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Lipschitz-Free Spaces over Cantor Sets and Approximation Properties

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Abstract. Let $K = 2^{\mathbb{N}}$ be the Cantor set, let $\mathcal{M}$ be the set of all metrics d on K that give its usual (product) topology, and equip $\mathcal{M}$ with the topology of uniform convergence, where the metrics are regarded as functions on K^2. We prove that the set of metrics $d \in \mathcal{M}$ for which the Lipschitz-free space $\mathcal{F}(K, d)$ has the metric approximation property is a residual $F_{\sigma\delta}$ set in $\mathcal{M}$, and that the set of metrics $d \in \mathcal{M}$ for which $\mathcal{F}(K, d)$ fails the approximation property is a dense meager set in $\mathcal{M}$. This answers a question posed by G. Godefroy.

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1. Introduction

Lipschitz-free spaces have become an active research area in Banach space theory in recent years. For a metric space (M, d) and a point $x_0 \in M$, we define the Banach space $\mathrm{Lip}_0(M, x_0)$ consisting of all real-valued Lipschitz functions f on M that vanish at x_0, equipped with the norm

$$\|f\| := \sup_{x,y \in M, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

For any $x \in M$, we define the bounded linear functional $\delta_x \in \mathrm{Lip}_0(M, x_0)^*$ by $\delta_x(f) = f(x)$, $f \in \mathrm{Lip}_0(M, x_0)$. The closed linear span of the set $\{\delta_x : x \in M\}$ is called the *Lipschitz-free space* $\mathcal{F}(M, x_0)$. It is well known that $\mathrm{Lip}_0(M, x_0)$ is isometric to the dual space of $\mathcal{F}(M, x_0)$ and that the Banach space structure of both spaces does not depend on the choice of the base point x_0. We will hereafter write $\mathrm{Lip}_0(M)$ and $\mathcal{F}(M)$ without specifying the base point, and call $\mathcal{F}(M)$ simply the free space over M. In the book [20], Weaver provides a comprehensive introduction to Lipschitz and Lipschitz-free spaces. In it, the latter are called Arens-Eells spaces and are denoted by Æ(M).

One direction of research in this area has been the approximation properties of free spaces. Results involving the approximation property (AP) appear in [12–14]. Results involving the bounded approximation property (BAP) appear in [8,11,15], and metric approximation property (MAP) results appear in [3–5,7,11,12,17,19]. More details can be found in the introduction to [19] and in [10].

In particular, we mention that if M is a sufficiently 'thin' totally disconnected metric space then $\mathcal{F}(M)$ has the MAP. For example, if M is a countable proper metric space (i.e. where all closed balls are compact), then $\mathcal{F}(M)$ has the MAP [4]. Also, as a corollary of [20, Corollary 4.39], $\mathcal{F}(M)$ has the MAP when M is compact and uniformly disconnected (this notion is defined at the beginning of Sect. 2). Moreover, as a corollary of [15, Proposition 2.3] and [1, Theorem B], $\mathcal{F}(M)$ has the MAP when M is a subset of a finite-dimensional normed space and is purely 1-unrectifiable, which is equivalent to the condition that M contains no bi-Lipschitz image of a compact subset of $\mathbb{R}$ of positive measure. On the other hand, in [12], G. Godefroy and N. Ozawa constructed a compact convex subset C of a separable Banach space such that $\mathcal{F}(C)$ fails the AP. Also, by a result in [13], there exists a metric space M homeomorphic to the Cantor space such that $\mathcal{F}(M)$ fails the AP.

Given a compact metric space M with metric d and a Lipschitz function $f\colon M \to \mathbb{R}$, we say that f is *locally flat* [20, Definition 4.1] if for every $x \in M$,

$$\lim_{\epsilon \to 0} \mathrm{Lip}(f|_{B_\epsilon(x)}) = 0,$$

where $B_\epsilon(x) = \{y \in M : d(y, x) < \epsilon\}$. The *Little Lipschitz space* $\mathrm{lip}_0(M)$ is the subspace of $\mathrm{Lip}_0(M)$ consisting of all locally flat Lipschitz functions. By [20, Corollary 4.5], $\mathrm{lip}_0(M)$ is a Banach space. It often happens that $\mathrm{lip}_0(M) = \{0\}$, for example when M is a connected smooth submanifold of $\mathbb{R}^N$. We say that $\mathrm{lip}_0(M)$ *separates points uniformly* [20, Definition 4.10] if there exists $a \in (0, 1]$ such that for every $x, y \in M$ there is $f \in \mathrm{lip}_0(M)$ such that $\mathrm{Lip}(f) \leq 1$ and $|f(x) - f(y)| = ad(x, y)$. By [1, Theorem A, Theorem B], $\mathrm{lip}_0(M)$ separates points uniformly if and only if $\mathrm{lip}_0(M)$ is an isometric predual to $\mathcal{F}(M)$, which holds if and only if M is purely 1-unrectifiable. If M is compact and purely 1-unrectifiable, then by [2, Proposition 3.5], if $\mathcal{F}(M)$ has the MAP then $\mathrm{lip}_0(M)$ has the MAP. But also, by the remarks after [9, Problem 6.5], if $\mathrm{lip}_0(M)$ has the MAP then $\mathcal{F}(M)$ has the MAP as well.

In [9], G. Godefroy surveys various aspects of the theory of free spaces, including the lifting property for separable Banach spaces, approximation properties of free spaces, and norm attainment of Lipschitz functions and operators. Regarding the bounded approximation property of the free space over a compact set M, he states a very useful criterion (see the end of Sect. 2) in terms of 'almost-extension' operators from Lipschitz spaces over finite subsets of M to $\mathrm{Lip}_0(M)$. In the last section he states a number of open problems, several of which concern approximation properties of free spaces.

Let $K = 2^{\mathbb{N}}$ be the Cantor set, equipped with the usual (product) topology. Define $\mathcal{M} \subseteq C(K^2)$ to be the space of all metrics d on K compatible with its topology. We equip $\mathcal{M}$ with the metric induced by the usual (supremum) norm of $C(K^2)$. The set $\mathcal{M}$ is a G_δ subset of $C(K^2)$ (a proof is provided at the beginning of Sect. 2), and is therefore a Polish (i.e. separable and completely metrisable) space. For $d \in \mathcal{M}$ we write $\mathrm{Lip}_0(K,d)$ and $\mathcal{F}(K,d)$ for the Lipschitz space and free space over the metric space (K,d), respectively. We define the following subsets of $\mathcal{M}$:

$$\begin{aligned} \mathcal{A}^\lambda &= \{d \in \mathcal{M} \ : \ \mathcal{F}(K,d) \text{ has the } \lambda\text{-BAP}\}, \text{ for } \lambda \in [1,\infty), \\ \mathcal{A}_f &= \{d \in \mathcal{M} \ : \ \mathcal{F}(K,d) \text{ fails the AP}\}, \\ \mathcal{P} &= \{d \in \mathcal{M} \ : \ (K,d) \text{ is purely 1-unrectifiable}\}. \end{aligned}$$

In [9, Problem 6.6], Godefroy asks what the topological nature of the set A_f is (it is nonempty by [13, Corollary 2.2]). Also, more precisely, he asks whether the set A_f is residual in $\mathcal{M}$.

In this paper, we investigate the topological properties the subsets of $\mathcal{M}$ defined above. The main results are the following.

Theorem 1.1. *The set $\mathcal{A}^1$ is a residual $F_{\sigma\delta}$ set in $\mathcal{M}$.*

Proposition 1.2. *The set $\mathcal{A}_f$ is a dense meager set in $\mathcal{M}$.*

In particular, these results indicate that it might be difficult to use a Baire category argument to show that $\mathcal{A}_f$ is nonempty (note that the proof of this fact in [13] uses other techniques).

The paper is organised as follows. Section 2 concerns notation and preliminary results. In Sect. 3 we give the proofs of Theorem 1.1 and Proposition 1.2. Furthermore, we prove that $\mathcal{P}$ is a dense G_δ set and $\mathcal{M} \setminus \mathcal{P}$ is dense. As a corollary, we obtain that the set of metrics d for which $\mathcal{F}(M)$ is a dual space to $\mathrm{lip}_0(M)$ and both $\mathcal{F}(M)$ and $\mathrm{lip}_0(M)$ have the MAP is residual. Finally, in Sect. 4, we construct a family (d_α) of metrics on K of size continuum, such that there is no algebra isomorphism between $\mathrm{Lip}_0(K,d_\alpha)$ and $\mathrm{Lip}_0(K,d_\beta)$ whenever $\alpha \neq \beta$. This should be compared with [13, Corollary 2.3], wherein it is shown that there exists an family (d_α) of metrics on K of size $\aleph_1$, such that $\mathcal{F}(K,d_\alpha)$ is not isomorphic to $\mathcal{F}(K,d_\beta)$ for $\alpha \neq \beta$.

2. Notation and Preliminary Results

For a metric space (A,d), $x \in A$ and $r > 0$ we write $B_r^d(x) = \{y \in A : d(x,y) < r\}$. If $C \subseteq A$ then write $B_r^d(C) = \{y \in A : d(y,C) < r\}$, where $d(y,C) = \inf\{d(y,x) : x \in C\}$. For a real-valued Lipschitz function f on (A,d), $\mathrm{Lip}_d(f)$ denotes the Lipschitz constant of f with respect to d. If (B,e) is another metric space and $f\colon A \to B$ then the Lipschitz constant of f is denoted by $\mathrm{Lip}_{d,e}(f)$. We call (A,d) and (B,e) *proportional* if there exists a surjection $f\colon A \to B$ and $c > 0$ such that $d(x,y) = ce(f(x),f(y))$ for all $x,y \in A$. The space (A,d) is called *uniformly disconnected* [20, Proposition 4.12] if there exists $r \in (0,1]$ such that for any distinct $x,y \in A$ there are complementary clopen sets $C,D \subseteq A$ such that $x \in C, y \in D$ and $d(C,D) \geq rd(x,y)$, where $d(C,D) = \inf\{d(z,t) : z \in C, t \in D\}$.

For $d \in \mathcal{M}$, $K_1,K_2 \subseteq K$, and $x \in K$ we put $D(K_1,K_2) = \sup\{d(x,y) : x \in K_1, y \in K_2\}$, and $D(x,K_1) = D(\{x\},K_1)$. We write $D(K_1)$ or $\mathrm{diam}_d(K_1)$ for $D(K_1,K_1)$. If S is a nonempty set then we call a finite family $\{S_1,\ldots,S_n\}$

of nonempty subsets of S a *partition* of S if $S_i \cap S_j = \emptyset$ whenever $i \neq j$ and $\bigcup_{i=1}^n S_i = S$. If X is a Banach space then B_X denotes the closed unit ball of X. If Y is another Banach space then $X \simeq Y$ means that X is isomorphic to Y.

We will first provide a short proof of the fact that $\mathcal{M}$ is a G_δ set in $C(K^2)$. Let μ be the canonical metric on K:

$$\mu(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-n} & \text{if } x \neq y \text{ and } n \in \mathbb{N} \text{ is minimal, such that } x(n) \neq y(n). \end{cases}$$

Define

$$A = \{f \in C(K^2) : f(x,x) = f(x,y) - f(y,x) = 0 \leq f(x,y) + f(y,z) - f(x,z) \\ \text{for all } x,y,z \in K\},$$

and for $n \in \mathbb{N}$, define

$$B_n = \{f \in C(K^2) \ : \ f(x,y) > 0 \text{ whenever } \mu(x,y) \geq 2^{-n}\}.$$

The set A is clearly closed and hence G_δ in $C(K^2)$, and the sets B_n are open by compactness. Therefore, the set $A \cap \bigcap_{n=1}^\infty B_n$ is G_δ. Pick any $d \in A \cap \bigcap_{n=1}^\infty B_n$ and observe that d is a metric on K. Let $\mathcal{T}_d$ be the topology on K induced by d. Since for any $x \in K$, the function $d(x,\cdot)$ is continuous on K, $B_r^d(x)$ is open in the topology of K for any $r > 0$. Thus, $\mathcal{T}_d$ is a coarser topology than the one on K. Therefore, any closed subset C of K is compact in $\mathcal{T}_d$, and is, therefore, closed, because $\mathcal{T}_d$ is Hausdorff. Thus, $\mathcal{T}_d$ agrees with the topology on K, so $d \in \mathcal{M}$. As d was arbitrary, $\mathcal{M} = A \cap \bigcap_{n=1}^\infty B_n$, and so $\mathcal{M}$ is G_δ.

The following is a crucial lemma that allows us to make a small change to a given metric on K in a very flexible way.

Lemma 2.1. *Suppose that $d \in \mathcal{M}, \epsilon > 0, \delta \in (0,1]$ and K' is a nonempty clopen subset of K. Let $\{K_1', \dots, K_n'\}$ be an arbitrary partition of K' into clopen sets satisfying $D(K_i') < \frac{\epsilon}{2}$, and let $e_1, \dots, e_n \in \mathcal{M}$ be arbitrary. Then, there exists $\tilde{d} \in \mathcal{M}$ such that $\|d - \tilde{d}\|_\infty < \epsilon$, d and $\tilde{d}$ agree on $K \backslash K'$, $(K_i', \tilde{d})$ is proportional to (K, e_i), and $\tilde{D}(K_i') \leq \delta$ for $i = 1, \dots, n$.*

Proof. Let $d \in \mathcal{M}, \epsilon > 0$, and $K' \subseteq K$ be given. Suppose that $\{K_1', \dots, K_n'\}$ is a partition of K' satisfying

$$D(K_i') < \frac{\epsilon}{2}, \text{ for } i = 1, \dots, n, \tag{2.1}$$

and let $e_1, \dots, e_n \in \mathcal{M}$ be arbitrary. As each K_i' is homeomorphic to K, we can find a metric d_i on K_i' compatible with its topology such that (K_i', d_i) is isometric to $\left(K, \frac{\min(\delta, D(K_i'))}{E_i(K)} e_i\right)$. Hence,

$$D_i(K_i') = \min(\delta, D(K_i')). \tag{2.2}$$

Now, define $\tilde{d}\colon K^2 \to [0,\infty)$ by

$$\tilde{d}(x,y) = \begin{cases} d(x,y), & \text{if } x,y \in K \setminus K', \\ D(x,K_i'), & \text{if } x \in K \setminus K' \text{ and } y \in K_i', \\ D(y,K_i'), & \text{if } x \in K_i' \text{ and } y \in K \setminus K', \\ d_i(x,y), & \text{if } x,y \in K_i' \text{ for some } i, \\ D(K_i',K_j') & \text{if } x \in K_i', y \in K_j' \text{ for } i \text{ and } j \text{ such that } i \neq j. \end{cases}$$

We will now show that $\tilde{d}$ is a metric on K. It is clearly symmetric and satisfies $\tilde{d}(x,y) = 0$ if and only if $x = y$. To show the triangle inequality, pick $x,y,z \in K$. If $x,y,z \in K\setminus K'$ or $x,y,z \in K_i'$ for some i then the triangle inequality follows from the triangle inequality for the metric d or d_i, respectively. We now consider the remaining cases:

Case 1: $x,y \in K\setminus K'$ and $z \in K_i'$ for some i.

We have

$$\tilde{d}(x,y) = d(x,y) \leq d(x,z) + d(z,y) \leq \tilde{d}(x,z) + \tilde{d}(z,y).$$

By compactness, there exists $z' \in K_i'$ such that $\tilde{d}(x,z) = d(x,z')$. Thus,

$$\tilde{d}(x,z) = d(x,z') \leq d(x,y) + d(y,z') \leq \tilde{d}(x,y) + \tilde{d}(y,z).$$

We can similarly show $\tilde{d}(y,z) \leq \tilde{d}(y,x) + \tilde{d}(x,z)$.

Case 2: $x \in K\setminus K'$ and $y,z \in K_i'$ for some i.

As $\tilde{d}(x,y) = \tilde{d}(x,z)$, we have $\tilde{d}(x,y) \leq \tilde{d}(x,z) + \tilde{d}(z,y)$ and $\tilde{d}(x,z) \leq \tilde{d}(x,y) + \tilde{d}(y,z)$. Now let $y',z' \in K_i'$ be such that $D(K_i') = d(y',z')$. We have

$$\begin{aligned} \tilde{d}(y,z) &= d_i(y,z) \leq D(K_i') && \text{by (2.2)} \\ &= d(y',z') \leq d(x,y') + d(z',x) \leq \tilde{d}(x,y) + \tilde{d}(z,x). \end{aligned}$$

Case 3: $x \in K\setminus K', y \in K_i'$ and $z \in K_j'$ with $i \neq j$.

Let $y' \in K_i'$ be such that $\tilde{d}(x,y) = d(x,y')$. We have

$$\tilde{d}(x,y) = d(x,y') \leq d(x,z) + d(z,y') \leq \tilde{d}(x,z) + \tilde{d}(z,y).$$

Similarly, $\tilde{d}(x,z) \leq \tilde{d}(x,y) + \tilde{d}(y,z)$. Now let $y_1 \in K_i'$ and $z_1 \in K_j'$ be such that $\tilde{d}(y,z) = d(y_1,z_1)$. We have

$$\tilde{d}(y,z) = d(y_1,z_1) \leq d(y_1,x) + d(x,z_1) \leq \tilde{d}(y,x) + \tilde{d}(x,z).$$

Case 4: $x,y \in K_i'$ and $z \in K_j'$, where $i \neq j$.

Let $x',y' \in K_i'$ be such that $d(x',y') = D(K_i')$. We have

$$\begin{aligned} \tilde{d}(x,y) &= d_i(x,y) \leq D(K_i') && \text{by (2.2)} \\ &= d(x',y') \leq d(x',z) + d(z,y') \leq \tilde{d}(x,z) + \tilde{d}(z,y). \end{aligned}$$

Also

$$\tilde{d}(x,z) = \tilde{d}(y,z) \leq \tilde{d}(x,y) + \tilde{d}(y,z),$$

and similarly $\tilde{d}(y,z) \leq \tilde{d}(y,x) + \tilde{d}(x,z)$.

Case 5: $x \in K_i', y \in K_j', z \in K_l'$ and i,j and l are distinct.

Let $x' \in K_i'$ and $y' \in K_j'$ be such that $\tilde{d}(x,y) = d(x',y')$. We have

$$\tilde{d}(x,y) = d(x',y') \leq d(x',z) + d(z,y') \leq \tilde{d}(x,z) + \tilde{d}(z,y).$$

Therefore, $\tilde{d}$ is a metric on K. Also, $\tilde{d}$ is continuous on K^2 because it is continuous on each set of the form $K_i' \times K_j'$, $K_i' \times (K \backslash K')$ and $(K \backslash K') \times K_j'$. Therefore, similarly as in the proof that $\mathcal{M}$ is G_δ, we conclude that $\tilde{d} \in \mathcal{M}$. To show $\|\tilde{d} - d\|_\infty < \epsilon$, pick $x, y \in K$. If $x, y \in K \backslash K'$ then $\tilde{d}(x,y) = d(x,y)$. If $x \in K \backslash K'$ and $y \in K_i'$ for some i, then let $y' \in K_i'$ be such that $\tilde{d}(x,y) = d(x,y')$. Then

$$|\tilde{d}(x,y) - d(x,y)| = |d(x,y') - d(x,y)| \leq d(y,y') < \frac{\epsilon}{2},$$

by (2.1). If $x, y \in K_i'$ for some i then

$$|\tilde{d}(x,y) - d(x,y)| \leq \tilde{d}(x,y) + d(x,y) = d_i(x,y) + d(x,y) < \epsilon,$$

by (2.1) and (2.2). Finally, if $x \in K_i', y \in K_j'$ for $i \neq j$ then let $x' \in K_i', y' \in K_j'$ be such that $\tilde{d}(x,y) = d(x',y')$. We have

$$|\tilde{d}(x,y) - d(x,y)| = |d(x',y') - d(x,y)| \leq d(x,x') + d(y,y') < \epsilon,$$

by (2.1). The last assertion of the lemma follows from (2.2). □

Remark 2.2. Note that the metric $\tilde{d}$ satisfies $\tilde{d}(x,y) \geq d(x,y)$ whenever $x, y \in K$ and x, y do not both belong to K_i' for any $i = 1, \ldots, n$.

Corollary 2.3. *Let $d \in \mathcal{M}, \epsilon > 0$, and $K' \subseteq K$ be a clopen subset. Then, there exists a partition $\{K_1, \ldots, K_n\}$ of K' consisting of clopen sets such that, for any arbitrary $e_1, \ldots, e_n \in \mathcal{M}$, there exists a metric $\tilde{d} \in \mathcal{M}$ such that $\|\tilde{d} - d\|_\infty < \epsilon$, $\tilde{d}$ and d agree on $K \setminus K'$, and $(K_i, \tilde{d})$ is proportional to (K, e_i) for each $i = 1, \ldots, n$.*

Proof. Let $d \in \mathcal{M}, \epsilon > 0$ and K' be given. Using the compactness of K' and the fact that each point in K' has a local base for its topology consisting of clopen neighbourhoods, we can find a cover $C_1, \ldots, C_p$ of K' such that C_i is clopen and $D(C_i) < \frac{\epsilon}{2}$ for each $i = 1, \ldots, p$. Now inductively define $K_1' = C_1$ and $K_{i+1}' = C_{i+1} \backslash \bigcup_{j=1}^{i} K_j'$. By dismissing any empty K_i' we obtain a partition $\{K_1', \ldots, K_n'\}$ of K' (where $n \leq p$), consisting of clopen sets satisfying $D(K_i') < \frac{\epsilon}{2}$ for each $i = 1, \ldots, n$. Now, if $e_1, \ldots, e_n \in \mathcal{M}$ are arbitrary, the corollary follows from an application of Lemma 2.1 with $\delta = 1$. □

We will also need the following results.

Lemma 2.4. *Suppose that $d \in \mathcal{M}$ and $(d_n)_n \subseteq \mathcal{M}$ are such that $d_n \to d$ uniformly. Let (M, ρ) be a compact metric space, $L > 0$ and $h_n \colon K \to M$ be functions such that $\mathrm{Lip}_{d_n,\rho}(h_n) \leq L$ for each $n \in \mathbb{N}$. Then there exists a subsequence $(h_{n_i})_i$ of $(h_n)_n$ and a function $h \colon K \to M$ such that $\mathrm{Lip}_{d,\rho}(h) \leq L$ and $h_{n_i} \to h$ uniformly on K as $i \to \infty$. Furthermore,*

(1) if $J > 0$ and h_n is bilipschitz with $\mathrm{Lip}_{\rho,d_n}(h_n^{-1}) \leq J$ for all n, then h is bilipschitz with $\mathrm{Lip}_{\rho,d}(h^{-1}) \leq J$,

(2) if h_n is surjective for all n, then h is surjective.

Proof. We will show that the functions $(h_n)_n$ are d-ρ-equicontinuous. Pick $\epsilon > 0$, and pick $n \in \mathbb{N}$ such that $\|d_k - d\|_\infty < \frac{\epsilon}{2L}$ for $k \geq n$. Then, for $k \geq n$ and $x, y \in K$ such that $d(x, y) < \frac{\epsilon}{2L}$,

$$\rho\Big(h_k(x), h_k(y)\Big) \leq L d_k(x, y) \leq L\left(\frac{\epsilon}{2L} + d(x, y)\right) < \epsilon.$$

Therefore, $(h_n)_n$ is equicontiuous on (K, d). By the Arzelà-Ascoli theorem, there exists a continuous $h\colon K \to M$ and a subsequence $(h_{n_i})_i$ of $(h_n)_n$ such that $h_{n_i} \to h$ uniformly. For $x, y \in K$,

$$\rho\Big(h(x), h(y)\Big) = \lim_{i\to\infty} \rho\Big(h_{n_i}(x), h_{n_i}(y)\Big) \leq \lim_{i\to\infty} L d_{n_i}(x, y) = L d(x, y).$$

Hence, $\mathrm{Lip}_{d,\rho}(h) \leq L$.

If $J > 0$ and h_n is bilipschitz with $\mathrm{Lip}_{\rho,d_n}(h_n^{-1}) \leq J$ for all $n \in \mathbb{N}$, then, similarly as previously we can show that h is bilipschitz with $\mathrm{Lip}_{\rho,d}(h^{-1}) \leq J$.

Now assume the h_n are surjective. Pick $y \in M$, and let $\epsilon > 0$. Choose $i \in \mathbb{N}$ such that $\|h - h_{k_i}\|_\infty < \epsilon$. If $h_{k_i}(x) = y$, then $|h(x) - y| < \epsilon$. As ϵ was arbitrary, y is a limit point of $h(K)$. Since $h(K)$ is compact, $y \in h(K)$ and h is surjective. □

Proposition 2.5. *Let $d \in \mathcal{M}$, x_0 be the base point of (K, d) and $\{K_1, \ldots, K_n\}$ be a partition of K consisting of clopen sets. If $K_i' = K_i \cup \{x_0\}$ for $i = 1, \ldots, n$ then*

$$\mathcal{F}(K, d) \simeq \mathcal{F}(K_1', d) \oplus_1 \ldots \oplus_1 \mathcal{F}(K_n', d).$$

Proof. Let x_0 be the base point of each K_i' as well. We define the bounded linear operator $T\colon \mathrm{Lip}_0(K, \tilde{d}) \to \mathrm{Lip}_0(K_1', d) \oplus_\infty \ldots \oplus_\infty \mathrm{Lip}_0(K_n', d)$ by

$$T(f) = (f|_{K_1'}, \ldots, f|_{K_n'}).$$

It is easy to see that T is surjective. Pick $x, y \in K, x \neq y$. If $x, y \in K_i$ for some i then clearly $\frac{|f(x)-f(y)|}{d(x,y)} \leq \|T(f)\|$ for each $f \in \mathrm{Lip}_0(K, d)$. Otherwise, $d(x, y) > \min_{i\neq j} d(K_i, K_j) =: b$, so

$$\frac{|f(x) - f(y)|}{d(x, y)} \leq b^{-1}\Big(|f(x)| + |f(y)|\Big) \leq 2b^{-1} D(K)\|T(f)\|.$$

Therefore, $\|f\| \leq \max\Big(1, 2b^{-1}D(K)\Big)\|T(f)\|$ for each $f \in \mathrm{Lip}_0(K, d)$, which shows that T is an isomorphism. It is not hard to see that T is the adjoint of the operator $T_*\colon \mathcal{F}(K_1', d) \oplus_1 \ldots \oplus_1 \mathcal{F}(K_n', d) \to \mathcal{F}(K, d)$,

$$T_*(\nu_1, \ldots, \nu_n) = \nu_1 + \ldots + \nu_n,$$

(the spaces $\mathcal{F}(K_i', d)$) are seen as subspaces of $\mathcal{F}(K, d)$, by [20, Theorem 3.7]). Therefore, T_* is the required isomorphism. □

Theorem 2.6. *(Grothendieck) If X is a separable dual Banach space with the AP then X has the MAP.*

A proof of the previous theorem can be found in e.g. [16, Theorem 1.e.15]. Finally, in this section, we give a useful criterion, due to Godefroy, for the λ-BAP of free spaces over compact metric spaces in terms of 'almost-extension' operators [9, Theorem 3.2] (also [8, Theorem 1]). We will need one of the several equivalent conditions for the λ-BAP stated in [9, Theorem 3.2]. For a finite subset $M' \subseteq M$ of a compact metric space M, and $\epsilon > 0$, we say that M' is ϵ-dense in M if for every $y \in M$ there is an $x \in M'$ such that $d(y, x) < \epsilon$. Note that if $(M_n)_n$ is an increasing sequence (with respect to inclusion) of finite subsets of M such that $\bigcup_{n=1}^{\infty} M_n$ is dense in M, then there exists a sequence $(\epsilon_n)_n$ of positive numbers tending to 0 such that M_n is ϵ_n-dense in M for all $n \in \mathbb{N}$.

Theorem 2.7. *Let M be a compact metric space and $(M_n)_n$ be a sequence of finite ϵ_n-dense subsets of M with $\lim_{n\to\infty} \epsilon_n = 0$. Then, $\mathcal{F}(M)$ has the λ-BAP if and only if there is a sequence $(T_n)_n$ of operators $T_n \colon \mathrm{Lip}_0(M_n) \to \mathrm{Lip}_0(M)$ such that $\|T_n\| \leq \lambda$ for all n and*

$$\lim_{n\to\infty} \sup_{f \in B_{\mathrm{Lip}_0(M_n)}} \|T_n(f)|_{M_n} - f\|_\infty = 0.$$

3. Topological Properties of $\mathcal{A}^\lambda$, $\mathcal{A}_f$ and $\mathcal{P}$

In this section we give our main results. We will first prove that $\mathcal{A}^\lambda$ is an $F_{\sigma\delta}$ set in $\mathcal{M}$ for any $\lambda \geq 1$. Fix a dense sequence of distinct elements $(x_n)_{n=1}^{\infty}$ in K. For $n \in \mathbb{N}$, define $A_n = \{x_1, \ldots, x_n\}$, and for $n \in \mathbb{N}, \lambda \geq 1$ and $\epsilon > 0$, define

$$\begin{aligned}\mathcal{B}_{n,\epsilon}^{\lambda} = \{d \in \mathcal{M} \ :\ & \text{there exists } T\colon \mathrm{Lip}_0(A_n, d) \to \mathrm{Lip}_0(K, d) \text{ such that} \\ & \|T\| \leq \lambda \text{ and } \sup\nolimits_{f \in B_{\mathrm{Lip}_0(A_n,d)}} \|(Tf)|_{A_n} - f\|_\infty \leq \epsilon\}.\end{aligned}$$

Proposition 3.1. *The set $\mathcal{B}_{n,\epsilon}^{\lambda}$ is closed in $\mathcal{M}$ for each $n \in \mathbb{N}, \epsilon > 0$ and $\lambda \geq 1$.*

Proof. Fix some $n \in \mathbb{N}, \epsilon > 0$ and $\lambda \geq 1$. Let $(d_k)_k$ be a sequence in $\mathcal{B}_{n,\epsilon}^{\lambda}$ converging to $d \in \mathcal{M}$. For each $k \in \mathbb{N}$, pick some $T_k \colon \mathrm{Lip}_0(A_n, d_k) \to \mathrm{Lip}_0(K, d_k)$ such that $\|T_k\| \leq \lambda$ and

$$\sup_{f \in B_{\mathrm{Lip}_0(A_n, d_k)}} \|(T_k f)|_{A_n} - f\|_\infty \leq \epsilon.$$

Fix an $m \in \mathbb{N}, m > n$. We consider the sequence of operators $(S_k)_{k=1}^{\infty}$, where $S_k \colon \mathrm{Lip}_0(A_n, d) \to \mathrm{Lip}_0(A_m, d)$ is defined by $S_k(f) = T_k(f)|_{A_m}$. As $d_k \to d$ uniformly and A_m is finite, it is not hard to see that $(\|S_k\|)_{k=1}^{\infty}$ is a bounded sequence. Therefore, by compactness, $(S_k)_k$ has a subsequence $(S_{k_j})_j$ which converges to an operator $P_m \colon \mathrm{Lip}_0(A_n, d) \to \mathrm{Lip}_0(A_m, d)$. For any $f \in \mathrm{Lip}_0(A_n, d)$, we have $\mathrm{Lip}_d(f) = \lim_{j\to\infty} \mathrm{Lip}_{d_{k_j}}(f)$ and for $x, y \in A_m$,

$$\begin{aligned} &|P_m(f)(x) - P_m(f)(y)| \\ &= \lim_{j\to\infty} |S_{k_j}(f)(x) - S_{k_j}(f)(y)| = \lim_{j\to\infty} |T_{k_j}(f)(x) - T_{k_j}(f)(y)| \\ &\le \limsup_{j\to\infty} \mathrm{Lip}_{d_{k_j}}(T_{k_j}(f)) d_{k_j}(x, y) \le \limsup_{j\to\infty} \|T_{k_j}\| \,\mathrm{Lip}_{d_{k_j}}(f) d_{k_j}(x, y) \\ &\le \lambda \,\mathrm{Lip}_d(f) d(x, y). \end{aligned}$$

Therefore, $\|P_m\| \le \lambda$. It is also clear that

$$\sup_{f\in B_{\mathrm{Lip}_0(A_n,d)}} \|(P_m(f))|_{A_n} - f\|_\infty \le \epsilon.$$

Now, fix a basis $f_1, \ldots, f_{n-1}$ of $\mathrm{Lip}_0(A_n, d)$ with $\|f_i\| = 1$ for all $i \in \{1, \ldots, n-1\}$. For each i and $m > n$ extend the function $P_m(f_i)$ to (K, d) without increasing its Lipschitz constant, by McShane's extension theorem [20, Theorem 1.33]. Now for $i \in \{1, \ldots, n-1\}$, consider the sequence $(P_m(f_i))_{m=n+1}^\infty \subseteq \lambda B_{\mathrm{Lip}_0(K,d)}$. By the Arzelà-Ascoli theorem, we can obtain a subsequence $(P_{m_j})_{j=1}^\infty$ of $(P_m)_{m=n+1}^\infty$ such that $P_{m_j}(f_i)$ converges uniformly to some continuous function $T(f_i)$ on K, for all $i \in \{1, \ldots, n-1\}$. Since for $x, y \in K$,

$$|T(f_i)(x) - T(f_i)(y)| = \lim_{j\to\infty} |P_{m_j}(f_i)(x) - P_{m_j}(f_i)(y)| \le \lambda d(x, y),$$

we conclude that $T(f_i) \in \lambda B_{\mathrm{Lip}_0(K,d)}$ for all $i \in \{1, \ldots, n-1\}$. We extend T by linearity to $\mathrm{Lip}_0(A_n, d)$.

For each $p \in \mathbb{N}$ and each $i = \{1, \ldots, n-1\}$, we have

$$T(f_i)(x_p) = \lim_{j\to\infty} P_{m_j}(f_i)(x_p).$$

Therefore, by linearity, $T(f)(x_p) = \lim_{j\to\infty} P_{m_j}(f)(x_p)$ for each $f \in \mathrm{Lip}_0(A_n, d)$. Now, let $p, l \in \mathbb{N}$, $p < l$. Then $x_p, x_l \in A_{m_j}$ whenever $m_j \ge l$, hence

$$\begin{aligned} |(Tf)(x_p) - (Tf)(x_l)| &= \lim_{j\to\infty} |(P_{m_j} f)(x_p) - (P_{m_j} f)(x_l)| \\ &\le \lambda \,\mathrm{Lip}_d(f) d(x_p, x_l). \end{aligned}$$

From the fact that $(x_p)_{p=1}^\infty$ is dense in K follows that $\mathrm{Lip}_d(T(f)) \le \lambda \,\mathrm{Lip}_d(f)$, and so $\|T\| \le \lambda$. Also, it is not hard to see that

$$\sup_{f\in B_{\mathrm{Lip}_0(A_n,d)}} \|(T(f))|_{A_n} - f\|_\infty \le \epsilon.$$

Therefore, $d \in \mathcal{B}^\lambda_{n,\epsilon}$ and $\mathcal{B}^\lambda_{n,\epsilon}$ is closed. □

Remark 3.2. In the second part of the previous proof, extending $P_m(f_i)$ by McShane's extension theorem is done for convenience rather than necessity. Alternatively, we can define Tf only on the set $\{x_n : n \in \mathbb{N}\}$ and then extend to K using the density of $\{x_n : n \in \mathbb{N}\}$.

Proposition 3.3. *The set $\mathcal{A}^\lambda$ is $F_{\sigma\delta}$ in $\mathcal{M}$ for any $\lambda \ge 1$.*

Proof. According to Theorem 2.7 with $M = K$, $M_n = A_n$, $n \in \mathbb{N}$, $\mathcal{F}(K, d)$ has the λ-BAP if and only if there exists a sequence $(\alpha_n)_n$ of positive real numbers converging to 0 such that $d \in \mathcal{B}^{\lambda}_{n,\alpha_n}$ for all $n \in \mathbb{N}$. This is equivalent to

$$\forall k \in \mathbb{N}\ \exists n_0 \in \mathbb{N}\ \forall n > n_0, \quad d \in \mathcal{B}^{\lambda}_{n,k^{-1}},$$

or

$$d \in \bigcap_{k\in\mathbb{N}} \bigcup_{n_0\in\mathbb{N}} \bigcap_{n>n_0} \mathcal{B}^{\lambda}_{n,k^{-1}}.$$

Now the statement follows from Proposition 3.1. □

For $a \in 2^{<\mathbb{N}}$ (that is, a is a finite sequence of 0 s and 1 s) and $n \in \mathbb{N}$, let $l(a)$ denote the length of a, and define $R_n = \{a \in 2^{<\mathbb{N}} \ :\ l(a) = n\}$, and $C_a = \{x \in K \ :\ x|_{\{1,2,\dots,l(a)\}} = a\}$. Each C_a is a clopen subset of K with $\operatorname{diam}_\mu(C_a) = 2^{-l(a)-1}$, and, for each $n \in \mathbb{N}$, K is the disjoint union of the sets $\{C_a \ :\ a \in R_n\}$. For each $a \in 2^{<\mathbb{N}}$, let $r_a = (a(1), \dots, a(l(a)), 0, 0, \dots) \in K$. For $d \in \mathcal{M}$ and $n \in \mathbb{N}$, we define $T^d_n\colon \operatorname{Lip}_0(\{r_a : a \in R_n\}, d) \to \operatorname{Lip}_0(K, d)$ by $T^d_n(f)(x) = f(r_{x|_{\{1,2,\dots,n\}}})$. Also define

$$\chi^d_n := \max_{a,b\in R_n, a\neq b} \frac{D(C_a, C_b)}{d(C_a, C_b)}.$$

Lemma 3.4. *The operator T^d_n satisfies $\|T^d_n\| \le \chi^d_n$.*

Proof. Let $f \in \operatorname{Lip}_0(\{r_a : a \in R_n\}, d)$, $\operatorname{Lip}(f) \le 1$, and $x, y \in K$. If $x, y \in C_a$ for some $a \in R_n$ then $T^d_n(f)(x) = T^d_n(f)(y)$. If $x \in C_a, y \in C_b$ where $a, b \in R_n, a \neq b$, then

$$\frac{|T^d_n(f)(x) - T^d_n(f)(y)|}{d(x, y)} = \frac{|f(r_a) - f(r_b)|}{d(x, y)} \le \frac{d(r_a, r_b)}{d(x, y)} \le \frac{D(C_a, C_b)}{d(C_a, C_b)}.$$

Therefore $\|T^d_n\| \le \chi^d_n$. □

Define $U_n = \{d \in \mathcal{M} \ :\ \chi^d_n < 1 + \frac{1}{n}\}$ for each $n \in \mathbb{N}$. It is clear that U_n is open in $\mathcal{M}$.

Proposition 3.5. *For each $n_0 \in \mathbb{N}$, $\bigcup_{n=n_0}^{\infty} U_n$ is dense and open in $\mathcal{M}$.*

Proof. Let $n_0 \in \mathbb{N}$. Obviously $\bigcup_{n=n_0}^{\infty} U_n$ is open. Pick $d \in \mathcal{M}$ and $\epsilon > 0$. By compactness of K, we can find $n \ge n_0$ such that $d(x, y) < \frac{\epsilon}{2}$ whenever $\mu(x, y) < 2^{-n}$, $x, y \in K$. We then have $D(C_a) < \frac{\epsilon}{2}$ for all $a \in R_n$ (strict inequality holds by compactness of C_a). Apply Lemma 2.1 to $d, \epsilon, \delta = 1$, $K' = K$, the partition $\{C_a : a \in R_n\}$ of K', and $e_i = \mu$ for all i. We obtain a metric $\widetilde{d}$ such that $\|\widetilde{d} - d\|_\infty < \epsilon$. By the definition of $\widetilde{d}$ in Lemma 2.1, for $a, b \in R_n, a \neq b$,

$$\widetilde{D}(C_a, C_b) = D(C_a, C_b) = \widetilde{d}(C_a, C_b).$$

Hence $\chi^{\widetilde{d}}_n = 1$, and so $\widetilde{d} \in U_n$. □

We are now ready to prove the main results.

Proof of Theorem 1.1. By Proposition 3.3, it suffices to prove that $\mathcal{A}^1$ is residual, and by Proposition 3.5, it suffices to prove

$$\bigcap_{n_0=1}^{\infty} \bigcup_{n=n_0}^{\infty} U_n \subseteq \mathcal{A}^1.$$

If d is in the set on the left-hand side, then there exists an increasing sequence of natural numbers $n_1 < n_2 < \ldots$ such that $d \in U_{n_i}$ for all i. Then, $\|T^d_{n_i}\| < 1 + \frac{1}{n_i}$ for all i, by Lemma 3.4. It is easy to see that the last condition of Theorem 2.7 holds for $M = K$, the subsets $M_i = \{r_a : a \in R_{n_i}\}$, and the operators $\frac{T^d_{n_i}}{\|T^d_{n_i}\|}$ for $i \in \mathbb{N}$, since $(T^d_{n_i}(f))|_{M_i} = f$ for all $f \in \mathrm{Lip}_0(M_i, d)$, and $\|T^d_{n_i}\| \to 1$. As $\bigcup_{i=1}^{\infty} M_i$ is dense in K, by Theorem 2.7 we obtain $d \in \mathcal{A}^1$. □

Proof of Proposition 1.2. The fact that $\mathcal{A}_f$ is meager follows from Theorem 1.1. To show it is dense, let $d \in \mathcal{M}$ and $\epsilon > 0$. By [13, Corollary 2.2] there exists $d' \in \mathcal{A}_f$. According to Corollary 2.3 with $K' = K$ and $e_i = d'$ for all i, there exists $\tilde{d} \in \mathcal{M}$ and a partition $\{K_1, \ldots, K_n\}$ of K consisting of clopen sets such that $\|\tilde{d} - d\|_\infty < \epsilon$ and $(K_i, \tilde{d})$ is proportional to (K, d'). According to Proposition 2.5,

$$\mathcal{F}(K, \tilde{d}) \simeq \mathcal{F}(K_1', \tilde{d}) \oplus_1 \ldots \oplus_1 \mathcal{F}(K_n', \tilde{d}),$$

where $K_i' = K_i \cup \{x_0\}$ and x_0 is the base point of K. Since proportional metric spaces have isometrically isomorphic free spaces, $\mathcal{F}(K_1, \tilde{d})$ fails the AP. As $\mathcal{F}(K_1, \tilde{d})$ has codimension 1 in $\mathcal{F}(K_1', \tilde{d})$, we have that $\mathcal{F}(K_1', \tilde{d})$ fails the AP as well. Hence, $\mathcal{F}(K, \tilde{d})$ fails the AP. □

We now state and prove some properties of the set $\mathcal{P}$ of metrics d for which (K, d) is purely 1-unrectifiable. Denote by Δ the Lebesgue measure on $\mathbb{R}$.

Proposition 3.6. *The set $\mathcal{P}$ is G_δ in $\mathcal{M}$.*

Proof. For $m, k \in \mathbb{N}$ define

$$V_{m,k} = \Big\{ d \in \mathcal{M} \ : \ \text{there exists a compact } K' \subseteq K \text{ and } h : K' \to \mathbb{R}, \\ \text{such that } m^{-1} d(x,y) \le |h(x) - h(y)| \le m d(x,y) \\ \text{for all } x, y \in K', \text{ and } \Delta(h(K')) \ge k^{-1} \Big\}.$$

We will prove that $V_{m,k}$ is closed. Let $(d_n)_{n\in\mathbb{N}} \subseteq V_{m,k}$ converge uniformly to $d \in \mathcal{M}$, and let K_n' and $h_n \colon K_n' \to \mathbb{R}$ be the compact set and bilipschitz function, respectively, associated to d_n for each $n \in \mathbb{N}$. By taking a subsequence, we can assume that the K_n' converge to a compact set $K' \subseteq K$ (in the Vietoris topology of $\mathbb{R}$). We extend each function h_n to a function (again denoted by h_n) on K by [20, Theorem 1.33], while preserving its Lipschitz constant with respect to d_n. We can assume, by translation, that $h_n(0, 0, \ldots) = 0$ for each n. As the diameters of (K, d_n) are uniformly bounded, the sets $h_n(K)$ are all contained in some fixed bounded interval. By Lemma 2.4, there exists a subsequence $(h_{n_i})_i$ converging uniformly to a function $h\colon K \to \mathbb{R}$ with $\mathrm{Lip}_d(h) \leq m$.

Now, let $x, y \in K'$ be arbitrary and let $(x_i)_i, (y_i)_i \subseteq K$ converge to x and y, respectively, and be such that $x_i, y_i \in K'_{n_i}$ for each $i \in \mathbb{N}$. Let $\epsilon > 0$ and pick i such that

$$d(x, x_i), d(y, y_i), \|h - h_{n_i}\|_\infty, \|d - d_{n_i}\|_\infty < \epsilon.$$

Then

$$\begin{aligned} |h(x) - h(y)| &\geq |h(x_i) - h(y_i)| - |h(x) - h(x_i)| - |h(y) - h(y_i)| \\ &\geq |h(x_i) - h(y_i)| - 2m\epsilon \geq |h_{n_i}(x_i) - h_{n_i}(y_i)| - (2 + 2m)\epsilon \\ &\geq m^{-1} d_{n_i}(x_i, y_i) - (2 + 2m)\epsilon \\ &\geq m^{-1} d(x_i, y_i) - \left(2 + 2m + m^{-1}\right)\epsilon \\ &\geq m^{-1}(d(x, y) - d(x, x_i) - d(y, y_i)) - \left(2 + 2m + m^{-1}\right)\epsilon \\ &\geq m^{-1} d(x, y) - \left(2 + 2m + 3m^{-1}\right)\epsilon. \end{aligned}$$

As ϵ was arbitrary, we get $|h(x) - h(y)| \geq m^{-1} d(x, y)$. This means that h satisfies the bilipschitz condition in the definition of $V_{m,k}$ on the set K'.

To show that $\Delta(h(K')) \geq k^{-1}$, pick $\epsilon > 0$ and set $U = B_\epsilon^{|\cdot|}(h(K'))$. We have that $h^{-1}(U)$ is an open set in K containing K'. Choose i such that $K'_{n_i} \subseteq h^{-1}(U)$ and $\|h - h_{n_i}\|_\infty < \epsilon$. Then

$$h_{n_i}(K'_{n_i}) \subseteq B_\epsilon^{|\cdot|}(h(K'_{n_i})) \subseteq B_\epsilon^{|\cdot|}(U) \subseteq B_{2\epsilon}^{|\cdot|}(h(K')).$$

Hence, $\Delta(B_{2\epsilon}^{|\cdot|}(h(K'))) \geq \Delta(h_{n_i}(K'_{n_i})) \geq k^{-1}$. As ϵ was arbitrary, we get that $\Delta(h(K')) \geq k^{-1}$. Therefore, $d \in V_{m,k}$ and so $V_{m,k}$ is closed. As $\mathcal{M} \backslash \mathcal{P} = \bigcup_{m,k \in \mathbb{N}} V_{m,k}$, we have that $\mathcal{M} \backslash \mathcal{P}$ is F_σ, and so $\mathcal{P}$ is G_δ. □

Proposition 3.7. *The set $\mathcal{P}$ is dense in $\mathcal{M}$.*

Proof. Let $d \in \mathcal{M}$ and $\epsilon > 0$ be arbitrary. It is not hard to see that (K, μ) is uniformly disconnected. By [20, Corollary 4.39 (ii)], $\mathcal{F}(K, \mu)$ is a dual space. Then by [1, Theorem B], (K, μ) is purely 1-unrectifiable. According to Corollary 2.3 applied to d, ϵ, $K' = K$ and $e_i = \mu$ for all i, there exists a partition $\{K_1, \ldots, K_n\}$ of K consisting of clopen sets, and a metric $\widetilde{d} \in \mathcal{M}$, such that $\|\widetilde{d} - d\|_\infty < \epsilon$ and $(K_i, \widetilde{d})$ is proportional to (K, μ) for all i. Therefore, each $(K_i, \widetilde{d})$ is purely 1-unrectifiable. Hence, it is not hard to see that $(K, \widetilde{d})$ is also purely 1-unrectifiable. □

Proposition 3.8. *The set $\mathcal{M} \setminus \mathcal{P}$ is dense in $\mathcal{M}$.*

Proof. Let $d \in \mathcal{M}$ and $\epsilon > 0$ be arbitrary. Let $d' \in \mathcal{M}$ be such that (K, d') is isometric to a Cantor subset of $\mathbb{R}$ of positive measure. Then, clearly (K, d') is not purely 1-unrectifiable. According to Corollary 2.3 with $K = K'$ and $e_i = d'$ for all i, there exists $\widetilde{d} \in \mathcal{M}$ such that $\|d - \widetilde{d}\|_\infty < \epsilon$ and $(K, \widetilde{d})$

contains a clopen subset proportional to (K, d'). Then, $(K, \widetilde{d})$ is not purely 1-unrectifiable. $\square$

Corollary 3.9. *The set of metrics d for which $\mathcal{F}(K, d)$ is the dual space of $\mathrm{lip}_0(K, d)$ and both $\mathrm{lip}_0(M)$ and $\mathcal{F}(M)$ have the MAP is residual in $\mathcal{M}$.*

Proof. Let G be the set of metrics in question. Clearly, $G \subseteq \mathcal{A}^1$, and by [1, Theorem B], $G \subseteq \mathcal{P}$. Also, if $d \in \mathcal{P} \cap \mathcal{A}^1$ then by [2, Proposition 3.5], $\mathrm{lip}_0(K, d)$ has the MAP, so $d \in G$. Therefore, $G = \mathcal{P} \cap \mathcal{A}^1$ and the corollary follows from Proposition 3.6, Proposition 3.7, and Theorem 1.1. $\square$

4. Lipschitz Equivalence Classes

In this section, we consider subsets of $\mathcal{M}$ consisting of Lipschitz-equivalent metrics, and describe some of their basic topological properties. For any $d \in \mathcal{M}$, we consider the 'Lipschitz equivalence class' of metrics

$$E_d = \{d' \in \mathcal{M} \;:\; \text{there exists a surjection } h : K \to K \text{ and } n > 0 \text{ such that } n^{-1} d(x, y) \le d'(h(x), h(y)) \le n d(x, y) \text{ for all } x, y \in K\}.$$

Proposition 4.1. *There is a family $(d_\alpha) \subseteq \mathcal{M}$ of size continuum such that $E_{d_\alpha} \cap E_{d_\beta} = \emptyset$ whenever $\alpha \ne \beta$.*

Proof. For each $\lambda \in (0, 1)$, let $d_\lambda \in \mathcal{M}$ be such that (K, d_λ) is isometric to the λ-middle-thirds Cantor subset of $\mathbb{R}$. By [6, Exercise 2.14], the Hausdorff dimension of (K, d_λ) for each $\lambda \in (0, 1)$ is $\frac{\log 2}{\log \frac{2}{1-\lambda}}$. By [6, Corollary 2.4], there is no bilipschitz surjection from (K, d_λ) to $(K, d_{\lambda'})$ for different λ and λ', and the statement follows. $\square$

Proposition 4.2. *For each $d \in \mathcal{M}$, E_d is a dense meager F_σ set in $\mathcal{M}$.*

Proof. Fix $d \in \mathcal{M}$. If $\hat{d} \in \mathcal{M}$ and $\epsilon > 0$ are arbitrary, we will show that there exists a metric $\widetilde{d} \in E_d$ which is ϵ-close to $\hat{d}$. We apply Corollary 2.3 to $\hat{d}$, $K' = K$, and ϵ, to get a partition $K_1, \ldots, K_n$ of K consisting of clopen sets. Now if $e_i \in \mathcal{M}$ is such that (K, e_i) is isometric to (K_i, d) for $i = 1, \ldots, n$, then by the same corollary, we obtain a metric $\widetilde{d} \in \mathcal{M}$, such that $\|\hat{d} - \widetilde{d}\|_\infty < \epsilon$, and $(K_i, \widetilde{d})$ is proportional to (K, e_i), and hence to (K_i, d), for all i. It is not hard to see that then $\widetilde{d} \in E_d$. Therefore, E_d is dense in $\mathcal{M}$.

Now for $n \in \mathbb{N}$, consider the set

$$E_d^n = \{d' \in \mathcal{M} \;:\; \text{there exists a surjection } h\colon K \to K \text{ such that } n^{-1} d(x, y) \le d'(h(x), h(y)) \le n d(x, y) \text{ for all } x, y \in K\}.$$

Suppose that $(d_k)_{k=1}^\infty$ is a sequence in E_d^n converging to $d_0 \in \mathcal{M}$. Let $h_k\colon K \to K$ be the surjection associated with d_k, for each $k \in \mathbb{N}$. By Lemma 2.4, there exists a surjective function $h\colon K \to K$ satisfying the condition in the definition of E_d^n with d_0 for d'. This shows that $d_0 \in E_d^n$ and so E_d^n is closed. Since $E_d = \bigcup_{n \in \mathbb{N}} E_d^n$, E_d is F_σ. Moreover, E_d^n is nowhere dense because if $e \notin E_d$ then E_e is dense in $\mathcal{M}$ and $E_e \cap E_d = \emptyset$. Therefore E_d is meager. $\square$

For a bounded metric space M, we define $\mathrm{Lip}(M)$ (resp. $\mathrm{Lip}(M,\mathbb{C})$) to be the space of all real (resp. complex)-valued Lipschitz functions on M equipped with the norm $\|f\| = \mathrm{Lip}(f) + \|f\|_\infty$ (the Lipschitz constant of a complex-valued function is defined by the same supremum as for a real-valued function). Under pointwise multiplication of functions, $\mathrm{Lip}(M)$ and $\mathrm{Lip}(M,\mathbb{C})$ are Banach algebras. Note that by [20, Lemma 1.28], $\mathrm{Lip}(M,\mathbb{C})$ is isomorphic to $\mathrm{Lip}(M) \oplus \mathrm{Lip}(M)$ as a Banach space. We can also view $\mathrm{Lip}_0(M)$ as an algebra under pointwise multiplication, however the Lipschitz constant is not submultiplicative. Instead, it satisfies $\mathrm{Lip}(fg) \leq \|f\|_\infty \mathrm{Lip}(g) + \|g\|_\infty \mathrm{Lip}(f)$, which implies $\|fg\| \leq 2\mathrm{diam}(M)\|f\|\|g\|$. Therefore the product is continuous, which implies that there is an equivalent norm which is submultiplicative. If X and Y are two algebras over the same field ($\mathbb{R}$ or $\mathbb{C}$), then we call a map $h\colon X \to Y$ an *algebra isomorphism* if h is linear, bijective, and respects multiplication, i.e. $h(xy) = h(x)h(y)$ for all $x, y \in X$.

Corollary 4.3. *There exists a family* $(d_\alpha) \subseteq \mathcal{M}$ *of size continuum such that there is no algebra isomorphism between* $\mathrm{Lip}_0(K, d_\alpha)$ *and* $\mathrm{Lip}_0(K, d_\beta)$ *whenever* $d_\alpha \neq d_\beta$.

Proof. Suppose that $d_1, d_2 \in \mathcal{M}$ are such that there exists an algebra isomorphism $h\colon \mathrm{Lip}_0(K, d_1) \to \mathrm{Lip}_0(K, d_2)$. The map $\widetilde{h}\colon \mathrm{Lip}(K, d_1) \to \mathrm{Lip}(K, d_2)$, given by $\widetilde{h}(\lambda \mathbf{1}_K + f) = \lambda \mathbf{1}_K + h(f)$, is an algebra isomorphism, where $\lambda \in \mathbb{R}$, $\mathbf{1}_K$ is the constant function on K equal to 1, and $f \in \mathrm{Lip}_0(K, d_1)$. Then the map $\hat{h}\colon \mathrm{Lip}((K, d_1), \mathbb{C}) \to \mathrm{Lip}((K, d_2), \mathbb{C})$, given by $\hat{h}(f + ig) = \widetilde{h}(f) + i\widetilde{h}(g)$, is again an algebra isomorphism. Finally, by [18, Theorem 5.1], there exists a bilipschitz surjection from (K, d_1) to (K, d_2), and the corollary follows from Proposition 4.1. □

We do not know (in ZFC) whether there exists a family $(d_\alpha) \subseteq \mathcal{M}$ of size continuum such that $\mathcal{F}(K, d_\alpha)$ is not linearly isomorphic to $\mathcal{F}(K, d_\beta)$ for $\alpha \neq \beta$.

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Declarations

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