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The Complete Solution of the Diophantine Equation $\left(F_{n+1}^{(k)}\right)^x - \left(F_{n-1}^{(k)}\right)^x = F_m^{(k)}$

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To the memory of Fred Howard.

Abstract. The well-known Fibonacci sequence has several generalizations, among them, the k-generalized Fibonacci sequence denoted by $F^{(k)}$. The first k terms of this generalization are $0, \ldots, 0, 1$ and each one afterward corresponds to the sum of the preceding k terms. For the Fibonacci sequence the formula $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$ holds for every $n \geq 1$. In this paper, we study the above identity on the k-generalized Fibonacci sequence terms, completing the work done by Bensella et al. (On the exponential Diophantine equation $(F_{m+1}^{(k)})^x - (F_{m-1}^{(k)})^x = F_n^{(k)}$, 2022. arxiv:2205.13168).

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1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the classical Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. A familiar identity related to these numbers is given by the formula

$$F_{n+1}^2 - F_{n-1}^2 = F_{2n}, \tag{1}$$

which holds for all integers n if we extend the Fibonacci sequence to all integer indices using its recurrence formula.

Patel and Chaves [18] investigated an analogue of (1) in higher powers, namely the Diophantine equation

$$F_{n+1}^x - F_{n-1}^x = F_m, \tag{2}$$

obtaining the following result.

Theorem 1. *The Eq.* (2) *on* (n, m, x) *has only the non-negative integral solutions* $(n, 2n, 2), (1, 1, x), (1, 2, x), (n, 0, 0)$.

Let $k \geq 2$ be an integer. A generalization of the Fibonacci sequence, called the *k-generalized Fibonacci sequence*, denoted by $F^{(k)} := \{F_n^{(k)}\}_{n \geq -(k-2)}$, is given by the linear recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad \text{for all} \quad n \geq 2, \tag{3}$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We refer to each $F_n^{(k)}$ as the nth *k-generalized Fibonacci number*. Note that for $k = 2$, we have $F_n^{(2)} = F_n$, the usual nth Fibonacci number.

Recently, Bensella et al. [1] studied an analogue of Diophantine equation (2) in k-generalized Fibonacci numbers, showing that

$$\left(F_{n+1}^{(k)}\right)^x - \left(F_{n-1}^{(k)}\right)^x = F_m^{(k)}, \tag{4}$$

has no positive integral solutions (k, n, m, x) with $x \geq 2$ under the restriction that $3 \leq k \leq \min\{n, \log x\}$. In this paper, we revisit the above Diophantine equation, remove the restriction from [1] and find parametric families of solutions. Namely, we establish the following result.

Main Theorem *The only solutions* (k, n, m, x) *of the Diophantine equation* (4) *with* $k \geq 3$, $n \geq 4$, $m \geq 2$ *and* $x \geq 2$ *are*

$$(2^{\ell+1} + 3\ell - 7, 4, 3 \cdot 2^\ell + 3\ell - 7, 2^\ell + \ell - 3), \quad \ell \geq 2;$$
$$(2^\ell + 2\ell - 4, 5, 2^{\ell+1} + 2\ell - 4, 2^{\ell-1} + (\ell - 3)/2), \ \ell \geq 3, \ \ell \text{ odd}.$$

Solutions with at least one small index or exponent (so, which do not satisfy the inequalities from the hypothesis of our Main Theorem) are called *trivial* and appear in Theorem 3.

Before we move into details, let us do a brief description of the method. First, we use lower bounds for linear forms in logarithms of algebraic numbers to get a polynomial bound for n, m and x in terms of k. When k is small, we use a variation of a result due to Dujella and Pethő on continued fractions to lower the bounds into a computationally feasible range. When k is large, we use the fact that the dominant root of the k-generalized Fibonacci sequence is exponentially close to 2 and substitute it into our calculations, this way we get a simpler linear form in logarithms which allows us to bound k and complete our calculations.

The calculations were done with Mathematica and the running time was about one hour on 25 computers.

2. Preliminary Results

2.1. On *k*-Fibonacci Numbers

The k-generalized Fibonacci sequence and its terms have an extensive list of properties. Here, we only present some of them, namely those strictly necessary to address our problem. For more related data, we invite the reader to consult [1,10], where one finds some background for this paper, as well as [2,3,5,7,9,11,13,16,17,19].

Lemma 1. *Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be all the zeroes of the characteristic polynomial $\Psi_k(z) = z^k - z^{k-1} - \cdots - z - 1$ for the sequence $F^{(k)}$.*

(i) Ψ_k is irreducible over $\mathbb{Q}[z]$ with one real zero outside the unit circle, named $\alpha := \alpha_1$ (sometimes denoted by $\alpha(k)$ to emphasize its dependence on k), which satisfies $2(1-2^{-k}) < \alpha < 2$.

(ii) The first $k+1$ non-zero terms in $F^{(k)}$ are powers of two, namely

$$F_1^{(k)} = 1 \quad \text{and} \quad F_n^{(k)} = 2^{n-2} \quad \text{for all} \quad 2 \le n \le k+1.$$

Also, $F_{k+2}^{(k)} = 2^k - 1$ and, moreover,

$$F_n^{(k)} < 2^{n-2} \quad \text{for all} \quad n \ge k+2. \tag{5}$$

(iii) Let $f_k(z) := (z-1)/(2+(k+1)(z-2))$. For all $n \ge 1$ and $k \ge 2$,

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{\,n-1} = f_k(\alpha)\alpha^{n-1} + e_k(n) \tag{6}$$

with $|e_k(n)| < 1/2$.

(iv) For all $k \ge 2$ and $i = 2, \ldots, k$

$$f_k(\alpha) \in [1/2, 3/4] \qquad \text{and} \qquad |f_k(\alpha_i)| < 1. \tag{7}$$

Thus, $f_k(\alpha)$ is not an algebraic integer, for any $k \ge 2$.

(v) For all $k \ge 19$,

$$f_k(\alpha) > \frac{1}{2} + \frac{k-1}{2^{k+2}} \qquad \text{and} \qquad f_k(\alpha)\alpha > 1 + \frac{k}{2^{k+2}}.$$

(vi) For all $n \ge 1$ and $k \ge 2$,

$$\alpha^{n-2} \le F_n^{(k)} \le \alpha^{n-1}. \tag{8}$$

(vii) The sequences $(F_n^{(k)})_{n\ge1}$, $(F_n^{(k)})_{k\ge2}$ and $(\alpha(k))_{k\ge2}$ are non decreasing.

(viii) For all $n \ge 3$ and $k \ge 3$, we have $F_{n-1}^{(k)}/F_{n+1}^{(k)} \le 3/7$.

Since our equation involves the powers $(F_r^{(k)})^x$ for $r = n \pm 1$, we need the following lemma.

Lemma 2. *Let $k \ge 2$, x, r positive integers. Then*

$$(F_r^{(k)})^x = f_k(\alpha)^x \alpha^{(r-1)x}(1+\eta_r), \quad \text{with} \quad |\eta_r| < \frac{xe^{x/\alpha^{r-1}}}{\alpha^{r-1}}.$$

Proof. By Binet's formula (6) and inequalities (7), it follows that

$$(F_r^{(k)})^x = f_k(\alpha)^x \alpha^{(r-1)x}\left(1 + \frac{e_k(r)}{f_k(\alpha)\alpha^{r-1}}\right)^x.$$

Then, if we take

$$\eta_r := \left(1 + \frac{e_k(r)}{f_k(\alpha)\alpha^{r-1}}\right)^x - 1 = \sum_{j=1}^{x} \binom{x}{j}\left(\frac{e_k(r)}{f_k(\alpha)\alpha^{r-1}}\right)^j;$$

we get

$$|\eta_r| < \sum_{j=1}^{x} \frac{(x/\alpha^{r-1})^j}{j!} < \frac{x}{\alpha^{r-1}} \sum_{j\geq 1} \frac{(x/\alpha^{r-1})^{j-1}}{(j-1)!} \leq \frac{x e^{x/\alpha^{r-1}}}{\alpha^{r-1}},$$

where we have used the fact that (7) implies $|e_k(r)| \leq 1/2 \leq f_k(\alpha)$. □

The following identity is due to Cooper and Howard (see [6]).

Lemma 3. *For $k \geq 2$ and $n \geq k+2$, we have*

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\ell-1} C_{n,j}\, 2^{n-(k+1)j-2},$$

where

$$\ell := \left\lfloor \frac{n+k}{k+1} \right\rfloor \quad \text{and} \quad C_{n,j} := C_{n,j}^{(k)} = (-1)^j \left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right]$$

with the classical convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of a or b is negative and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

The following lemmas are consequences of the previous one and will be essential to effectively solve the Diophantine equation (4).

Lemma 4. *If $k+2 \leq r < 2^{ck}$ for some $c \in (0,1)$, then the following estimates hold:*

(i) $F_r^{(k)} = 2^{r-2}\left(1 - \frac{r-k}{2^{k+1}} + \frac{f(r,k)}{2^{2k+3}} + \zeta(r,k)\right)$, *with* $|\zeta(r,k)| < \frac{4\,r^3}{2^{3k+3}}$, *where* $f(r,k) := \delta(r-2k+1)(r-2k-2)$ *and*

$$\delta := \begin{cases} 0, & \text{if } r \leq 2k+2, \\ 1, & \text{if } r > 2k+2. \end{cases}$$

Even more,

(ii) $F_r^{(k)} = 2^{r-2}(1+\zeta')$, *with* $|\zeta'| < \dfrac{2r}{2^k} < \dfrac{2}{2^{k(1-c)}}$.

(iii) $F_r^{(k)} = 2^{r-2}\left(1 - \dfrac{r-k}{2^{k+1}} + \zeta''\right)$, *with* $|\zeta''| < \dfrac{4r^2}{2^{2k+2}} < \dfrac{1}{2^{2k(1-c)}}$.

Proof. By Cooper and Howard's identity in Lemma 3, we can write

$$F_r^{(k)} = 2^{r-2}\left(1 - \frac{C_{r,1}}{2^{k+1}} + \frac{\delta C_{r,2}}{2^{2k+2}} + \zeta(r,k)\right),$$

where $C_{r,1} = r-k,\ C_{r,2} = (r-2k+1)(r-2k-2)/2$ and

$$\begin{aligned} |\zeta(r,k)| &\leq \sum_{j=3}^{\lfloor \frac{r+k}{k+1} \rfloor - 1} \frac{|C_{r,j}|}{2^{(k+1)j}} < \sum_{j\geq 3} \frac{2\,r^j}{2^{(k+1)j}\,j!} \\ &< \frac{2\,r^3}{2^{3k+3}} \sum_{j\geq 3} \frac{(r/2^{k+1})^{j-3}}{(j-3)!} < \frac{2\,r^3}{2^{3k+3}} e^{r/2^{k+1}}. \end{aligned}$$

Since $r < 2^{ck}$ with $c \in (0,1)$, then $e^{r/2^{k+1}} < 2$. Thus, we get

$$|\zeta(r,k)| < \frac{4\,r^3}{2^{3k+3}},$$

which corresponds to (i). Further, since we have

$$\frac{r-k}{2^{k+1}} < \frac{r}{2^{k+1}} \qquad \text{and} \qquad \frac{f(r,k)}{2^{2k+3}} < \frac{r^2}{2^{2k+3}} < \frac{r}{2^{k+3}},$$

we can conclude (ii) and (iii). □

As a consequence of the previous lemma, we have the following result.

Lemma 5. *For integers $x \geq 1$, $k \geq 2$, $i \in \{-1,1\}$, $n+i \geq k+2$ and $\max\{n+i,x\} < 2^{ck}$ for some $c \in (0,1/2)$, the estimate*

$$\left(F_{n+i}^{(k)}\right)^x = 2^{(n+i-2)x}\left(1 - \frac{x(n+i-k)}{2^{k+1}} + \xi_i\right)$$

holds with

$$|\xi_i| < \frac{24(nx)^2}{2^{2k+2}} < \frac{6}{2^{2k(1-2c)}}.$$

Proof. By item (iii) of Lemma 4 with $r = n+i$ and $c \in (0,1/2)$, we have

$$F_{n+i}^{(k)} = 2^{n+i-2}\left(1 - \frac{n+i-k}{2^{k+1}} + \zeta_i''\right), \text{ with } |\zeta_i''| < \frac{4(n+i)^2}{2^{2k+2}} < \frac{1}{2^{2k(1-c)}}.$$

Hence,

$$\left(F_{n+i}^{(k)}\right)^x = 2^{(n+i-2)x}\left(1 - \frac{n+i-k}{2^{k+1}} + \zeta_i''\right)^x.$$

We now use the binomial theorem to analyze

$$\left(1 - \frac{n+i-k}{2^{k+1}} + \zeta_i''\right)^x.$$

We put $A_i := -(n+i-k)/2^{k+1}$, so $|A_i| < (n+i)/2^{k+1}$. Since $n+i < 2^{ck}$, we get that

$$|\zeta_i''| < \frac{2}{2^{(1-c)k}}\left(\frac{n+i}{2^{k+1}}\right).$$

Further, let

$$\xi_i := (1 + A_i + \zeta_i'')^x - 1 - xA_i.$$

Thus,

$$
\begin{aligned}
|\xi_i| &\le x|\zeta_i''| + \sum_{j=2}^{x} \binom{x}{j} |A_i + \zeta_i''|^j \\
&< \frac{4x(n+i)^2}{2^{2k+2}} + \sum_{j=2}^{x} \left(x\left(|A_i| + |\zeta_i''|\right)\right)^j \\
&< \frac{4x(n+i)^2}{2^{2k+2}} + \left(\frac{1.5x(n+i)}{2^{k+1}}\right)^2 \sum_{j\ge 0} \left(\frac{1.5x(n+i)}{2^{k+1}}\right)^j \\
&< \frac{x^2(n+i)^2}{2^{2k+2}} \left(\frac{4}{x} + 9\right) = \frac{13x^2(n+i)^2}{2^{2k+2}} < \frac{24(nx)^2}{2^{2k+2}}.
\end{aligned}
$$

In the above, we used that

$$
x(|A_i| + |\zeta_i''|) < \frac{x(n+i)}{2^{k+1}} \left(1 + \frac{1}{2^{(1-c)k}}\right) < \frac{1.5x(n+i)}{2^{k+1}}.
$$

Furthermore, since the above number is smaller than $\frac{1.5}{2^{(1-2c)k+1}} < \frac{3}{4}$ it follows that the sum of the geometrical progression is at most

$$
\sum_{j\ge 0} \left(\frac{1.01x(n+i)}{2^{k+1}}\right)^j < \frac{1}{1 - \frac{1.01x(n+i)}{2^{k+1}}} < 4.
$$

In addition to this, we used for the last inequality that $n \ge 3$, $i \le 1$, so

$$
(n+i)^2 \le (n+1)^2 \le (4/3)^2 n^2 \quad \text{and} \quad 13 \times (4/3)^3 < 24.
$$

□

Finally, we present some relations between the variables in the Eq. (4).

Lemma 6. *Let (n, m, x, k) be a solution of the Diophantine equation (4) with $n, m \ge k+1,\ x \ge 1$ and $k \ge 2$ and assume that*

$$
k > 2\log_2(m+k).
$$

Then, one of the following holds:

(i) $n - 1 \equiv -1,\ 0, 1$ *or* $2 \pmod{k+1}$. *Thus,* $m \equiv 1$ *or* $2 \pmod{k+1}$.

(ii) *Letting* r_1 *be the residue of* n *by division with* $k+1$, *with* $3 \le r_1 \le k-1$, *we have* $(r_1 - 2)x \le 2\log_2(m+k) + k - 1$.

Proof. Let us take

$$
u := \frac{m+k}{k+1} + k - 2, \quad v := k+1; \quad w := \frac{2(m+k)}{k+1} + k - 1.
$$

By relation (3), we have the identity $F_n^{(k)} = 2F_{n-1}^{(k)} - F_{n-k-1}^{(k)}$ for all $n \ge k+1$. Reducing modulo 2 we get that $F_n^{(k)} \equiv F_{n-k-1}^{(k)} \pmod 2$, so $F^{(k)}$ is periodic modulo 2 with period $k+1$. Since $F_1^{(k)} = F_2^{(k)} = 1$ and $F_j^{(k)} = 2^{j-2} \equiv 0 \pmod 2$ for $j = 3, \ldots, k+1$ it follows that $F_n^{(k)}$ is even except when $n \equiv 1, 2 \pmod{k+1}$. So, assuming $F_{n+1}^{(k)}$ and $F_{n-1}^{(k)}$ are both even, we get that $n - 1 \not\equiv -1, 0, 1, 2, \pmod{k+1})$. We write $n - 1 = (k+1)s_1 + r_1$

with $r_1 \in \{3, \ldots, k-1\}$. Then $n-1+k = (k+1)(s_1+1) + r_1 - 1$ and $\lfloor (n-1+k)/(k+1) \rfloor - 1 = s_1$. We recall that by Cooper and Howard's formula (Lemma 3),

$$F_{n-1}^{(k)} = 2^{n-3} + \sum_{j=1}^{s_1} C_{n-1,j} 2^{n-(k+1)j-3}.$$

Note that, the exponent of 2 in the last term above is $n-(k+1)s_1-3 = r_1-2$. Thus, $2^{r_1-2} \mid F_{n-1}^{(k)}$. Similarly, since $n+1 = (k+1)s_1 + r_1 + 2$, we have that $(n+1)+k = (k+1)(s_1+1) + r_1 + 1$, so, we have that $2^{r_1} \mid F_{n+1}^{(k)}$. Hence,

$$2^{(r_1-2)x} \mid \left(F_{n+1}^{(k)}\right)^x - \left(F_{n-1}^{(k)}\right)^x = F_m^{(k)}. \tag{9}$$

We now use Cooper and Howard's formula

$$F_m^{(k)} = 2^{m-2} + \sum_{j=1}^{s-1} C_{m,j} 2^{m-(k+1)j-2},$$

where $s = \lfloor (m+k)/(k+1) \rfloor (\geq 2)$. Write $m+k = s(k+1)+r$, where $0 \leq r \leq k$. Then the last term of the above sum has exponent of 2 equal to

$$\begin{aligned} m-(s-1)(k+1)-2 &= s(k+1)+r-k-(s-1)(k+1)-2 \\ &= (k+1)-k+r-2 \\ &= r-1. \end{aligned}$$

Therefore, we have

$$2^{k+r-1} \mid 2^{m-2} + \sum_{j=1}^{s-2} C_{m,j} 2^{m-(k+1)j-2},$$

so

$$F_m^{(k)} = 2^{k+r-1} M + 2^{r-1} C_{m,s-1}$$

with some integer M. We proceed to calculate an upper bound for $\nu_2(C_{m,s-1})$. We have

$$\begin{aligned} C_{m,s-1} &= (-1)^{s-1} \left(\binom{m-(s-1)k}{s} - \binom{m-(s-1)k-2}{s-2} \right) \\ &= (-1)^{s-1} \left(\binom{s+r}{s-1} - \binom{s+r-2}{s-3} \right). \end{aligned}$$

Then,

$$\begin{aligned} C_{m,s-1} &= \binom{s+r-2}{s-3} \frac{((s+r)(s+r-1)-(s-1)(s-2))}{(s-2)(s-1)} \\ &= \binom{s+r-2}{s-3} \frac{(r+1)(2s+r-2)}{(s-1)(s-2)} \\ &= \binom{s+r-2}{r+1} \frac{(r+1)(2s+r-2)}{(s-2)(s-1)} \\ &\mid \binom{s+r-2}{r} (2s+r-2) \end{aligned}$$

where $\mid$ denote the symbol of divisibility.

Thus, we have

$$\begin{aligned}\nu_2(C_{m,s-1}) &\le \nu_2\binom{s+r-2}{r} + \nu_2(2s+r-2)\\ &\le \log_2(s+r-2) + \log_2(2s+r-2).\end{aligned}$$

In the first inequality we have used Kummer's inequality (see [14]), which states that the 2-adic valuation[1] of $\binom{v}{w}$, is equal to the number of carries in the sum between w and $v-w$ in base 2, in particular

$$\nu_2\binom{v}{w} = \sum_{i=1}^{\infty}\left(\left\lfloor\frac{v}{2^i}\right\rfloor - \left\lfloor\frac{w}{2^i}\right\rfloor - \left\lfloor\frac{v-w}{2^i}\right\rfloor\right) \le \log_2 v,$$

as the above sum contains at most $\lfloor \log_2 v\rfloor$ nonzero terms and each nonzero term is equal to 1.

Since

$$s+r-2 \le 2s+r-2 < s(k+1)+r < m+k,$$

we get that $\nu_2(C_{m,s-1}) \le 2\log_2(m+k)$. Thus, if $k > 2\log_2(m+k)$, then

$$\begin{aligned}\nu_2(F_m^{(k)}) = \nu_2(2^{k+r-1}M + C_{m,s-1}2^{r-1}) &= \nu_2(C_{m,s-1}2^{r-1})\\ &\le 2\log_2(m+k)+r-1.\end{aligned}$$

Since $\nu_2(F_m^{(k)}) \ge (r_1-2)x$ according to (9) and $r \le k$, we get (ii). $\square$

3. Tools

3.1. Linear Forms in Logarithms

Since we use a Baker-type lower bound for a nonzero linear form in logarithms of algebraic numbers, in this section we present the necessary concepts and results.

Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers $m(z) := a_0\prod_{i=1}^{d}(z-\eta_i) \in \mathbb{Z}[z]$, where the leading coefficient a_0 is positive and the η_i's are the conjugates of η. The logarithmic height of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\max\{|\eta_i|,1\}\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q)=1$ and $q>0$, we then have $h(\eta) = \log\max\{|p|,q\}$.

For example, since $\Psi_k(z)$ is the minimal primitive polynomial of α, we have $\mathbb{Q}(\alpha) = \mathbb{Q}(f_k(\alpha))$. Besides, by (7), we have that $|f_k(\alpha_i)| \le 1$ for all $i = 1,\ldots,k$ and $k \ge 2$. Thus,

$$h(\alpha) = (\log\alpha)/k \quad \text{and} \quad h(f_k(\alpha)) < 2\log k, \quad \text{for all} \quad k \ge 2. \tag{10}$$

See [5] for further details concerning the proof of (10).

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[1] $\nu_2(m)$ is the exponent of 2 in the factorizatión of $m \neq 0$ in primes factors.

The following properties of $h(\cdot)$ will be used in the following sections:

$$h(\eta \pm \gamma) \le h(\eta) + h(\gamma) + \log 2, \qquad h(\eta\gamma^{\pm 1}) \le h(\eta) + h(\gamma),$$
$$h(\eta) = h(\eta^{(i)}), \qquad h(\eta^s) = |s| h(\eta) \quad (s \in \mathbb{Z}).$$

Our main tool is a lower bound for a non-zero linear form in logarithms of algebraic numbers due to Matveev [15]:

Theorem 2. (Matveev's theorem) *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \quad \textit{and} \quad B \ge \max\{|b_1|, \ldots, |b_t|\}.$$

Let $A_i \ge \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ for $i = 1, \ldots, t$. If $\Lambda \ne 0$, then

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B) A_1 \cdots A_t).$$

3.2. Analytical Arguments

Note that, for $\gamma_1, \ldots, \gamma_t$ real algebraic numbers,

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \quad \text{and} \quad \Gamma := b_1 \log \eta_1 + \cdots + b_t \log \eta_t,$$

are such that $\Lambda = e^{\Gamma} - 1$. Therefore, it is a straight-forward exercise to show that $|\Gamma| < (1-c)^{-1}|\Lambda|$, when $|\Lambda| < c$, for all constant c in $(0, 1)$. We use this argument several times without mentioning it.

On the other hand, in some specific parts of our work, we need the following analytic result, which correspond to Lemma 7 from [12].

Lemma 7. *If $r \ge 1$ and $T > (4r^2)^r$, then*

$$\frac{y}{(\log y)^r} < T, \qquad \text{implies} \qquad y < 2^r T (\log T)^r.$$

3.3. Reduction by Continued Fractions

The application of the results from the previous subsections give us some large bounds on the integers variables of the Diophantine equation (4). Since these bounds are very large, we use some results from the theory of continued fractions and geometry of numbers to reduce them. Here, we present these reduction techniques.

For the treatment of homogeneous linear forms in two integer variables we use a classical theorem of Legendre.

Lemma 8. *Let M be a positive integer and $P_1/Q_1, P_2/Q_2, \ldots$ the convergents of the continued fraction $[a_0, a_1, \ldots]$ for τ. Let N be a positive integer such that $M < Q_{N+1}$. If $a_M := \max\{a_\ell : 0 \le \ell \le N+1\}$, then the inequality*

$$\left|\tau - \frac{v}{u}\right| > \frac{1}{(a_M + 2)u^2},$$

holds for all pairs (u, v) of integers with $0 < u < M$.

Next, we present a slight variation of the result of Dujella and Pethő (Lemma 5a in [8]). For a real number X, we use

$$||X|| := \min\{|X - n| : n \in \mathbb{Z}\}$$

to denote the distance from X to its nearest integer.

Lemma 9. *Let M and Q be positive integers such that $Q > 6M$, and A, B, τ, μ be real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := ||\mu Q|| - M||\tau Q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < A \cdot B^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(AQ/\varepsilon)}{\log B}.$$

In practical applications, Q is the denominator of a continued fraction convergent for τ.

4. Some Considerations

Recall that we are working with integers $k \geq 2$, $n \geq 1$, $m \geq 2$ and $x \geq 1$. First, we present some trivial solutions of Eq. (4).

Theorem 3. *The trivial solutions (k, n, m, x) of Diophantine equation (4) are*

$$(2, n, n, 1), \quad (2, n, 2n, 2) \quad \text{for all} \quad n \geq 1;$$
$$(k, 1, 2, x), \quad (k, 2, 2, 1), \quad (k, 2, k+2, k), \quad \text{for all} \quad k \geq 2 \text{ and } x \geq 1;$$

and

$$(2x, 3, 2x+2, x), \quad \text{for all} \quad x \geq 2.$$

Proof. When $k = 2$, according to Theorem 1, the Diophantine equation (2) only has the solutions $(n, m, x) = (n, n, 1)$ or $(n, 2n, 2)$, for all $n \geq 1$ or $(1, 2, x)$ for all $x \geq 1$.

Now, let us consider some particular values of n:

- If $n = 1$, then (4) corresponds to $F_m^{(k)} = 1$, which has the solution $m = 2$ for all $k \geq 2$ and $x \geq 1$.
- If $n = 2$, then (4) corresponds to $F_m^{(k)} = 2^x - 1$, which has only the solutions $(m, x) = (2, 1)$ and $(k+2, k)$ for all $k \geq 3$ by the results from [4].
- If $n = 3$, then, when $k \geq 3$, we have that Eq. (4) corresponds to $F_m^{(k)} = 2^{2x} - 1$, which has only the solutions $(k, m, x) = (x, 2x+2, x)$ for all $x \geq 2$, again by the results from [4].

Hence, from now on we may assume that $k \geq 3$ and $n \geq 4$. Note that by inequalities (8), we have that

$$\alpha^{m-2} \leq F_m^{(k)} = \left(F_{n+1}^{(k)}\right)^x - \left(F_{n-1}^{(k)}\right)^x < \left(F_{n+1}^{(k)}\right)^x \leq \alpha^{nx},$$

and

$$\alpha^{(n-1)x-1} < \alpha^{(n-1)x} - \alpha^{(n-2)x} \leq \left(F_{n+1}^{(k)}\right)^x - \left(F_{n-1}^{(k)}\right)^x = F_m^{(k)} \leq \alpha^{m-1}.$$

Thus, we can conclude that

$$(n-1)x < m < nx + 2. \tag{11}$$

So, if we take $x = 1$, the previous inequality implies that $m = n$ or $n+1$, which do not provide solutions for Eq. (4). $\square$

The previous arguments reduce our problem to find the solutions (k, n, m, x) of the Diophantine equation

$$\left(F_{n+1}^{(k)}\right)^x - \left(F_{n-1}^{(k)}\right)^x = F_m^{(k)},$$

with $k \geq 3$, $n \geq 4$, $m \geq 2$ and $x \geq 2$.

4.1. Non-zero Linear Forms in Logarithms

In this section, we show that the linear forms in logarithms of algebraic numbers, to which we apply Theorem 2, are non-zero. Let us take

$$\begin{aligned}
\Lambda_1 &:= f_k(\alpha)\alpha^{m-1}2^{-(n-1)x} - 1;\\
\Lambda_2 &:= (f_k(\alpha))^{-1}\alpha^{-(m-1)}2^{(n-3)x}(2^{2x} - 1) - 1;\\
\Lambda_3 &:= f_k(\alpha)\alpha^{m-1}\left(F_{n+1}^{(k)}\right)^{-x} - 1;\\
\Lambda_4 &:= f_k(\alpha)^{1-x}\alpha^{m-1-nx} - 1;\\
\Lambda_5 &:= f_k(\alpha)^{x-1}\alpha^{nx-(m-1)}(1 - \alpha^{-2x}) - 1,
\end{aligned}$$

$\mathbb{K} = \mathbb{Q}(\alpha)$ and $D = [\mathbb{K} : \mathbb{Q}] = k$. With the previous notation, we prove the following result.

Lemma 10. $\Lambda_i \neq 0$ *for all* $i \in \{1, 2, 3, 4, 5\}$.

Proof. For $i = 1, 2, 3, 4$ and 5, if $\Lambda_i = 0$, then

$$f_k(\alpha) \in \left\{\alpha^{1-m}2^{(n-1)x}, \alpha^{1-m}2^{(n-3)x}(2^{2x} - 1), \alpha^{1-m}\left(F_{n+1}^{(k)}\right)^x\right\}$$

and

$$f_k(\alpha)^{x-1} \in \left\{\alpha^{m-1-nx}, \alpha^{nx-(m-1)}(1 - \alpha^{-2x})\right\}.$$

All these cases imply that $f_k(\alpha)$ would be an algebraic integer since α is a unit in $\mathcal{O}_{\mathbb{K}}$, which contradicts (iv) from Lemma 1. $\square$

5. The Case $n \leq k$

By (ii) in Lemma 1, the equation is

$$F_m^{(k)} = 2^{(n-1)x} - 2^{(n-3)x}. \tag{12}$$

Since $n \geq 4$ and

$$(n-1)x - (n-3)x = 2x > 1,$$

it is clear that $F_m^{(k)}$ cannot be a power of 2. Thus, by (ii) of Lemma 1, $m \geq k + 2$. Using inequalities (5) and (8) in (12), we have

$$2^{(n-1)x-0.2} < F_m^{(k)} < 2^{m-2} \quad \text{and} \quad 2^{(m-2)/2} < \alpha^{m-2} < F_m^{(k)} < 2^{(n-1)x},$$

which gives

$$(n-1)x + 1.8 \leq m \leq 2(n-1)x + 2. \tag{13}$$

We next establish some additional relations between the variables in the Diophantine equation (12).

Lemma 11. *Let $4 \le n \le k$. If (k, n, m, x) is a solution of (12) with $x \ge 2$, then*

$$x < 3.2 \times 10^{11} k^4 (\log k)^2 \log m. \tag{14}$$

and

$$m < 3.4 \times 10^{29} k^9 (\log k)^6. \tag{15}$$

Proof. The Binet formula (6) implies that

$$|\Lambda_1| = \left| f_k(\alpha)\alpha^{m-1} 2^{-(n-1)x} - 1 \right| < \frac{2}{2^{2x}}. \tag{16}$$

We apply Theorem 2 on inequality (16) with $t := 3$, $\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$, $B := m$ and

$$\begin{array}{lll} (\gamma_1, b_1) := (f_k(\alpha), 1), & (\gamma_2, b_2) := (\alpha, m-1), & (\gamma_3, b_3) := (2, -(n-1)x); \\ A_1 := 2k \log k, & A_2 := 0.7, & A_3 := 0.7k. \end{array}$$

We have

$$\exp\left(-4.4 \times 10^{11} \times k^4 (\log k)^2 \log m\right) < |\Lambda_1| < 2^{-(2x-1)}.$$

We conclude that

$$x < 3.2 \times 10^{11} k^4 (\log k)^2 \log m,$$

as we wanted to prove.

Returning to Eq. (12), we also include $2^{(n-3)x}$ into the above linear form in logarithms, getting

$$|\Lambda_2| = \left| (f_k(\alpha))^{-1} \alpha^{-(m-1)} 2^{(n-3)x} (2^{2x} - 1) - 1 \right| < \frac{2}{\alpha^m}. \tag{17}$$

We apply again Theorem 2 with $t := 4$, the same $\mathbb{K}$, D and B, and

$$\begin{array}{ll} (\gamma_1, b_1, A_1) := (f_k(\alpha), -1, 2k \log k), & (\gamma_2, b_2, A_2) := (\alpha, -(m-1), 0.7), \\ (\gamma_3, b_3, A_3) := (2, (n-2)x, 0.7k), & (\gamma_4, b_4, A_4) := (2^{2x} - 1, 1, 1.4kx). \end{array}$$

The conclusion of Theorem 2 together with the above inequality (17) yield

$$\begin{aligned} m &< 6.7 \times 10^{13} k^4 (kx) (\log k)^2 (\log m) \\ &< 2.2 \times 10^{25} k^9 (\log k)^4 (\log m)^2, \end{aligned}$$

where we have used inequality (14). Hence, by Lemma 7 with $(y, r) := (m, 2)$ and $T := 2.2 \times 10^{25} k^9 (\log k)^4$, we have

$$m < 3.4 \times 10^{29} k^9 (\log k)^6,$$

which corresponds to inequality (15). □

5.1. When $k \leq 600$

We start by looking for an upper bound on x. So, we take

$$\Gamma_1 := (m-1)\log\alpha - (n-1)x\log 2 + \log(f_k(\alpha)).$$

Since $x \geq 2$, by inequality (16) we have that

$$0 < |(m-1)\log\alpha - (n-1)x\log 2 + \log(f_k(\alpha))| < \frac{3}{2^{2x}}.$$

So, if we set

$$\tau_k := (\log\alpha)/\log 2, \quad \mu_k := \log(f_k(\alpha))/\log 2, \quad A := 5, \quad B := 2,$$

we then get

$$0 < |(m-1)\tau_k - (n-1)x + \mu_k| < AB^{-2x}. \tag{18}$$

For each $k \in [4, 600]$, we consider $M := 3.4 \times 10^{29} k^9 (\log k)^6$, which is an upper bound to $m-1$, according to inequality (15). A computer search shows that

$$\max_{k\in[4,600]} \left\{ \lfloor \log\left(AQ^{(k)}/\varepsilon_k \right) / \log B \rfloor \right\} \leq 1190.$$

Hence, by Lemma 9, we can conclude that $x \leq 595$.

Now that we have bounded x, let us fix it in $[2, 595]$ and consider

$$\Gamma_2 := (m-1)\log\alpha - (n-3)x\log 2 - \log\left(\frac{2^{2x}-1}{f_k(\alpha)} \right).$$

Using inequality (17) in its logarithmic form, we obtain a similar inequality to (18), namely

$$0 < |(m-1)\tau_k - (n-3)x + \mu_{k,x}| < AB_k^{-m}, \tag{19}$$

where we put

$$\tau_k := \frac{\log\alpha}{\log 2}, \quad \mu_{k,x} := -\frac{\log((2^{2x}-1)/f_k(\alpha))}{\log 2}, \quad A := 5 \quad \text{and} \quad B_k := \alpha.$$

Therefore, for $k \in [4, 600]$ and $x \in [2, 595]$, we apply Lemma 9 to inequality (19) using $M := 3.4 \times 10^{29} k^9 (\log k)^6$. With computational support, we obtain

$$\max_{k\in[4,600],\ x\in[2,595]} \left\{ \lfloor \log\left(AQ^{(k,x)}/\varepsilon_{k,x} \right) / \log B_k \rfloor \right\} \leq 1520.$$

Thus, by Lemma 9, we have that $m \leq 1520$.

In summary, for $n \leq k$, the integer solutions (k, n, m, x) of (12) must satisfy $k \in [4, 600]$, $x \in [2, 595]$, $m \in [k+2, 1520]$ and, by (13)

$$n \in [4, N] \quad \text{with} \quad N := \min\{k, 1 + \lfloor (m-1)/x \rfloor\}.$$

However, a computational search in the above range for solutions of the Diophantine equation (12) gave us only those that we indicated in the statement of the Main Theorem (the first family for $\ell \in [2, 8]$ and the second family for $\ell \in [3, 9]$ and odd).

5.2. When $k > 600$

Given that

$$m < 3.4 \times 10^{29} k^9 (\log k)^6 < 2^{k/2},$$

by equality (12) and Lemma 4, (ii) with $c = 1/2$ for $r = m$, we get

$$|2^{(n-1)x} - 2^{(n-3)x} - 2^{m-2}| < \frac{2^{m-1}}{2^{k/2}}.$$

Thus, dividing by 2^{m-2}, we get

$$|1 + 2^{(n-3)x-(m-2)} - 2^{(n-1)x-(m-2)}| < \frac{2}{2^{k/2}}.$$

By (13), we have $(n-1)x - (m-2) \leq 0$. If $(n-1)x - (m-2) \leq -1$, then in the above inequality the left-hand side is larger than $1/2$, which contradicts the fact that $k > 600$. So, $m - 2 = (n-1)x$.

Now, by equality (12) and Lemma 4, (iii) with $c = 1/2$ for $r = m$, we have

$$\left| \frac{m-k}{2^{k+1}} - \frac{1}{2^{2x}} \right| < \frac{4m^2}{2^{2k+2}}.$$

Assume that the left-hand side is non-zero. Then

$$\left| \frac{m-k}{2^{k+1}} - \frac{1}{2^{2x}} \right| \geq \frac{1}{2^{\max\{2x, k+1\}}}.$$

If $2x \leq k+1$, the left-hand side above is $\geq 1/2^{k+1}$. If $2x \geq k+2$, then

$$\frac{m-k}{2^{k+1}} - \frac{1}{2^{2x}} \geq \frac{1}{2^{k+1}} - \frac{1}{2^{k+2}} = \frac{1}{2^{k+2}}.$$

Hence, the inequality

$$\left| \frac{m-k}{2^{k+1}} - \frac{1}{2^{2x}} \right| \geq \frac{1}{2^{k+2}}$$

holds in all cases. Thus, we get

$$\frac{1}{2^{k+2}} \leq \frac{4m^2}{2^{2k+2}},$$

which leads to

$$2^{k/2} < 2m < 6.8 \times 10^{29} k^9 (\log k)^6.$$

However, this implies $k < 400$, a contradiction. In summary, we have

$$2^{m-2} = 2^{(n-1)x} \qquad \text{and} \qquad 2^{2x}(m-k) = 2^{k+1}.$$

Now, if $m > 2k+2$ then by Lemma 4, (i) with $c = 1/2$ for $r = m$, we obtain

$$\left| \frac{f(m,k)}{2^{2k+3}} \right| < \frac{4m^3}{2^{3k+3}}, \qquad \text{with} \qquad f(m,k) = (m-2k+1)(m-2k-2),$$

which lead us to a contradiction on $k > 600$. Hence, $m \leq 2k+2$. Thus, by Lemma 3, our equation becomes

$$2^{(n-1)x} - 2^{(n-3)x} = F_m^{(k)} = 2^{m-2} - 2^{m-k-3}(m-k).$$

We get $2^{m-k-3}(m-k) = 2^{(n-3)x}$. But also $m-k = 2^{k+1-2x}$. Since $m \leq 2k+2$, we have that

$$k + 1 - 2x = \log_2(m-k) \leq \log_2(k+2),$$

so

$$x \geq \frac{k+1-\log_2(k+2)}{2}.$$

However,

$$n-1 = \frac{m-2}{x} \leq \frac{2k}{(k+1-\log_2(k+2))/2} < 5 \qquad \text{for} \qquad k > 600,$$

showing that $n \leq 5$. Hence, $n \in \{4,5\}$. Further, let $\ell := k+1-2x$. Then $m = k + 2^\ell$ so $\ell \geq 1$. Next, we have

$$x = \frac{m-2}{n-1} = \frac{k+2^\ell-2}{n-1}.$$

Finally, since also $m-k = 2^{(n-3)x-(m-k-3)}$, we get that

$$\begin{aligned} \ell = (n-3)x - (m-k-3) &= \frac{(n-3)}{(n-1)}(k+2^\ell-2) - 2^\ell + 3 \\ &= \frac{(n-3)}{(n-1)}k - \frac{2^{\ell+1}}{n-1} + \frac{n+3}{n-1}. \end{aligned}$$

This gives

$$k = \frac{2^{\ell+1}-(n+3)+(n-1)\ell}{n-3} = \begin{cases} 2^{\ell+1}-7+3\ell & \text{if } n=4; \\ 2^\ell - 4 + 2\ell & \text{if } n=5. \end{cases}$$

Since $m \leq 2k+2$, we have $2^\ell = m-k \leq k+2$. When $n=4$, we get $2^\ell \leq k+2 = 2^{\ell+1}+3\ell-5$, and the above inequality holds when $\ell \geq 1$. When $n=5$, we get $2^\ell \leq k+2 = 2^\ell + 2\ell - 2$, and again the above inequality holds for all $\ell \geq 1$. So, we have

$$m = k + 2^\ell \qquad \text{and} \qquad x = \frac{k+1-\ell}{2},$$

which give us the parametric families of solutions

$$\begin{aligned} (k,n,m,x) = &(2^{\ell+1}+3\ell-7, 4, 3\cdot 2^\ell + 3\ell - 7, 2^\ell + \ell - 3), \\ &(2^\ell + 2\ell - 4, 5, 2^{\ell+1}+2\ell-4, 2^{\ell-1}+(\ell-3)/2) \end{aligned}$$

indicated in the statement of the Main Theorem. In the first case $\ell \geq 9$ (to insure $k > 600$), while in the second case $\ell \geq 10$ (again, to insure $k > 600$) and ℓ must be odd to insure that x is an integer.

6. The Case $n > k$

Here, as before, we need to establish some relations between the variables in our equation. The following result gives us an inequality for x in terms of k and n.

Lemma 12. *Let* (k, n, m, x) *be an integral solution of* (4) *with* $n > k \geq 3$ *and* $x \geq 2$*, then*

$$x < 7.2 \times 10^{15} n k^4 (\log k)^2 \log n. \tag{20}$$

Proof. Equation (4) can be rewritten as

$$f_k(\alpha)\alpha^{m-1} - \left(F_{n+1}^{(k)}\right)^x = -\left(F_{n-1}^{(k)}\right)^x - e_k(m). \tag{21}$$

Dividing both sides of equation (21) by $(F_{n+1}^{(k)})^x$ and taking absolute values, we get

$$|\Lambda_3| = \left| f_k(\alpha)\alpha^{m-1}\left(F_{n+1}^{(k)}\right)^{-x} - 1 \right| < 2\left(\frac{F_{n-1}^{(k)}}{F_{n+1}^{(k)}}\right)^x < \frac{2}{2.3^x}, \tag{22}$$

where we used Lemma 1, $(viii)$.

We apply Theorem 2 with the parameters $t := 3$,

$$(\gamma_1, b_1, A_1) := (f_k(\alpha), 1, 2k \log k), \qquad (\gamma_2, b_2, A_2) := (\alpha, m-1, 0.7),$$
$$(\gamma_3, b_3, A_3) := (F_{n+1}^{(k)}, -x, 0.7nk)$$

and $\mathbb{K}, D, B$ as for Λ_1.

Now, Theorem 2 combined with inequality (22) yields

$$\begin{aligned} x &< 1.15 \times 10^{14} n k^4 (\log k)^2 \log m \\ &< 1.2 \times 10^{14} n k^4 (\log k)^2 \log(nx), \end{aligned} \tag{23}$$

where we used the fact that $m < nx + 2$, which follows from (11).

We next extract from (23) an upper bound for x depending on n and k. Multiplying by n both sides of the inequality (23) we obtain

$$nx < 1.2 \times 10^{14} n^2 k^4 (\log k)^2 \log(nx),$$

Taking $y := nx$ and $T := 1.2 \times 10^{14} n^2 k^4 (\log k)^2$, by Lemma 7 and the fact that $n > k$,

$$nx < 7.2 \times 10^{15} n^2 k^4 (\log k)^2 \log n.$$

It remains to divide by n both sides of the previous inequality. □

We now work under the assumption that $n > 700$ to find an upper bound for n, m and x in terms of k only.

Lemma 13. *Let* (k, n, m, x) *be an integral solution of* (4) *with* $n > \max\{k, 700\}$*. Then the following inequalities*

$$\begin{aligned} &n < 8.6 \times 10^{24} k^6 (\log k)^6, \qquad x < 1.8 \times 10^{30} k^7 (\log k)^6, \\ &m < 1.5 \times 10^{43} k^{10} (\log k)^9 \end{aligned} \tag{24}$$

hold.

Proof. Given that $n > k$, from (20), we have that

$$x < 7.2 \times 10^{15} n^5 (\log n)^3. \tag{25}$$

Thus, for $i = \pm 1$,

$$\frac{x}{\alpha^{n+i-1}} < \frac{7.2 \times 10^{15} n^5 (\log n)^3}{\alpha^{n-2}} < \frac{1}{\alpha^{0.8n}}, \qquad \text{since } n > 700.$$

Then, by Lemma 2, we can write

$$\left(F_{n+i}^{(k)}\right)^x = f_k(\alpha)^x \alpha^{(n+i-1)x}(1+\eta_n), \quad \text{with} \quad |\eta_n| < \frac{2}{\alpha^{0.8n}}. \tag{26}$$

We now use (26) to rewrite the Eq. (4) as

$$\left| f_k(\alpha)\alpha^{m-1} - f_k(\alpha)^x \alpha^{nx}(1-\alpha^{-2x}) \right| < 2|\eta_n| f_k(\alpha)^x \alpha^{nx}(1-\alpha^{-2x}) + \frac{1}{2}. \tag{27}$$

Dividing both sides of the previous inequality by $f_k(\alpha)^x \alpha^{nx}$, we conclude that

$$\begin{aligned} \left| f_k(\alpha)^{1-x}\alpha^{m-1-nx} - (1-\alpha^{-2x}) \right| &< 2|\eta_n|(1-\alpha^{-2x}) + \frac{1}{2 f_k(\alpha)^x \alpha^{nx}} \\ &< |\eta_n| + \frac{1}{2}\left(\frac{1}{\alpha^{n-2}}\right)^x \\ &< \frac{2}{\alpha^{0.8n}}, \end{aligned}$$

where we have used the fact that $1-\alpha^{-2x} < 1/2$, $f_k(\alpha)\alpha^n > \alpha^{n-2}$ and $(n-2)x + 1 \geq 0.8n$ for all $n > 700$, $k \geq 3$ and $x \geq 2$. Hence,

$$|\Lambda_4| = \left| f_k(\alpha)^{1-x}\alpha^{m-1-nx} - 1 \right| < \frac{2}{\alpha^{0.8n}} + \frac{1}{\alpha^{2x}} < \frac{3}{\alpha^{\kappa}}, \tag{28}$$

with $\kappa := \min\{0.8n, 2x\}$.

We apply again Theorem 2 with the parameters $t := 2$,

$$(\gamma_1, b_1, A_1) := (f_k(\alpha), 1-x, 2k\log k), \quad (\gamma_2, b_2, A_2) := (\alpha, m-1-nx, 0.7),$$

and $\mathbb{K}$ and D as before. Moreover, we can take $B := x$, since $|m-1-nx| \leq x$ by inequality (11).

The conclusion of Theorem 2 and the inequality (28) yield, after taking logarithms, the following upper bound for κ:

$$\kappa < 9 \times 10^9\, k^3 (\log k)^2 \log x. \tag{29}$$

If $\kappa = 0.8n$, then from (29),

$$n < 1.2 \times 10^{10} k^3 (\log k)^2 \log x,$$

and using the inequality (25), we obtain that

$$\begin{aligned} n &< 1.2 \times 10^{10} k^3 (\log k)^2 \left(\log(7.2 \times 10^{15}) + 5\log n + 3\log\log n\right) \\ &< 10^{10} k^3 (\log k)^2 (12 \log n) \\ &< 1.5 \times 10^{11} k^3 (\log k)^2 \log n, \end{aligned}$$

since $n > 700$. Hence, we apply Lemma 7 with $T := 1.5 \times 10^1 k^3 (\log k)^2$ and $(y, r) := (n, 1)$ to obtain an upper bound on n depending only on k.

Further, inserting the resulting bound on n in terms of k in (20) and using the inequality (11), we have that

$$\begin{aligned} &n < 8.1 \times 10^{12} k^3 (\log k)^3, \qquad x < 1.8 \times 10^{30} k^7 (\log k)^6, \\ &m < 1.5 \times 10^{43} k^{10} (\log k)^9. \end{aligned} \tag{30}$$

If $\kappa = 2x$, then by (29) and Lemma 7 with $T := 4.5 \times 10^9 k^3 (\log k)^2$ and $(y, r) := (x, 1)$, we get

$$x < 2.2 \times 10^{11} k^3 (\log k)^3. \tag{31}$$

Furthermore, given that $x \leq 0.45n$ we have by Lemma 2, that for $i = \pm 1$ $x/\alpha^{n+i-1} < 0.45n/\alpha^{n-2} < 1/\alpha^{0.98n}$ since $n > 700$. Thus,

$$\left(F_{n+i}^{(k)}\right)^x = f_k(\alpha)^x \alpha^{(n+i-1)x}(1+\eta_n), \quad \text{with} \quad |\eta_n| < \frac{1}{\alpha^{0.98n}}.$$

We return to the inequality (27) and dividing both sides by $f_k(\alpha)\alpha^{m-1}$, we obtain

$$\begin{aligned} |f_k(\alpha)^{x-1}\alpha^{nx-(m-1)}(1-\alpha^{-2x}) - 1| &< 2|\eta_n| f_k(\alpha)^{x-1}\alpha^{nx-(m-1)}(1-\alpha^{-2x}) \\ &\quad + \frac{1}{2 f_k(\alpha)\alpha^{m-1}} \\ &< \frac{2\alpha (f_k(\alpha)\alpha)^{x-1}}{\alpha^{0.98n}}(1-\alpha^{-2x}) + \frac{1}{\alpha^{m-1}} \\ &< 4\left(\frac{(3/2)^{0.8n}}{\alpha^{0.98n}}\right) + \frac{1}{\alpha^{0.38n}} \\ &< \frac{2}{\alpha^{0.38n}}, \end{aligned}$$

where we used the inequalities:

$$\begin{aligned} &x \leq 0.8n, \quad nx - (m-1) \leq x, \quad m - 1 > 0.38n, \\ &f_k(\alpha)\alpha < 3/2, \quad 4(3/2)^{0.8n}/\alpha^{0.98n} < 1/\alpha^{0.38n} \ \text{ and } \ 1 - \alpha^{-2x} < 1, \end{aligned}$$

valid for $n > 700$, $x \geq 2$ and $k \geq 3$. In conclusion, we have shown that

$$|\Lambda_5| = |f_k(\alpha)^{x-1}\alpha^{nx-(m-1)}(1-\alpha^{-2x}) - 1| < \frac{2}{\alpha^{0.38n}}. \tag{32}$$

Here, we apply again Theorem 2 with the parameters $t := 3$,

$$(\gamma_1, b_1, A_1) := (f_k(\alpha), x-1, 2k\log k), \quad (\gamma_2, b_2, A_2) := (\alpha, nx-(m-1), 0.7),$$
$$(\gamma_3, b_3, A_3) := (1-\alpha^{-2x}, 1, 2x)$$

and $\mathbb{K}$ and D as before. Moreover, again we can take $B := x$. Combining the conclusion of Theorem 2 with inequality (32), we get

$$n < 1.3 \times 10^{12} \times k^3 (\log k)^2 x \log x. \tag{33}$$

By (31), we have $x < 2.2 \times 10^{11} k^3 (\log k)^3$, therefore

$$\log x < \log(2.2 \times 10^{11}) + 3\log k + 3\log\log k < 30 \log k$$

since $k \geq 3$. Hence, returning to inequality (33) and taking into account that $m < nx + 2$, we have in summary

$$n < 8.6 \times 10^{24} k^6 (\log k)^6, \qquad x < 2.2 \times 10^{11} k^3 (\log k)^3,$$
$$m < 2 \times 10^{36} k^9 (\log k)^9. \tag{34}$$

Comparing inequalities (30) and (34), we get that

$$n < 8.6 \times 10^{24} k^6 (\log k)^6, \qquad x < 1.8 \times 10^{30} k^7 (\log k)^6,$$
$$m < 1.5 \times 10^{43} k^{10} (\log k)^9,$$

as we wanted to show. □

The inequalities in Lemma 13 were obtained under the assumptions that $n > 700$. However, when $n \leq 700$ the inequalities (11) and (25) yield smaller upper bounds for x and m in terms of k.

6.1. When $k \leq 700$

Here, we prove the following result where we establish some computational ranges to search for the integral solutions of equation (4).

Lemma 14. *Let (k, n, m, x) be an integral solution of Diophantine equation (4) with $n > k$, $k \leq 700$ and $x \geq 2$. Then $m \in [M_0, M_1]$ with*

$$M_0 := \lceil (n-1)x + 1.8 \rceil \quad \text{and} \quad M_1 := 2(n-1)x + 2. \tag{35}$$

Furthermore, if $n > 700$, then $n \leq 1810$ and $x \leq 1260$, otherwise $x \leq 1150$.

Proof. Note that the range for m is given by inequality (13). Now, let us start assuming $n > 700$, which allows us to use the inequalities of Lemma 13 in order to obtain upper bounds on n, x and m. Taking into account the inequality (28), we take

$$\Gamma_4 := (x-1)\log(f_k(\alpha)^{-1}) + (m - 1 - nx)\log\alpha.$$

Then, using the analytic argument of Sect. 3.2, we get

$$|\Gamma_4| < \frac{6}{\alpha^\kappa}, \qquad \text{with} \quad \kappa = \min\{0.8n, 2x\}.$$

Dividing the above inequality by $(x-1)\log\alpha$, we obtain

$$\left| \frac{\log(f_k(\alpha)^{-1})}{\log\alpha} - \frac{nx + 1 - m}{x - 1} \right| < \frac{6}{\alpha^\kappa (x-1)\log\alpha} < \frac{10}{\alpha^\kappa (x-1)}. \tag{36}$$

Now, we need to distinguish two cases:

- **Case** $m = 1 + nx$. Here, the inequality (36) correspond to

$$\left| \frac{\log(f_k(\alpha)^{-1})}{\log\alpha} \right| < \frac{10}{\alpha^\kappa (x-1)}.$$

  A quick computational search shows that the left-hand side of the previous inequality is greater than 0.7 for all $k \in [3, 700]$. Thus, since $x \geq 2$, we get

$$0.7 < \frac{10}{\alpha^\kappa}. \tag{37}$$

Now, since $\kappa = \min\{0.8n, 2x\}$, if we assume that $\kappa = 0.8n$, then, by inequality (37), we get $n \leq 6$, a contradiction with our assumption about n. Therefore, we have $\kappa = 2x$, which together with inequality (37) implies $x = 2$. Now, by inequality (32) with $m = 1 + nx$ and using $k \in [3, 700]$, we get

$$0.5 < |\log(f_k(\alpha)) + \log\left(1 - \alpha^{-4}\right)| < 4/\alpha^{0.38n},$$

where the left-hand side was found using computations. However, this inequality implies $n \leq 11$, again a contradiction.

- **Case** $m \neq 1 + nx$: Here we apply Lemma 8 to inequality (36) using $k \in [3, 700]$. To do it, let us take $\tau_k := \log(f_k(\alpha)^{-1})/\log\alpha$. So, by inequality (24), we look for the integer t_k such that

$$Q_{t_k}^{(k)} > 1.8 \times 10^{30} k^7 (\log k)^6 > x - 1,$$

and take $a_M := \max\{a_i^{(k)} : 0 \leq i \leq t_k,\ 3 \leq k \leq 700\}$. Then, by Lemma 8, we have that

$$\left|\tau_k - \frac{nx - (m-1)}{x-1}\right| > \frac{1}{(a_M + 2)(x-1)^2}. \tag{38}$$

Hence, combining the inequalities (36) and (38), and taking into account that $a_M + 2 < 1.1 \times 10^{208}$ (confirmed by computations), we obtain

$$\alpha^{\kappa} < 1.1 \times 10^{209} x.$$

If $\kappa = 0.8n$, since $n > 700 \geq k$, by inequality (25) we have

$$\alpha^{0.8n} < 8.6 \times 10^{224} n^5 (\log n)^3,$$

which implies

$$n \leq 1010. \tag{39}$$

Thus, let us consider

$$\Gamma_3 := \log f_k(\alpha) + (m-1)\log\alpha - x\log F_{n+1}^{(k)}, \tag{40}$$

with $k \in [3, 700]$ and $n \in [701, 1010]$. Note that, by inequality (22), we have

$$|\Gamma_3| < \frac{4}{2.3^x}.$$

Dividing both sides by $\log F_{n+1}^{(k)}$, we get

$$\left|(m-1)\left(\frac{\log\alpha}{\log F_{n+1}^{(k)}}\right) - x + \frac{\log f_k(\alpha)}{\log F_{n+1}^{(k)}}\right| < \frac{3}{2.3^x} \tag{41}$$

where we used that $\log F_{n+1}^{(k)} \geq \log F_5^{(3)} = \log 7$. In order to apply Lemma 9, we take

$$\gamma_{k,n} := \frac{\log\alpha}{\log F_{n+1}^{(k)}}, \qquad \mu_{k,n} := \frac{\log f_k(\alpha)}{\log F_{n+1}^{(k)}} \qquad A := 3 \quad \text{and} \quad B := 2.3,$$

for $k \in [3, 700]$ and $n \in [701, 1010]$ with $M := 3.7 \times 10^{33} k^7 (\log k)^6$, thanks to inequalities (13) and (24). We obtain

$$\max_{k\in[3,700],\ n\in[701,1010]} \left\{ \left\lfloor \log\left(AQ^{(k,n)}/\varepsilon_{k,n}\right) / \log B \right\rfloor \right\} \leq 1260,$$

which, by Lemma 9, implies

$$x \leq 1260. \tag{42}$$

Now, if $\kappa = 2x$, then $\alpha^{2x} < 1.1 \times 10^{209} x$, which implies

$$x \leq 360. \tag{43}$$

So, we go back to inequality (32) and take

$$\Gamma_5 := (x-1)\log(f_k(\alpha)) + (nx - (m-1))\log\alpha + \log\left(1 - \alpha^{-2x}\right).$$

Since $|\Gamma_5| < 4/\alpha^{0.38n}$, dividing by $\log\alpha$, we obtain

$$\left|(x-1)\frac{\log(f_k(\alpha))}{\log\alpha} + \frac{\log\left(1-\alpha^{-2x}\right)}{\log\alpha} - (m-1-nx)\right| < \frac{7}{\alpha^{0.38n}}.$$

We take

$$\tau_{k,x} := (x-1)\frac{\log(f_k(\alpha))}{\log\alpha} + \frac{\log\left(1-\alpha^{-2x}\right)}{\log\alpha}.$$

We have that

$$\min_{k\in[3,700],\ x\in[2,360]} \|\tau_{k,x}\| < |\tau_{k,x} - (m-1-nx)| < \frac{7}{\alpha^{0.38n}}.$$

Computationally, we found that the minimum on the left-hand of the previous inequality is at least 7×10^{-207}. Therefore, we get

$$n \leq 1810. \tag{44}$$

To sum up, if $n > 700$, then by inequalities (39), (42), (43) and (44), all the positive integral solutions (k, n, m, x) of Eq. (4) satisfy $n \leq 1810$ and $x \leq 1260$.

Finally, let us consider $n \leq 700$. Since in this section we are working with $k \leq 700$ and $n > k$, it is clear that $k \leq 699$. So, let us use Γ_3 as we defined it in (40) to proceed as we did with (41). This time we take $k \in [3, 699]$ and $n \in [k+1, 700]$ with $M := 2.6 \times 10^{33} k^7 (\log k)^6$, which is given by inequalities (13) and (24). We get

$$\max_{k\in[3,699],\ n\in[k+1,700]} \left\{ \left\lfloor \log\left(AQ^{(k,x)}/\varepsilon_{k,x}\right) / \log B_k \right\rfloor \right\} \leq 1150,$$

which, by Lemma 9, implies $x \leq 1150$. $\square$

In conclusion, our problem is now reduced to a computational search for integral solutions of the Diophantine equation (4) in the ranges indicated by Lemma 14; i.e., in the ranges $k \in [3, 700]$, $m \in [M_0, M_1]$ (with the limits given by (35)),

$$n \in [701, 1810] \quad \text{and} \quad x \in [2, 1260],$$

or

$$n \in [k+1, 700] \quad \text{and} \quad x \in [2, 1150].$$

A computer search using Lemma 6 allow us to conclude that there are no integral solutions for Eq. (4) in these ranges.

6.2. When $k > 700$

From now on, we assume that $k > 700$. We show that there are no such solutions. We have, from (24), that

$$n + i < 8.7 \times 10^{24} k^6 (\log k)^6 < 2^{0.24k}, \qquad m < 1.5 \times 10^{43} k^{10} (\log k)^9 < 2^{0.48k}.$$

We recall that $n \geq k+1$, so $m > k+1$ according to (11). By item (iii) of Lemma 4 (for m with $c := 0.48$) and Lemma 5 (for $n+i$ with $i \in \{\pm 1\}$ and $c := 0.24$), we conclude that

$$F_m^{(k)} = 2^{m-2}\left(1 - \frac{m-k}{2^{k+1}} + \zeta''\right), \qquad |\zeta''| < \frac{1}{2^{1.04k}};$$
$$\left(F_{n+i}^{(k)}\right)^x = 2^{(n+i-2)x}\left(1 - \delta_i \frac{x(n+i-k)}{2^{k+1}} + \delta_i \xi_i\right), \qquad |\xi_i| < \frac{6}{2^{1.04k}},$$

where $\delta_1 = 1$ for all $n \geq k+1$ and

$$\delta_{-1} = \begin{cases} 1, & \text{for } n \geq k+3; \\ 0, & \text{for } n \in \{k+1, k+2\}. \end{cases}$$

Now, let us take $M = \max\{(n-1)x, m-2\}$ and $N = \min\{(n-1)x, m-2\}$, so, we get

$$|2^{(n-1)x} - 2^{(n-3)x} - 2^{m-2}| \leq 2^M \left(\frac{6(1+\delta_i)}{2^{1.04k}} + \frac{1}{2^{1.04k}} + \frac{1+\delta_i}{2^{0.52k+1}} + \frac{1}{2^{0.52k+1}}\right)$$
$$< \frac{2^{M+2}}{2^{0.52k}}.$$

In the above, we used that $x(n+i-k) < x(n-1) < m < 2^{0.48k}$ for $i \in \{\pm 1\}$, where the second inequality follows from (11), in addition to $k > 700$. After dividing by 2^M becomes

$$1 - \frac{1}{2^{M-N}} - \frac{1}{2^{2x}} < \frac{1}{2^M}\left|2^{(n-1)x} - 2^{(n-3)x} - 2^{m-2}\right| < \frac{4}{2^{0.52k}}.$$

If $M > N$, then the left-hand side is at least $1/4$, so $2^{0.52k} < 16$, a contradiction since $k > 700$. Thus, $M = N$, or, equivalently, $(n-1)x = m-2$. We next get

$$\left|\frac{x(n-k+1)}{2^{k+1}} - \frac{m-k}{2^{k+1}} - \frac{1}{2^{2x}}\right| < \frac{8}{2^{1.04k}},$$

or

$$\left|\frac{x(n-k+1) - (m-k)}{2^{k+1}} - \frac{1}{2^{2x}}\right| < \frac{8}{2^{1.04k}}.$$

But

$$
\begin{aligned}
x(n-k+1)-(m-k) &= x(n-k+1)-(m-2)+(k-2) \\
&= x(n-k+1-(n-1))+k-2 \\
&= (k-2)(1-x) < 0.
\end{aligned}
$$

Thus,

$$\left|\frac{x(n-k+1)-(m-k)}{2^{k+1}} - \frac{1}{2^{2x}}\right| > \frac{(k-2)(x-1)}{2^{k+1}} > \frac{1}{2^{k+1}},$$

and we get

$$\frac{1}{2^{k+1}} < \frac{8}{2^{1.04k}},$$

or $2^{0.04k} < 16$, a contradiction with $k > 700$. Thus, we showed that our Diophantine equation has no solution in the range $n > k > 700$.

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Declarations

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References

[1] Bensella, H., Patel, B.K., Behloul, D.: On the exponential Diophantine equation $(F_{m+1}^{(k)})^x - (F_{m-1}^{(k)})^x = F_n^{(k)}$ (2022). arxiv:2205.13168

[2] Bravo, J., Luca, F.: Powers of two in generalized Fibonacci sequences. Rev. Colomb. Mat. **46**, 67–79 (2012)

[3] Bravo, J., Luca, F.: On a conjecture about repdigits in k-generalized Fibonacci sequences. Publ. Math. Debr. **82**, 623–639 (2013)

[4] Bravo, J.J., Gómez, C.A.: Mersenne k-Fibonacci numbers. Glas. Mat. **51**(71), 307–319 (2016)

[5] Bravo, J.J., Gómez, C.A., Luca, F.: Powers of two as sums of two k-Fibonacci numbers. Miskolc Math. Notes **17**(1), 85–100 (2016)

[6] Cooper, C., Howard, F.T.: Some identities for r-Fibonacci numbers. Fibonacci Q. **49**, 231–243 (2011)

[7] Dresden, G.P., Du, Z.: A simplified Binet formula for k-generalized Fibonacci numbers. J. Integer Seq. **17**, Article 14.4.7 (2014)

[8] Dujella, A., Pethő, A.: A generalization of a theorem of Baker and Davenport. Q. J. Math. Oxf. Ser. **49**(195), 291–306 (1998)

[9] Everest, G., Van der Poorten, A., Shparlinski, I., Ward, T.: Recurrence Sequences, Mathematical Surveys and Monographs, vol. 104. American Mathematical Society, Providence (2003)

[10] Gómez, C.A., Luca, F.: An exponential Diophantine equation related to the sum of powers of two consecutive k-generalized Fibonacci numbers. Colloq. Math. **137**(2), 171–188 (2014)

[11] Gómez, C.A., Luca, F.: Multiplicative independence in k-generalized Fibonacci sequences. Lith. Math. J. **56**(49), 503–517 (2016)

[12] Guzmán, S., Luca, F.: Linear combinations on factorials and S-units in a binary recurrence sequence. Ann. Math. Québec **38**(2), 169–188 (2014)

[13] Hua, L. K., Wang, Y. : Applications of Number Theory to Numerical Analysis, Translated from Chinese. Springer, Berlin; Kexue Chubanshe (Science Press), Beijing (1981)

[14] Kummer, E.E.: Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen. J. Reine Angew. Math. **1852**(44), 93–164 (1851). https://doi.org/10.1515/crll.1852.44.93

[15] Matveev, E.M.: An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. Izv. Math. **64**, 1217–1269 (2000)

[16] Miles, E.P., Jr.: Generalized Fibonacci numbers and associated matrices. Am. Math. Mon. **67**, 745–752 (1960)

[17] Miller, M.D.: Mathematical notes: on generalized Fibonacci numbers. Am. Math. Mon. **78**, 1108–1109 (1971)

[18] Patel, B.K., Chaves, A.P.: On the exponential Diophantine equation $F_{n+1}^x - F_{n-1}^x = F_m$, Mediterr. J. Math. **18**(5), Paper No, 187, 11 pp (2021)

[19] Wolfram, D.A.: Solving generalized Fibonacci recurrences. Fibonacci Q. **36**, 129–145 (1998)

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