $\{N_k\}$ is increasing. Furthermore, since $\{y_{N_k,i}\}\subset C_i$ for each k, we can suppose $A_iy_{N_k,i}\ne 0$ (otherwise, $y_{N_k,i}\in VI(C_i,A_i),\ i=0,1,2\ldots,N$) and let

$$u_{N_k,i}=\frac{A_iy_{N_k,i}}{\|A_iy_{N_k,i}\|^2}$$

Then, $\langle A_iy_{N_k,i},u_{N_k,i}\rangle = 1$ for each $k,\ i=0,1,2,\ldots,N$. From (4.29), we obtain

$$\langle A_iy_{N_k,i},T_ix+\Theta_{k,i}u_{N_k,i}-y_{N_k,i}\rangle \ge 0,\quad \forall\ T_ix\in C_i,\ i=0,1,2,\ldots,N.$$

By the pseudomonotonicity of A_i, we obtain

$$\langle A_i(T_ix+\Theta_{k,i}u_{N_k,i}),T_ix+\Theta_{k,i}u_{N_k,i}-y_{N_k,i}\rangle \ge 0,\quad \forall\ T_ix\in C_i, i=0,1,2,\\ \ldots,N.$$

which is equivalent to

$$\langle A_iT_ix, T_ix - y_{N_k,i}\rangle \geq \langle A_iT_ix - A_i(T_ix + \Theta_{k,i}u_{N_k,i}), T_ix + \Theta_{k,i}u_{N_k,i} - y_{N_k,i}\rangle - \Theta_{k,i}\langle A_iT_ix, u_{N_k,i}\rangle, \ \forall T_ix \in C_i, i = 0, 1, \ldots, N. \tag{4.30}$$

To complete the proof, we need to show that $\lim_{k\to\infty} \Theta_{k,i}u_{N_k,i} = 0$. Since $w_{n_k} \rightharpoonup z$ and T_i is a bounded linear operator for each $i = 0, 1, 2, \ldots, N$, we have $T_iw_{n_k} \rightharpoonup T_iz$, $\forall i = 0, 1, 2, \ldots, N$. Thus, from (4.23) we get $y_{n_k,i} \rightharpoonup T_iz$, $\forall\ i = 0, 1, 2, \ldots, N$. Since $\{y_{n_k,i}\} \subset C_i$, $i = 0, 1, 2, \ldots, N$, we have $T_iz \in C_i$. If $T_iz = 0$, $\forall\ i = 0, 1, 2, \ldots, N$, then $T_iz \in VI(C_i, A_i)$ $\forall\ i = 0, 1, 2, \ldots, N$, which implies that $z \in \Omega$. On the contrary, we suppose $T_iz \neq 0$, $\forall\ i = 0, 1, 2, \ldots, N$. Since A_i satisfies condition (3.1), we have for all $i = 0, 1, 2, \ldots, N$

$$0 < \|A_iT_iz\| \leq \liminf_{k\to\infty} \|A_iy_{n_k,i}\|.$$

Using the facts that $\{y_{N_k,i}\} \subset \{y_{n_k,i}\}$ and $\Theta_{k,i} \to 0$ as $k \to \infty$, $i = 0, 1, 2 \ldots, N$, we have

$$0 \leq \limsup_{k\to\infty} \|\Theta_{k,i}u_{N_k,i}\| = \limsup_{k\to\infty}\Big(\frac{\Theta_{k,i}}{\|A_iy_{n_k,i}\|}\Big) \leq \frac{\limsup_{k\to\infty} \Theta_{k,i}}{\liminf_{k\to\infty}\|A_iy_{n_k,i}\|} = 0,$$

which implies that $\limsup_{k\to\infty}\ \Theta_{k,i}u_{N_k,i} = 0$. Applying the facts that A_i is continuous, $\{y_{N_k,i}\}$ and $\{u_{N_k,i}\}$ are bounded and $\lim_{k\to\infty}\ \Theta_{k,i}u_{N_k,i} = 0$, from (4.30) we obtain

$$\liminf_{k\to\infty}\langle A_iT_ix, T_ix - y_{N_k,i}\rangle \geq 0, \quad \forall\ T_ix \in C_i,\ i = 0, 1, 2, \ldots, N.$$

From the last inequality, we obtain

$$\langle A_iT_ix, T_ix - T_iz\rangle = \lim_{k\to\infty}\langle A_iT_ix, T_ix - y_{N_k,i}\rangle = \liminf_{k\to\infty}\langle A_iT_ix, T_ix - y_{N_k,i}\rangle \geq 0,\ \forall\ T_ix \in C_i,\ i = 0, 1, 2, \ldots, N.$$

By Lemma 2.6, we have

$$T_iz \in VI(C_i, A_i),\ i = 0, 1, 2, \ldots, N,$$

which implies that

$$z \in T_i^{-1}\big(VI(C_i, A_i)\big),\ i = 0, 1, 2, \ldots, N,$$

Thus, we have $z \in \bigcap_{i=0}^{N} T_i^{-1}\big(VI(C_i, A_i)\big)$, which implies that $z \in \Omega$ as required. □

Lemma 4.6. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 under Assumption A. Then, the following inequality holds for all $p \in \Omega$:*

$$\|x_{n+1} - p\|^2 \leq (1 - \alpha_n)||x_n - p||^2 + \alpha_n\Big[3M_2(1 - \alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + 2\langle p, p - x_{n+1}\rangle\Big] - \xi_n(1 - \alpha_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1 - (\phi_{n,i} + \phi_i)]\|T_iw_n - z_{n,i}\|^2.$$

Proof. Let $p \in \Omega$. Then, by applying Lemma 2.3 together with the Cauchy-Schwartz inequality we have

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\le \|x_n - p\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 + 2\theta_n\|x_n - x_{n-1}\|\|x_n - p\| \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|(\theta_n\|x_n - x_{n-1}\| + 2\|x_n - p\|) \\
&\le \|x_n - p\|^2 + 3M_2\theta_n\|x_n - x_{n-1}\| \\
&= \|x_n - p\|^2 + 3M_2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|, \qquad (4.31)
\end{aligned}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{\|x_n - p\|, \theta_n\|x_n - x_{n-1}\|\} > 0$.
Next, by the definition of x_{n+1}, (4.13), (4.31) and applying Lemma 2.3 we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \xi_n)(w_n - p) + \xi_n(b_n - p) - \alpha_n p\|^2 \\
&\le \|(1 - \alpha_n - \xi_n)(w_n - p) + \xi_n(b_n - p)\|^2 - 2\alpha_n\langle p, x_{n+1} - p\rangle \\
&= (1 - \alpha_n - \xi_n)^2\|w_n - p\|^2 + \xi_n^2\|b_n - p\|^2 \\
&\quad + 2\xi_n(1 - \alpha_n - \xi_n)\langle w_n - p, b_n - p\rangle \\
&\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\le (1 - \alpha_n - \xi_n)^2\|w_n - p\|^2 + \xi_n^2\|b_n - p\|^2 \\
&\quad + 2\xi_n(1 - \alpha_n - \xi_n)\|w_n - p\|\|b_n - p\| \\
&+ 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\le (1 - \alpha_n - \xi_n)^2\|w_n - p\|^2 + \xi_n^2\|b_n - p\|^2 \\
&\quad + \xi_n(1 - \alpha_n - \xi_n)\left[\|w_n - p\|^2 + \|b_n - p\|^2\right] \\
&\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&= (1 - \alpha_n - \xi_n)(1 - \alpha_n)\|w_n - p\|^2 + \xi_n(1 - \alpha_n)\|b_n - p\|^2 \\
&\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\le (1 - \alpha_n - \xi_n)(1 - \alpha_n)\|w_n - p\|^2 + \xi_n(1 - \alpha_n)\Big[\|w_n - p\|^2 \\
&\quad - \sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1 - (\phi_{n,i} + \phi_i)]\|T_i w_n - z_{n,i}\|^2\Big] + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&= (1 - \alpha_n)^2\|w_n - p\|^2 - \xi_n(1 - \alpha_n) \\
&\quad \sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1 - (\phi_{n,i} + \phi_i)]\|T_i w_n - z_{n,i}\|^2 \\
&\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\le (1 - \alpha_n)^2||x_n - p||^2 + 3M_2\alpha_n(1 - \alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \\
&\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\quad - \xi_n(1 - \alpha_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1 - (\phi_{n,i} + \phi_i)]\|T_i w_n - z_{n,i}\|^2
\end{aligned}$$

$$\leq (1-\alpha_n)||x_n-p||^2 + \alpha_n\Big[3M_2(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + 2\langle p, p-x_{n+1}\rangle\Big] - \xi_n(1-\alpha_n)\sum_{i=0}^{N}\delta_{n,i}\eta_{n,i}[1-(\phi_{n,i}+\phi_i)]\|T_iw_n-z_{n,i}\|^2,$$

which is the required inequality. □

Theorem 4.7. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that Assumption A holds. Then, $\{x_n\}$ converges strongly to $\hat{x}\in\Omega$, where $\hat{x}=\min\{\|p\|:p\in\Omega\}$.*

Proof. Let $\hat{x}=\min\{\|p\|:p\in\Omega\}$, that is, $\hat{x}=P_\Omega(0)$. Then, from Lemma 4.6 we obtain

$$\begin{aligned}\|x_{n+1}-\hat{x}\|^2 &\leq (1-\alpha_n)||x_n-\hat{x}||^2 \\ &\quad+\alpha_n\Big[3M_2(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + 2\langle\hat{x},\hat{x}-x_{n+1}\rangle\Big]\\ &= (1-\alpha_n)||x_n-\hat{x}||^2+\alpha_n d_n,\end{aligned} \tag{4.32}$$

where $d_n = 3M_2(1-\alpha_n)^2\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + 2\langle\hat{x},\hat{x}-x_{n+1}\rangle$.
Now, we claim that the sequence $\{\|x_n-\hat{x}\|\}$ converges to zero. In view of Lemma 2.4, it suffices to show that $\limsup_{k\to\infty} d_{n_k}\leq 0$ for every subsequence $\{\|x_{n_k}-\hat{x}\|\}$ of $\{\|x_n-\hat{x}\|\}$ satisfying

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-\hat{x}\|-\|x_{n_k}-\hat{x}\|\right)\geq 0. \tag{4.33}$$

Suppose that $\{\|x_{n_k}-\hat{x}\|\}$ is a subsequence of $\{\|x_n-\hat{x}\|\}$ such that (4.33) holds. Again, from Lemma 4.6, we obtain

$$\begin{aligned}&\xi_{n_k}(1-\alpha_{n_k})\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_iw_{n_k}-z_{n_k,i}\|^2\\ &\qquad\leq (1-\alpha_{n_k})\|x_{n_k}-\hat{x}\|^2-\|x_{n_k+1}-\hat{x}\|^2\\ &\qquad\quad+\alpha_{n_k}\Big[3M_2(1-\alpha_{n_k})^2\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\\ &\qquad\quad+2\langle\hat{x},\hat{x}-x_{n_k+1}\rangle\Big].\end{aligned}$$

By (4.33), Remark 3.2 and the fact that $\lim_{k\to\infty}\alpha_{n_k}=0$, we have

$$\xi_{n_k}(1-\alpha_{n_k})\sum_{i=0}^{N}\delta_{n_k,i}\eta_{n_k,i}[1-(\phi_{n_k,i}+\phi_i)]\|T_iw_{n_k}-z_{n_k,i}\|^2\to 0, \quad k\to\infty.$$

Thus, we get

$$\lim_{k\to\infty}\|T_iw_{n_k}-z_{n_k,i}\|=0, \quad \forall i=0,1,2,\ldots,N. \tag{4.34}$$

It follows that

$$\|T_i^*(z_{n_k,i}-T_iw_{n_k})\|\leq\|T_i^*\|\|z_{n_k,i}-T_iw_{n_k}\|\to 0, \quad k\to\infty\ \forall i=0,1,2,\ldots,N. \tag{4.35}$$

By the definition of b_n and by applying (4.35), we obtain

$$\begin{aligned}\|b_{n_k} - w_{n_k}\| &= \|\sum_{i=0}^{N} \delta_{n_k,i}\big(w_{n_k} + \eta_{n_k,i}T_i^*(z_{n_k,i} - T_i w_{n_k})\big) - w_{n_k}\| \\ &\le \sum_{i=0}^{N} \delta_{n_k,i}\eta_{n_k,i}\|T_i^*(z_{n_k,i} - T_i w_{n_k})\| \to 0. \end{aligned} \tag{4.36}$$

From the definition of w_n and by Remark 3.2, we get

$$\|w_{n_k} - x_{n_k}\| = \theta_{n_k}\|x_{n_k} - x_{n_k-1}\| \to 0, \quad k \to \infty. \tag{4.37}$$

Next, from (4.36) and (4.37) we obtain

$$\|x_{n_k} - b_{n_k}\| \le \|x_{n_k} - w_{n_k}\| + \|w_{n_k} - b_{n_k}\| \to 0, \quad k \to \infty. \tag{4.38}$$

Applying (4.37), (4.38) and the fact that $\lim_{k\to\infty} \alpha_{n_k} = 0$ we obtain

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &= \|(1 - \alpha_{n_k} - \xi_{n_k})(w_{n_k} - x_{n_k}) + \xi_{n_k}(b_{n_k} - x_{n_k}) - \alpha_n x_{n_k}\| \\ &\le (1 - \alpha_{n_k} - \xi_{n_k})\|w_{n_k} - x_{n_k}\| + \xi_{n_k}\|b_{n_k} - x_{n_k}\| \\ &\quad +\alpha_{n_k}\|x_{n_k}\| \to 0, \quad k \to \infty. \end{aligned} \tag{4.39}$$

Since $\{x_n\}$ is bounded, $w_\omega(x_n) \neq \emptyset$. Let $x^* \in w_\omega(x_n)$ be an arbitrary element. Then, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. It follows from (4.37) that $w_{n_k} \rightharpoonup x^*$. Now, invoking Lemma 4.5 and applying (4.36) we have $x^* \in \Omega$. Since $x^* \in w_\omega(x_n)$ was chosen arbitrarily, it follows that $w_\omega(x_n) \subset \Omega$.

Next, by the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup q$ and

$$\limsup_{k\to\infty}\langle \hat{x}, \hat{x} - x_{n_k}\rangle = \lim_{j\to\infty}\langle \hat{x}, \hat{x} - x_{n_{k_j}}\rangle.$$

Since $\hat{x} = P_\Omega(0)$, it follows from the property of the metric projection that

$$\limsup_{k\to\infty}\langle \hat{x}, \hat{x} - x_{n_k}\rangle = \lim_{j\to\infty}\langle \hat{x}, \hat{x} - x_{n_{k_j}}\rangle = \langle \hat{x}, \hat{x} - q\rangle \le 0, \tag{4.40}$$

Hence, from (4.39) and (4.40) we obtain

$$\limsup_{k\to\infty}\langle \hat{x}, \hat{x} - x_{n_{k+1}}\rangle \le 0. \tag{4.41}$$

Now, by Remark 3.2 and (4.41) we have $\limsup_{k\to\infty} d_{n_k} \le 0$. Thus, by applying Lemma 2.4 it follows from (4.32) that $\{\|x_n - \hat{x}\|\}$ converges to zero, which completes the proof. □

5. Applications

5.1. Split Convex Minimization Problem with Multiple Output Sets

Let C be a nonempty, closed and convex subset of a real Hilbert space H. The convex minimization problem is formulated as finding a point $x^* \in C$, such that

$$g(x^*) = \min_{x\in C} g(x), \tag{5.1}$$

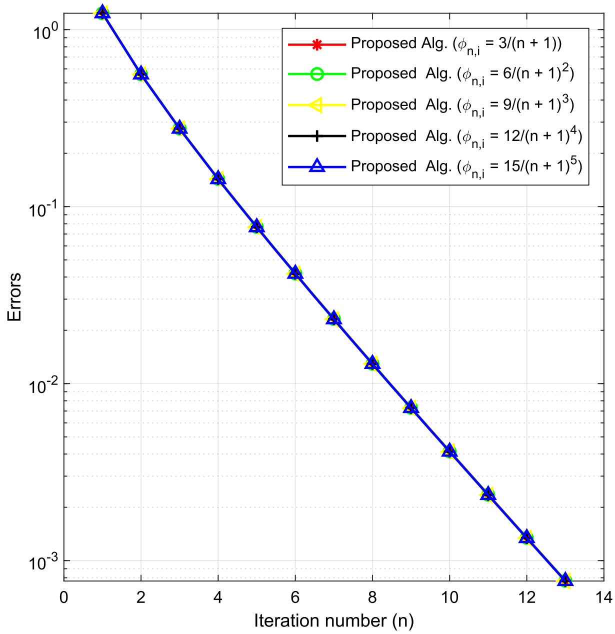

Figure 1. Experiment 6.5: $m = 25$

where g is a real-valued convex function. We denote the solution set of Problem (5.1) by $\arg\min g$.

Let C, C_i be nonempty, closed and convex subsets of real Hilbert spaces $H, H_i, i = 1, 2, ..., N$, respectively, and let $T_i : H \to H_i, i = 1, 2, ..., N$, be bounded linear operators with adjoints T_i^*. Let $g : H \to \mathbb{R}, g_i : H_i \to \mathbb{R}$ be convex and differentiable functions. Here, we apply our result to approximate the solution of the following *split convex minimization problem with multiple output sets* (SCMPMOS): Find $x^* \in C$ such that

$$x^* \in \Gamma := \arg\min g \cap \Big(\bigcap_{i=1}^{N} T_i^{-1}\big(\arg\min g_i\big)\Big) \neq \emptyset. \tag{5.2}$$

We need the following lemma to establish our next result.

Lemma 5.1. [36] *Let C be a nonempty, closed and convex subset of a real Banach space E. Let g be a convex function of E into $\mathbb{R}$. If g is Fréchet differentiable, then z is a solution of Problem (5.1) if and only if $z \in VI(C, \nabla g)$, where ∇g is the gradient of g.*

Now, by applying Theorem 4.7 and Lemma 5.1, we obtain the following strong convergence theorem for approximating the solution of the SCMPMOS (5.2) in Hilbert spaces.

Theorem 5.2. *Let C, C_i be nonempty, closed and convex subsets of real Hilbert spaces $H, H_i, i = 1, 2, ..., N$, respectively, and let $T_i : H \to H_i, i = 1, 2, ..., N$, be bounded linear operators with adjoints T_i^*. Let $g : H \to \mathbb{R}, g_i : H_i \to$*

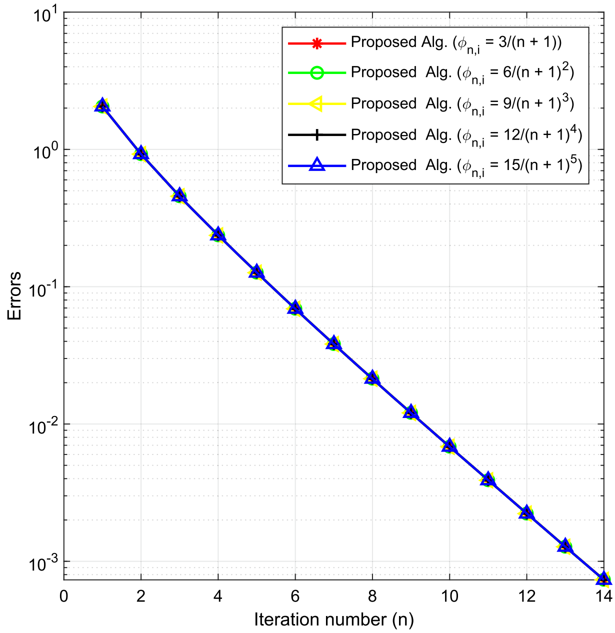

Figure 2. Experiment 6.5: $m = 50$

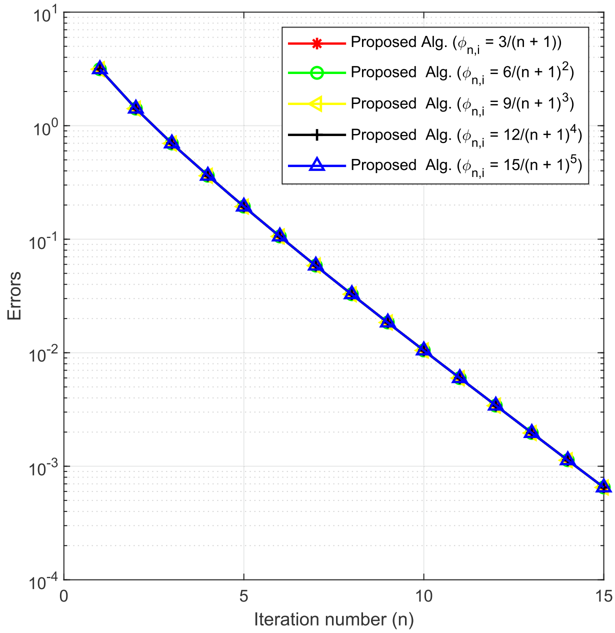

Figure 3. Experiment 6.5: $m = 100$

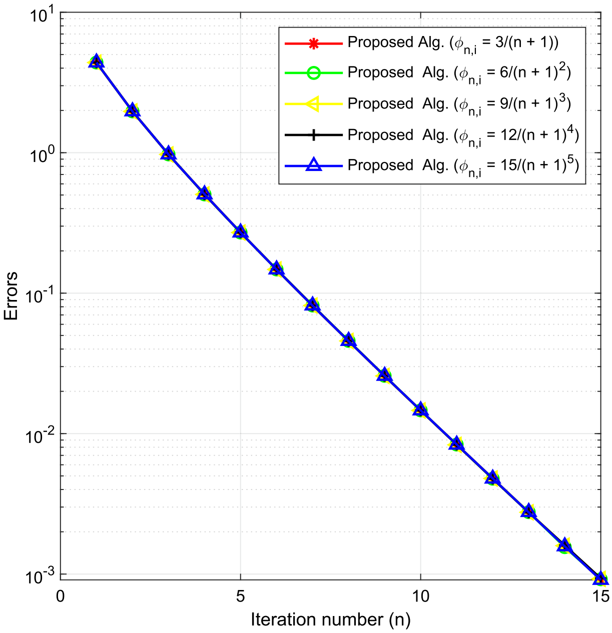

Figure 4. Experiment 6.5: $m = 2000$

Table 1. Numerical results for (Experiment 6.5)

Proposed Alg. 3.1	$m = 25$		$m = 50$		$m = 100$		$m = 200$	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
$\phi_{n,i} = \frac{3}{n+1}$	13	3.5072	14	4.0949	15	9.0474	15	5.1828
$\phi_{n,i} = \frac{6}{(n+1)^2}$	13	2.7786	14	3.3384	15	5.0823	15	4.4530
$\phi_{n,i} = \frac{9}{(n+1)^3}$	13	2.7132	14	3.3103	15	5.0511	15	4.4659
$\phi_{n,i} = \frac{12}{(n+1)^4}$	13	2.7295	14	3.3348	15	5.1468	15	4.4561
$\phi_{n,i} = \frac{15}{(n+1)^5}$	13	2.7467	14	3.2094	15	5.0072	15	4.4961

$\mathbb{R}$ *be fréchet differentiable convex functions such that* $\nabla g, \nabla g_i$ *are uniformly continuous. Suppose that Assumption A of Theorem 4.7 holds and the solution set* $\Gamma \neq \emptyset$. *Then, the sequence* $\{x_n\}$ *generated by the following algorithm converges strongly to* $\hat{x} \in \Gamma$, *where* $\hat{x} = \min\{\|p\| : p \in \Gamma\}$.

Proof. Since g_i , $i = 0, 1, 2, \ldots, N$ are convex, then ∇g_i are monotone [36] and thus pseudomonotone. Consequently, the result follows by applying Lemma 5.1 and setting $A_i = \nabla g_i$ in Theorem 4.7. □

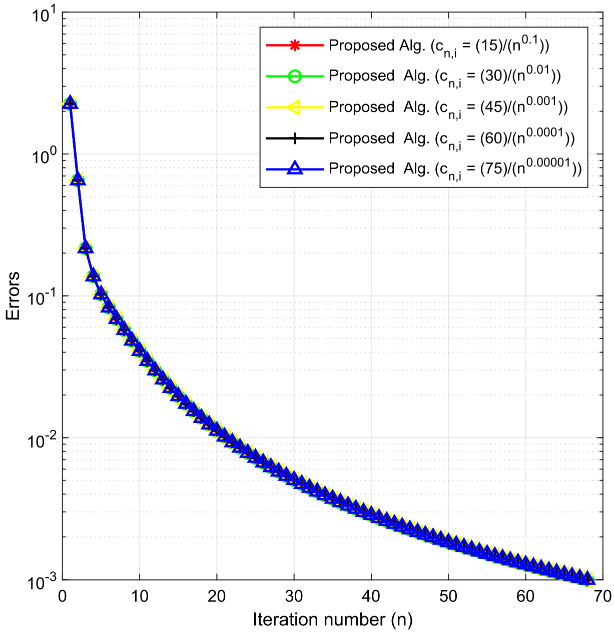

Figure 5. Experiment 6.6: $m = 10$

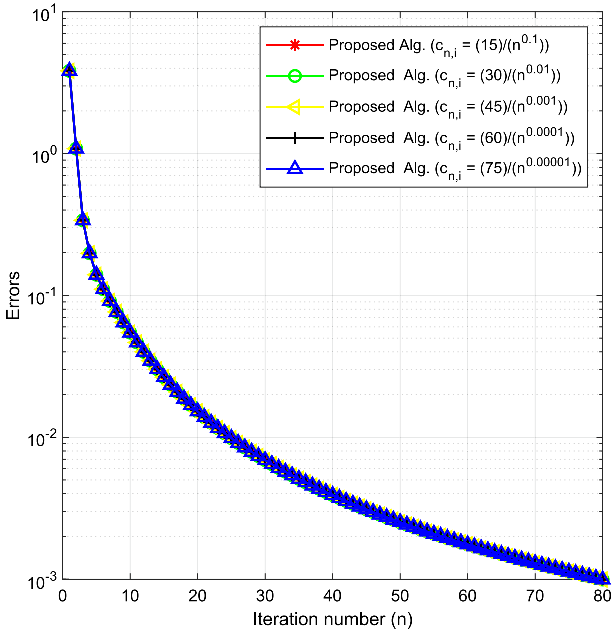

Figure 6. Experiment 6.6: $m = 20$

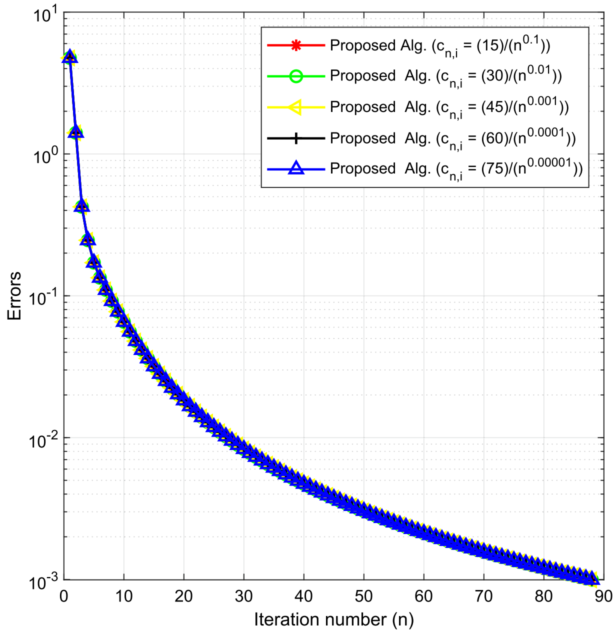

Figure 7. Experiment 6.6: $m = 30$

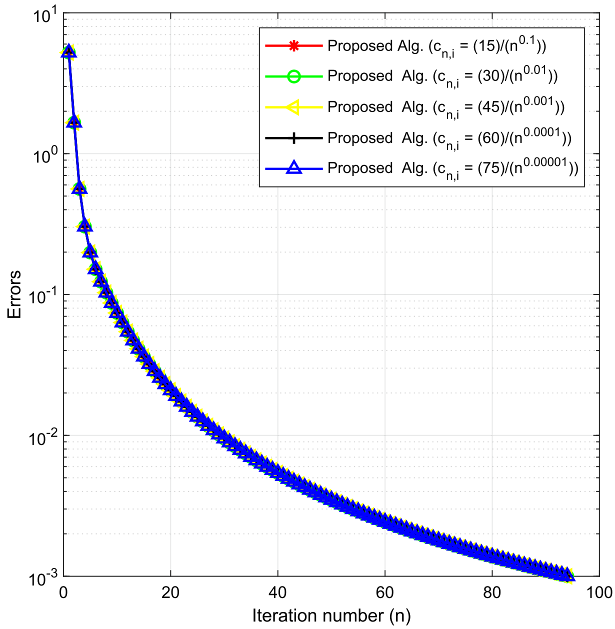

Figure 8. Experiment 6.6: $m = 40$

Algorithm 5.3.

Step 0. Select initial points $x_0, x_1 \in H$. Let $C_0 = C,\ T_0 = I^H,\ \nabla g_0 = \nabla g$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \le \theta_n \le \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta,\ \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$y_{n,i} = P_{C_i}(T_i w_n - \lambda_{n,i}\nabla g_i T_i w_n)$$

$$\lambda_{n+1,i} = \begin{cases} \min\{\frac{(c_{n,i}+c_i)\|T_i w_n - y_{n,i}\|}{\|\nabla g_i T_i w_n - \nabla g_i y_{n,i}\|},\ \lambda_{n,i} + \rho_{n,i}\}, & \text{if } \nabla g_i T_i w_n - \nabla g_i y_{n,i} \neq 0, \\ \lambda_{n,i} + \rho_{n,i}, & \text{otherwise.} \end{cases}$$

$$z_{n,i} = T_i w_n - \beta_{n,i} r_{n,i},$$

where

$$r_{n,i} = T_i w_n - y_{n,i} - \lambda_{n,i}(\nabla g_i T_i w_n - \nabla g_i y_{n,i})$$

and

$$\beta_{n,i} = \begin{cases} (k_i + k_{n,i})\frac{\langle T_i w_n - y_{n,i}, r_{n,i}\rangle}{\|r_{n,i}\|^2}, & \text{if } r_{n,i} \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Step 4. Compute

$$b_n = \sum_{i=0}^{N} \delta_{n,i}\big(w_n + \eta_{n,i} T_i^*(z_{n,i} - T_i w_n)\big),$$

where

$$\eta_{n,i} = \begin{cases} \frac{(\phi_{n,i}+\phi_i)\|T_i w_n - z_{n,i}\|^2}{\|T_i^*(T_i w_n - z_{n,i})\|^2}, & \text{if } \|T_i^*(T_i w_n - z_{n,i})\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 5. Compute

$$x_{n+1} = (1 - \alpha_n - \xi_n)w_n + \xi_n b_n.$$

Set $n := n + 1$ and return to **Step 1**.

5.2. Generalized Split Variational Inequality Problem

Finally, we apply our result to study the generalized split variational inequality problem (see [28]). Let C_i be nonempty, closed and convex subsets of real Hilbert spaces $H_i, i = 1, 2, ..., N$, and let $S_i : H_i \to H_{i+1}, i = 1, 2, ..., N-1$, be bounded linear operators, such that $S_i \neq 0$. Let $B_i : H_i \to H_i, i = 1, 2, ..., N$, be single-valued operators. The *generalized split variational inequality problem* (GSVIP) is formulated as finding a point $x^* \in C_1$ such that

$$\begin{aligned} x^* \in \Gamma := VI(C_1, B_1) \cap S_1^{-1}(VI(C_2, B_2)) \cap \dots \\ S_1^{-1}(S_2^{-1} \dots (S_{N-1}^{-1}(VI(C_N, B_N)))) \neq \emptyset; \end{aligned} \tag{5.3}$$

that is, $x^* \in C_1$ such that

$$x^* \in VI(C_1, B_1), S_1 x^* \in VI(C_2, B_2), \dots, S_{N-1}(S_{N-2} \dots S_1 x^*) \in VI(C_N, B_N).$$

Table 2. Numerical results for (Experiment 6.6)

Proposed Alg. 3.1	$m = 10$		$m = 20$		$m = 30$		$m = 40$	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
$c_{n,i} = \frac{15}{n^{0.1}}$	68	0.0692	80	0.0720	89	0.0971	94	0.1070
$c_{n,i} = \frac{30}{n^{0.01}}$	68	0.0456	80	0.0596	89	0.06613	94	0.1080
$c_{n,i} = \frac{45}{n^{0.001}}$	68	0.0559	80	0.0521	89	0.0746	94	0.1044
$c_{n,i} = \frac{60}{n^{0,0001}}$	68	0.0495	80	0.0546	89	0.0852	94	0.0940
$c_{n,i} = \frac{75}{n^{0.00001}}$	68	0.0398	80	0.0602	89	0.0699	94	0.1104

Observe that if we let $C = C_1, A = B_1, A_i = B_{i+1},\ 1 \leq i \leq N-1, T_1 = S_1, T_2 = S_2S_1,\ \ldots,$ and $T_{N-1} = S_{N-1}S_{N-2}\ldots S_1$, then the SVIPMOS (1.10) becomes the GSVIP (5.3). Hence, we obtain the following result for approximating the solution of GSVIP (5.3) when the cost operators are pseudomonotone and uniformly continuous.

Theorem 5.4. *Let C_i be nonempty, closed and convex subsets of real Hilbert spaces $H_i, i = 1,2,...,N$, and let $S_i : H_i \to H_{i+1}, i = 1,2,...,N-1$, be bounded linear operators with adjoints S_i^* such that $S_i \neq 0$. Let $B_i : H_i \to H_i,\ 1,2,...,N$ be uniformly continuous pseudomonotone operators satisfying condition (3.1), and suppose Assumption A of Theorem 4.7 holds and the solution set $\Gamma \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $\hat{x} \in \Gamma$, where $\hat{x} = \min\{\|p\| : p \in \Gamma\}$.*

6. Numerical experiments

In this section, we present some numerical experiments to illustrate the implementability of our proposed method (Proposed Alg. 3.1). For simplicity, in all the experiments we consider the case when $N = 4$. All numerical computations were carried out using Matlab version R2021(b).

In our computations, we choose $\alpha_n = \frac{1}{2n+3}, \epsilon_n = \frac{1}{(2n+3)^3}, \xi_n = \frac{(1-\alpha_n)}{2}, \theta = 0.99, \lambda_{1,i} = i + 1.2, c_i = 0.97, \phi_i = 0.98, k_i = 1.96, \rho_{n,i} = \frac{10}{n^2}, \delta_{n,i} = \frac{1}{5}$.

We consider the following test examples in both finite and infinite dimensional Hilbert spaces for our numerical experiments.

Example 6.1. Let $H_i = \mathbb{R}^m,\ i = 0, 1, \ldots, 4$, and let $A_i : \mathbb{R}^m \to \mathbb{R}^m$ be a linear operator defined by $A_i(x) = Sx + q$, where $q \in \mathbb{R}^m$ and $S = NN^T + Q + D, N$ is a $m \times m$ matrix, Q is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being nonnegative (thus S is positive symmetric definite). We let $C_i = \{x \in \mathbb{R}^m : -(i+2) \leq x_j \leq i+2, j = 1, ..., m\}$. In this example, we generate randomly all the entries of N, Q in $[-3, 3]$ while D is randomly generated in $[0, 3], q = 0$ and $T_i x = \frac{3x}{i+3}$.

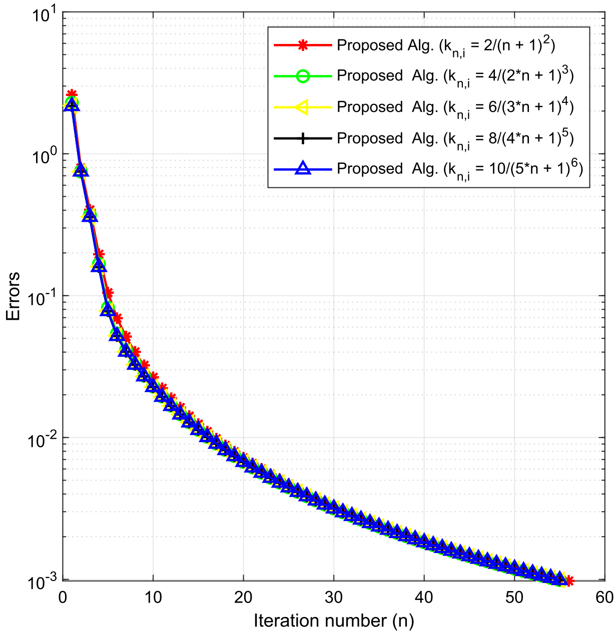

Figure 9. Experiment 6.7(1):CaseI

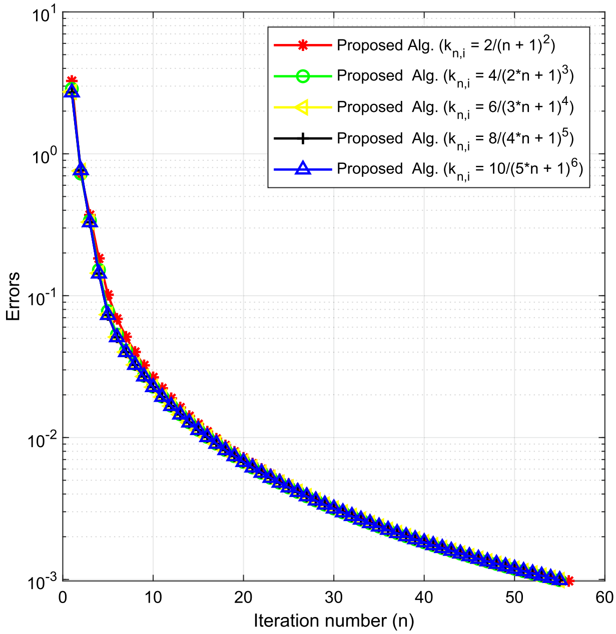

Figure 10. Experiment 6.7(1):Case2

Algorithm 5.5.

Step 0. Select initial points $x_0, x_1 \in H_1$. Let $S_0 = I^{H_1}$, $\hat{S}_{N-1} = S_{N-1}S_{N-2}\dots S_0$, $\hat{S}^*_{N-1} = S^*_0 S^*_1 \dots S^*_{N-1}$, $i = 1, 2, \dots, N$ and set $n = 1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n = \begin{cases} \min\left\{\theta, \ \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$y_{n,i} = P_{C_i}(\hat{S}_{i-1}w_n - \lambda_{n,i}B_i\hat{S}_{i-1}w_n)$$

$$\lambda_{n+1,i} = \begin{cases} \min\{\frac{(c_{n,i}+c_i)\|\hat{S}_{i-1}w_n - y_{n,i}\|}{\|B_i\hat{S}_{i-1}w_n - B_iy_{n,i}\|}, \ \lambda_{n,i} + \rho_{n,i}\}, & \text{if} \quad B_i\hat{S}_{i-1}w_n \\ & -B_iy_{n,i} \neq 0, \\ \lambda_{n,i} + \rho_{n,i}, & \text{otherwise.} \end{cases}$$

$$z_{n,i} = \hat{S}_{i-1}w_n - \beta_{n,i}r_{n,i},$$

where

$$r_{n,i} = \hat{S}_{i-1}w_n - y_{n,i} - \lambda_{n,i}(B_i\hat{S}_{i-1}w_n - B_iy_{n,i})$$

and

$$\beta_{n,i} = \begin{cases} (k_i + k_{n,i})\frac{\langle \hat{S}_{i-1}w_n - y_{n,i}, r_{n,i}\rangle}{\|r_{n,i}\|^2}, & \text{if} \quad r_{n,i} \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Step 4. Compute

$$b_n = \sum_{i=1}^N \delta_{n,i}\big(w_n + \eta_{n,i}\hat{S}^*_{i-1}(z_{n,i} - \hat{S}_{i-1}w_n)\big),$$

where

$$\eta_{n,i} = \begin{cases} \frac{(\phi_{n,i}+\phi_i)\|\hat{S}_{i-1}w_n - z_{n,i}\|^2}{\|\hat{S}^*_{i-1}(\hat{S}_{i-1}w_n - z_{n,i})\|^2}, & \text{if} \quad \|\hat{S}^*_{i-1}(\hat{S}_{i-1}w_n - z_{n,i})\| \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 5. Compute

$$x_{n+1} = (1 - \alpha_n - \xi_n)w_n + \xi_n b_n.$$

Set $n := n + 1$ and return to **Step 1**.

Example 6.2. For each $i = 0, 1, \dots, 4$, we define the feasible set $C_i = \mathbb{R}^m$, $T_ix = \frac{2x}{i+2}$ and $A_i(x) = Mx$, where M is a square $m \times m$ matrix given by

$$a_{j,k} = \begin{cases} -1, & \text{if} \quad k = m+1-j \quad \text{and} \quad k > j, \\ 1 & \text{if} \quad k = m+1-j \quad \text{and} \quad k \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

We note that M is a Hankel-type matrix with nonzero reverse diagonal.

Example 6.3. Let $H_i = \mathbb{R}^2$ and $C_i = [-1-i, 1+i]^2$, $i = 0, 1, \dots, 4$. We define $T_ix = \frac{4x}{i+4}$ and the cost operator $A_i : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$A_i(x, y) = (-xe^y, y), \quad (i = 0, 1, \dots, 4).$$

Table 3. Numerical results for Experiment 6.7(1)

Proposed Alg. 3.1	Case I		Case II	
	Iter.	CPU time	Iter.	CPU time
$\phi_{n,i} = \frac{3}{n+1}$	13	3.5072	14	4.0949
$\phi_{n,i} = \frac{6}{(n+1)^2}$	13	2.7786	14	3.3384
$\phi_{n,i} = \frac{9}{(n+1)^3}$	13	2.7132	14	3.3103
$\phi_{n,i} = \frac{12}{(n+1)^4}$	13	2.7295	14	3.3348
$\phi_{n,i} = \frac{15}{(n+1)^5}$	13	2.7467	14	3.2094

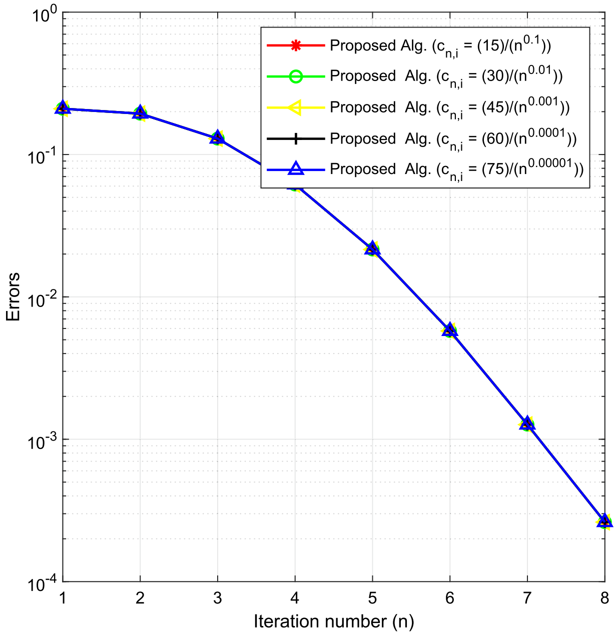

Figure 11. Experiment 6.7(2):Case I

We consider the next example in infinite dimensional Hilbert space.

Example 6.4. Let $H_i = (\ell_2(\mathbb{R}), \|\cdot\|_2), i = 0, 1, \ldots, 4$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \ldots, x_j, \ldots), x_j \in \mathbb{R} : \sum_{j=1}^{\infty} |x_j|^2 < \infty\}, ||x||_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$ for all $x \in \ell_2(\mathbb{R})$. Let $C_i := \{x = (x_1, x_2, \ldots, x_j, \ldots,) \in E : \|x\|_2 \leq i + 1\}$, and we define $T_i = \frac{5x}{i+5}$ and the cost operator $A_i : H_i \to H_i$ by $A_i x = (\frac{1}{\|x\|+s} + \|x\|)x, \quad (s > 0; i = 0, 1, \cdots 4)$. Then, A_i is uniformly continuous and pseudomonotone.

We test Examples 6.1, 6.2, 6.3 and 6.4 under the following experiments:

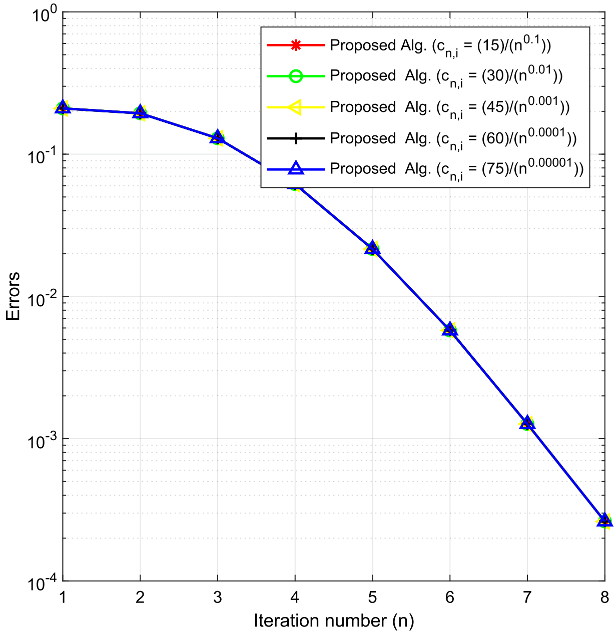

Figure 12. Experiment 6.7(2):Case II

Experiment 6.5. *In this experiment, we check the behavior of our method by fixing the other parameters and varying $\phi_{n,i}$ in Example 6.1. We do this to check the effects of this parameter and the sensitivity of our method to it.*
We consider $\phi_{n,i} \in \{\frac{3}{(n+1)}, \frac{5}{(n+1)^2}, \frac{7}{(n+1)^3}, \frac{9}{(n+1)^4}, \frac{11}{(n+1)^5}\}$ *with* $m = 25$, $m = 50$, $m = 100$ *and* $m = 200$.
Using $\|x_{n+1} - x_n\| < 10^{-3}$ *as the stopping criterion, we plot the graphs of* $\|x_{n+1} - x_n\|$ *against the number of iterations for each m.. The numerical results are reported in Figs.* 1, 2, 3, 4 *and Table* 1.

Experiment 6.6. *In this experiment, we check the behavior of our method by fixing the other parameters and varying $c_{n,i}$ in Example 6.2. We do this to check the effects of this parameter and the sensitivity of our method on it.*

We consider $c_{n,i} \in \{\frac{15}{n^{0.1}}, \frac{30}{n^{0.01}}, \frac{45}{n^{0.001}}, \frac{60}{n^{0.0001}}, \frac{75}{n^{0.00001}}\}$ *with* $m = 10$, $m = 20$, $m = 30$ *and* $m = 40$.

Using $\|x_{n+1} - x_n\| < 10^{-3}$ *as the stopping criterion, we plot the graphs of* $\|x_{n+1} - x_n\|$ *against the number of iterations in each case. The numerical results are reported in Figures 5, 6, 7, 8 and Table 2.*

Finally, we test Examples 6.3 and 6.4 under the following experiment:

Experiment 6.7. *In this experiment, we check the behavior of our method by fixing the other parameters and varying $k_{n,i}$ and $c_{n,i}$ in Examples 6.3 and 6.4. We do this to check the effects of these parameters and the sensitivity of our method on them.*

Table 4. Numerical results for Experiment 6.7 (2)

Proposed Alg. 3.1	Case I		Case II	
	Iter.	CPU time	Iter.	CPU time
$\phi_{n,i} = \frac{3}{n+1}$	8	0.0354	8	0.0327
$\phi_{n,i} = \frac{6}{(n+1)^2}$	8	0.0211	8	0.0198
$\phi_{n,i} = \frac{9}{(n+1)^3}$	8	0.0211	8	0.0195
$\phi_{n,i} = \frac{12}{(n+1)^4}$	8	0.0172	8	0.0192
$\phi_{n,i} = \frac{15}{(n+1)^5}$	8	0.0222	8	0.0189

(1) *We consider* $k_{n,i} \in \{\frac{2}{(n+1)}, \frac{4}{(2n+1)^2}, \frac{6}{(3n+1)^3}, \frac{8}{(4n+1)^4}, \frac{10}{(5n+1)^5}\}$ *with the following two cases of initial values* x_0 *and* x_1 :

Case I: $x_0 = (2, 3)$; $x_1 = (3, 4)$;

Case II: $x_0 = (1, 3)$; $x_1 = (2, 0)$.

Using $\|x_{n+1} - x_n\| < 10^{-4}$ *as the stopping criterion, we plot the graphs of* $\|x_{n+1} - x_n\|$ *against the number of iterations in each case. The numerical results are reported in Figs. 9, 10 and Table 3.*

(2) *We consider* $c_{n,i} \in \{\frac{15}{n^{0.1}}, \frac{30}{n^{0.01}}, \frac{45}{n^{0.001}}, \frac{60}{n^{0.0001}}, \frac{75}{n^{0.00001}}\}$ *with the following two cases of initial values* x_0 *and* x_1 :

Case I: $x_0 = (3, 1, \frac{1}{3}, \cdots)$; $x_1 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \cdots)$;

Case II: $x_0 = (2, 1, \frac{1}{2}, \cdots)$; $x_1 = (\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \cdots)$.

Using $\|x_{n+1} - x_n\| < 10^{-4}$ *as the stopping criterion, we plot the graphs of* $\|x_{n+1} - x_n\|$ *against the number of iterations in each case. The numerical results are reported in Figs. 11, 12 and Table 4.*

Remark 6.8. Using different initial values, cases of m and varying the key parameters in Examples 6.1–6.4, we obtained the numerical results displayed in Tables 1, 2 and 3 and Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. We noted the following from our numerical experiments:

(1) In all the examples, the choice of the key parameters $c_{n,i}$, $k_{n,i}$ and $\phi_{n,i}$ does not affect the number of iterations and no significant difference in the CPU time. Thus, our method is not sensitive to these key parameters for each initial value and case of m.

(2) The number of iterations for our method remains consistent in all the examples and so well-behaved.

7. Conclusion

In this paper, we studied the concept of split variational inequality problem with multiple output sets when the cost operators are pseudomonotone and uniformly continuous. We proposed a new Mann-type inertial projection and contraction method with self-adaptive step sizes for approximating the solution of the

problem in the framework of Hilbert spaces. Under some mild conditions on the control sequences and without prior knowledge of the operator norms, we obtained strong convergence result for the proposed algorithm. Finally, we applied our result to study certain classes of optimization problems and we presented several numerical experiments to illustrate the applicability of the proposed method.

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