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Some New Results on the Uniform Asymptotic Stability for Volterra Integro-differential Equations with Delays

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Abstract. In this work, we establish sufficient conditions of the uniform asymptotic stability (UAS) of solutions to second-order and third-order of Volterra integro-differential equations (VIDE) with delay. Here, we prove two new theorems on the UAS of the solutions of the considered VIDEs. Our approach is based on Lyapunov's second method. Our results improve and form a complement to some known recent results in the literature. Two illustrative examples are considered to support the results and two graphs are drawn to illustrate the asymptotic stability of the zero solution for the considered numerical equations. The obtained results are new and original.

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Keywords. VIDEs, UAS, delay differential equations (DDEs).

1. Introduction

The integro-differential equations (IDEs), which combine differential and integral equations, have attracted more attention in recent years. Applications in mathematics, physics, biology, and engineering all heavily rely on IDEs.

The equations known as the Volterra equations were studied in the early years of the 20th century by Italian mathematician Vito Volterra. In the 1930 s, Volterra showed that mathematical models for some seasonal diseases, e.g., influenza, are formulated as integral and differential equations. The use of VIDEs is widespread in the fields of biology, ecology, medicine, physics, and other sciences. To the best of our knowledge, it has been observed in a variety of physical applications, including the glass-forming process, heat transfer, the diffusion process generally, neutron diffusion, the coexistence of biological species with varying generation rates, and wind ripple in the desert.

One of the most crucial methods for researching the qualitative characteristics of solutions to ordinary, functional, and IDEs is Lyapunov's second method because this method is widely recognized as an excellent tool in the study of differential equations. Theoretically, this method is quite significant, and it is used in many different applications, see [24]. Lyapunov's second method is a sufficient condition to show the stability of systems, which means the system may still be stable even if one cannot find a Lyapunov-Krasovskii functional (LKF) candidate to conclude the system stability property.

There are many interesting results have been obtained in the literature to study the behaviour of solutions for DDE by Lyapunov's theory, see for example [4,10,15,16,22,25].

Besides, it is worth mentioning, that according to our observation from the literature, recently we found many exciting papers on the kind of VIDEs, for example [2,3,9–13,15–22].

In 2000, Zhang [25] investigated the uniform asymptotic stability for the linear scaler VIDE

$$\dot{x}(t) = Ax(t) + \int_0^t C(t-s)x(s)\mathrm{d}s,$$

where A ia a constant and $C : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function.

In 2015, Tunç [14] studied the stability and the boundedness of the zero solution of the non-linear VIDE with delay of the form

$$\dot{x}(t) = -a(t)f(x(t)) + \int_{t-\tau}^t B(t,s)g(x(s))\mathrm{d}s + p(t).$$

Recently, in 2022, Appleby and Reynold [1] studied the asymptotic stability of the scalar linear VIDE

$$\dot{x}(t) = -ax(t) + \int_0^t k(t-s)x(s)\mathrm{d}s,\ t > 0,\ x(0) = x_0.$$

Our goal for this paper is to create the sufficient conditions for the UAS of second and third-order VIDEs with delay for the following equations

$$\ddot{x} + f_1(x)\dot{x} + \int_0^t h_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1 = 0, \tag{1.1}$$

and

$$\dddot{x} + f_2(\dot{x})\ddot{x} + \alpha\dot{x} + \int_0^t h_2(t-s_2)v_2\left(\dot{x}(s_2)\right)\mathrm{d}s_2 = 0, \tag{1.2}$$

where $h_1, h_2 : [0,\infty) \to (-\infty,\infty)$ are continuous functions depend on the differences $t-s_1, t-s_2$, respectively, and $L^1(0,\infty)$, L^1 is the space of integrable Lebesgue functions, s_1, s_2 are time delays with $s_1, s_2 \leq t$, also there exist two functions $H_1, H_2 : [0,\infty) \to [0,\infty)$ such that $\dot{H}_1(t-s_1) = \frac{\mathrm{d}}{\mathrm{d}t}(H_1(t-s_1)) = -h_1(t-s_1)$, $\dot{H}_2(t-s_2) = \frac{\mathrm{d}}{\mathrm{d}t}(H_2(t-s_2)) = -h_2(t-s_2)$ with $\int_0^\infty |h_1(u)|\mathrm{d}u, \int_0^\infty |h_2(u)|du \in L^1[0,\infty)$ and $\int_t^\infty |H_2(u)|\mathrm{d}u, \int_t^\infty |H_2(u)|\mathrm{d}u \in L^1[0,\infty)$. The functions $f_1(x), f_2(y), v_1(x)$ and $v_2(y)$ are continuous scalar functions defined on $\mathbb{R}$ with $f_1(0) = f_2(0) = v_1(0) = v_2(0) = 0$.

Remark 1.1. We will give the following remarks:

1. Whenever, $\ddot{x}$ replaced by $\dot{x}$, $f_1(x)\dot{x}$ replaced by $Ax(t)$, and let $v_1(x) = x(t)$, in the integral term then (1.1) reduces to the equation that is considered in [25]. Thus, the stability and results obtained in (1.1) include and extend the previous results.
2. In [1], If we replaced the term $\ddot{x}$ by $\dot{x}$, $f_1(x)\dot{x}$ by $ax(t)$, and let $v_1(x) = x(t)$ in the integral term, then (1.1) reduces to the equation that considered in [1]. Then the stability results of this paper include and improve the stability result obtained in [1]. Then (1.1) and (1.2) generalize and improve the results obtained in [1,25].
3. As an application in physics, many models can be modeled by IDEs. For example, first, by the Kirchhoffs second law, the net voltage drop across a closed loop equals the voltage impressed $E(t)$. Thus, the standard closed electric RLC circuit can be governed IDE [5], second, an Abel-type Volterra integral equation describes the temperature distribution along the surface when the heat transfer to it is balanced by radiation from it. Finally, also, Abel-type Volterra integral equation determines the temperature in a semi-infinite solid, whose surface can dissipate heat by nonlinear radiation [23].

2. Main Results

Consider the general functional differential system

$$\dot{x} = F(t, x_t), \tag{2.1}$$

where, x_t represents a function from $[\alpha, t] \to \mathbb{R}^n$, $-\infty \le \alpha \le t_0$. For any $t \ge t_0$, by $(X(t), \|.\|)$, we shall mean the space of continuous functions $\phi : [\alpha, t] \to \mathbb{R}^n, \alpha > 0$, with$\|\phi\| = \sup_{\alpha \le s \le t} |\phi(s)|$, $s \in R$ and $|.|$ is any norm on $\mathbb{R}^n$. The symbol $X_H(t)$ denotes those $\phi \in X(t)$ with $\|\phi\| \le H$ for some $H > 0$.

Here, F is a continuous function of t for $t_0 \le t \le \infty$, whenever $x_t \in X_H(t)$ for $t_0 \le t \le \infty$, and takes closed bounded sets of $\mathbb{R} \times X(t)$ into bounded sets of $\mathbb{R}^n$.

Theorem 2.1. [7] *Let $V(t, x_t)$ be continuous functional and locally Lipschitz for*

$t_0 \le t < \infty$and $x_t \in X_H(t)$. Suppose there is a continuous function $\Phi : [0, \infty) \to [0, \infty)$ which is $L^1[0, \infty)$ and satisfies

(i) $W_1(|x|) \le V(t, x_t) \le W_2(|x|) + W_3\bigg(\int_\alpha^t \Phi(t-s) W_4(|x(s)|) \mathrm{d}s \bigg)$, *where* W_i; $\big(i = 1, 2, 3, 4\big)$ *are wedges;*

(ii) $\dot{V}_{(2.1)}(t, x_t) \le -W_5(|x|)$.

Then, the zero solution of (2.1) is uniformly asymptotically stable (UAS).

The following two theorems will be our main results for (1.1) and (1.2).

Theorem 2.2. *In addition to the basic assumptions given on the functions f_1, H_1 and v_1 for (1.1), we suppose that there are the non-negative constants $a_1,\ a_2,\ b_1,\ b_2,\ L_1,\ L,\ c_1,\ \beta_1,\ \beta_2$ and c_2, such that*

(i) $a_2 \le f_1(x) \le a_1,\ \ |f_1'(x)| \le c_1$ *and* $b_2 \le v_1(x) \le b_1,\ \ |v_1'(x)| \le c_2$.
(ii) $\int_0^\infty |H_1(u)|\mathrm{d}u = L < 1$ *and* $\int_t^\infty |H_1(u)|\mathrm{d}u \in L^1[0,\infty)$.
(iii) $0 < \beta_1 \le |H_1(0)| \le \beta_2$.
(iv) $\int_0^t |H_1(t-s_1)|\mathrm{d}s_1 + \int_t^\infty |H_1(u-t)|\mathrm{d}u = L_1$.

Then, the zero solution of (1.1) is UAS, provided that

$$\beta_1 b_2 c_1 + 2\beta_1 c_2^2 L_1 \ge c_1 b_1 L \quad \text{and} \quad 2a_2 \ge c_1 b_1 L.$$

Theorem 2.3. *Together with the fundamental conditions given on the functions f_2, H_2 and v_2 for (1.2), we assume that there exist the positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4,\ d_1, d_2, L, \theta_1, \theta_2$ and θ_3, so that the following assumptions are true*

(i) $\alpha_1 \le f_2(y) \le \alpha_2,\ |f_2'(y)| \le d_1$ *and* $\alpha_3 \le v_2(y) \le \alpha_4,\ |v_2'(y)| \le d_2$.
(ii) $\int_0^\infty |H_2(u)|\mathrm{d}u = L < 1$ *and* $\int_t^\infty |H_2(u)|\mathrm{d}u \in L^1[0,\infty)$.
(iii) $0 < \theta_1 \le |H_2(0)| \le \theta_2$.
(iv) $\int_0^t |H_2(t-s_2)|\mathrm{d}s_2 + \int_t^\infty |H_2(u-t)|\mathrm{d}u \le \frac{\theta_1\theta_3}{\theta_2}$.

Then, the zero solution of (1.2) is UAS, provided that

$$(1 + \alpha + 2\alpha_1) \ge d_2\theta_3.$$

3. Proof of Theorem 2.2.

Rewrite (1.1) as the following

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -f_1(x)y - H_1(0)v_1(x) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1. \end{aligned} \tag{3.1}$$

Define the LKF $V_1(t, x_t, y_t)$ as

$$\begin{aligned} V_1(t,x_t,y_t) = &\left(y - \int_0^t H_1(t-s_1)v_1(x(s_1))ds_1\right)^2 + 4H_1(0)\int_0^x v_1(\xi)d\xi \\ &+ \left(y + \int_0^x f_1(\xi)d\xi - \int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1\right)^2 \\ &+ 2H_1(0)\int_0^t\int_t^\infty |H_1(u-s_1)|\mathrm{d}u\, v_1^2(x(s_1))\mathrm{d}s_1. \end{aligned} \tag{3.2}$$

It can be written as

$$\begin{aligned} V_1 = \ & 2y^2 + 2\left(\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1\right)^2 - 4y\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1 \\ &+4H_1(0)\int_0^x v_1(\xi))\mathrm{d}\xi + \left(\int_0^x f_1(\xi)\mathrm{d}\xi\right)^2 + 2y\int_0^x f_1(\xi)\mathrm{d}\xi \end{aligned}$$

$$-2\int_0^x f_1(\xi)\mathrm{d}\xi\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1$$
$$+2H_1(0)\int_0^t\int_t^\infty |H_1(u-s_1)|\mathrm{d}u\, v_1^2(x(s_1))\mathrm{d}s_1. \tag{3.3}$$

Using the Schwarz inequality [8], we get

$$\begin{aligned}\left(\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1\right)^2 &= \left(\int_0^t |H_1(t-s_1)|^{\frac{1}{2}}|H_1(t-s_1)|^{\frac{1}{2}}v_1(x(s_1))\mathrm{d}s_1\right)^2\\ &\le \int_0^t |H_1(t-s_1)|\mathrm{d}s_1\int_0^t |H_1(t-s_1)|v_1^2(x(s_1))\mathrm{d}s_1.\end{aligned}$$

By using the inequality $|mn| \le \frac{1}{2}(m^2+n^2)$, and the previous inequality, we can write (3.3) as the following form

$$\begin{aligned}V_1 \le\ & 2y^2 + 2\int_0^t |H_1(t-s_1)|\mathrm{d}s_1\int_0^t |H_1(t-s_1)|v_1^2(x(s_1))\mathrm{d}s_1\\ &+ 2\int_0^t H_1(t-s_1)\big(v_1^2(x(s_1)) + y^2(t)\big)\mathrm{d}s_1 + 4H_1(0)\int_0^x v_1(\xi)\mathrm{d}\xi\\ &+ \left(\int_0^x f_1(\xi)d\xi\right)^2 + 2y\int_0^x f_1(s_1)\mathrm{d}s_1\\ &- 2\int_0^x f_1(s_1)\mathrm{d}s_1\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1\\ &+ 2H_1(0)\int_0^t\int_t^\infty |H_1(u-s_1)|\mathrm{d}u\, v_1^2(x(s_1))\mathrm{d}s_1.\end{aligned}$$

By the assumptions of Theorem 2.2, we have

$$\begin{aligned}V_1 \le & (2\beta_2c_2 + a_1 + a_1^2 + a_2L)x^2 + (2+2L+a_1)y^2\\ &+ W\left[\int_0^t\left\{(2+a_2+2L)|H_1(t-s_1)| + 2\beta_2\int_{t-s_1}^\infty H_1(u)\mathrm{d}u\right\}c_2^2x^2(s_1)\mathrm{d}s_1\right],\end{aligned}$$

where W is a wedge function.

Therefore, we have

$$(2+a_2+2L)c_2^2|H_1(t-s_1)| + 2c_2^2\beta_2\int_{t-s_1}^\infty H_1(u)\mathrm{d}u = \Phi(t-s_1).$$

Therefore, one can conclude that

$$V_1 \le \gamma_1(x^2+y^2) + W\left(\int_0^t \Phi(t-s_1)(x^2(s_1)+y^2(s_1))\mathrm{d}s_1\right),\ \ \gamma_1 > 0. \tag{3.4}$$

On the other hand

$$\begin{aligned} V_1 \geq & \left(y-\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1\right)^2 \\ & + \left(y+\int_0^x f_1(\xi)\mathrm{d}\xi - \int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1\right)^2 \\ \geq & \left(|y|-\int_0^t |H_1(t-s_1)||v_1(x(s_1))|\mathrm{d}s_1\right)^2 \\ & + \left(|y+\int_0^x f_1(\xi)\mathrm{d}\xi| - \int_0^t |H_1(t-s_1)||v_1(x(s_1))|\mathrm{d}s_1\right)^2. \end{aligned}$$

Since $\int_0^\infty |H_1(u)|\mathrm{d}u = L < 1$ and by the assumption (i) of Theorem 2.2, we conclude

$$V_1 \ \geq \left(|y| - c_2|x|\right)^2 + \left(|y| + \alpha_2|x| - c_2|x|\right)^2. \tag{3.5}$$

Thus, from (3.4) and (3.5), we conclude that the condition (i) of Theorem 2.1 is satisfied.

Now, by differentiating Eq. (3.2), we obtain

$$\begin{aligned} \frac{\mathrm{d}V_1}{\mathrm{d}t} =& 2\left(y-\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1\right)\left(-f_1(x)y - H_1(0)v_1(x)\right) \\ & + 2\left(y+\int_0^t f_1(\xi)\mathrm{d}\xi - \int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1\right)\left(-H_1(0)v_1(x)\right) \\ & + 4H_1(0)v_1(x)y + 2H_1(0)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\int_t^\infty |H_1(u-s_1)|\mathrm{d}u\, v_1^2(x(s_1))\mathrm{d}s_1. \end{aligned}$$

From Leibnitz rule [23] Pg. 17 and the identity [23] Pg. 17 and [6] Pg. 41, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\int_t^\infty |H_1(u-s_1)|\mathrm{d}u\, v_1^2(x(s_1))\mathrm{d}s_1 =& \int_t^\infty |H_1(u-t)|\mathrm{d}u\, v_1^2(x(t)) \\ & - \int_0^t |H_1(t-s_1)|v_1^2(x(s_1))\mathrm{d}s_1, \end{aligned}$$

then, we get

$$\begin{aligned} \frac{\mathrm{d}V_1}{\mathrm{d}t} =& -2f_1(x)y^2 - 2H_1(0)v_1(x)\int_0^x f_1(\xi)\mathrm{d}\xi + 2H_1(0)\int_t^\infty |H_1(u-t)|\mathrm{d}u\, v_1^2(x(t)) \\ & - 2H_1(0)\int_0^t |H_1(t-s_1)|v_1^2(x(s_1))\mathrm{d}s_1 + 4H_1(0)v_1(x)\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1 \\ & + 2f_1(x)y\int_0^t H_1(t-s_1)v_1(x(s_1))\mathrm{d}s_1. \end{aligned}$$

From the condition (i) and the inequality $|mn| \leq \frac{1}{2}(m^2+n^2)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}V_1}{\mathrm{d}t} \leq &- 2a_2y^2 - H_1(0)c_1b_2x^2 + 2H_1(0)\int_t^\infty |H_1(u-t)|\mathrm{d}u\, v_1^2(x(t))\\ &+ c_1b_1L(x^2+y^2) - 2H_1(0)\int_0^t |H_1(t-s_1)|v_1^2(x(s_1))\mathrm{d}s_1\\ &+ 2H_1(0)\int_0^t H_1(t-s_1)\big(v_1^2(x(s_1)) + v_1^2(x(t))\big)\mathrm{d}s_1.\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}\frac{\mathrm{d}V_1}{\mathrm{d}t} \leq &- 2a_2y^2 - H_1(0)c_1b_2x^2 + 2H_1(0)\bigg(\int_t^\infty |H_1(u-t)|\mathrm{d}u\\ &+ \int_0^t |H_1(t-s_1)|\mathrm{d}s_1\bigg)v_1^2(x(t)) + c_1b_1Lx^2 + c_1b_1Ly^2.\end{aligned}$$

Consider the conditions (i)–(iv) and $|H_1(0)| \geq \beta_1$, we have

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} \leq -\bigg\{\Big(\beta_1b_2c_1 + 2\beta_1c_2^2L_1 - c_1b_1L\Big)|x|^2 + \Big(2a_2 - c_1b_1L\Big)|y|^2\bigg\}.$$

Therefore, we conclude for $D_1 > 0$, that

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} \leq -D_1(|x|^2+|y|^2), \text{ for all } D_1 > 0, \tag{3.6}$$

where, $D_1 = \min\{\beta_1b_2c_1 + 2\beta_1c_2 - c_1b_1L, 2a_2 - c_1b_1\}$.
Thus, from (3.4), (3.5) and (3.6) all the assumptions of Theorem 2.1 are satisfied. Therefore the zero solution of (1.1) is UAS. Hence, the proof of Theorem 2.2 is now complete.

4. Proof of Theorem 2.3.

We can rewrite (1.2) as the following equivalent system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= -f_2(y)z - \alpha y - H_2(0)v_2(y) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2.\end{aligned}\tag{4.1}$$

Define the LKF $V_2(t,x_t,y_t,z_t)$ as

$$\begin{aligned}V_2(t,x_t,y_t,z_t) =&\bigg(z + \alpha x + \int_0^y f_2(\xi)\mathrm{d}\xi - \int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\bigg)^2\\ &+ H_2(0)\int_0^t\int_t^\infty |H_2(u-s_2)|\mathrm{d}u\, v_2^2(y(s_2))\mathrm{d}s_2.\end{aligned}\tag{4.2}$$

From Eq. (4.2), we get

$$\begin{aligned} V_2 =& \left(z+\alpha x+\int_0^y f_2(\xi)\mathrm{d}\xi\right)^2+\left(\int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\right)^2 \\ &-2\left(z+\alpha x+\int_0^y f_2(\xi)\mathrm{d}\xi\right)\left(\int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\right) \\ &+H_2(0)\int_0^t\int_t^\infty |H_2(u-s_2)|\mathrm{d}u\, v_2^2(y(s_2))\mathrm{d}s_2. \end{aligned}$$

Applying the condition (i) and the inequality $|mn| \leq \frac{1}{2}(m^2+n^2)$, we obtain

$$\begin{aligned} V_2 \leq& z^2+\alpha^2x^2+\alpha\left(x^2+z^2\right)+\alpha_2^2y^2+\alpha_2\left(y^2+z^2\right)+\alpha\alpha_2\left(x^2+y^2\right) \\ &+\left(\int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\right)^2+\int_0^t H_2(t-s_2)\left\{v_2^2(y(s_2))+z^2(t)\right\}\mathrm{d}s_2 \\ &+\alpha\int_0^t H_2(t-s_2)\left\{v_2^2(y(s_2))+x^2(t)\right\}\mathrm{d}s_2 \\ &+\alpha_1\int_0^t H_2(t-s_2)\left\{v_2^2(y(s_2))+y^2(t)\right\}\mathrm{d}s_2 \\ &+H_2(0)\int_0^t\int_t^\infty |H_2(u-s_2)|\mathrm{d}u\, v_2^2(y(s_2))\mathrm{d}s_2. \end{aligned}$$

Since $\int_0^\infty H_2(u)\mathrm{d}u = L$ and from condition (i), then we get

$$\begin{aligned} V_2 \leq& (1+\alpha+\alpha_2+L)\left(\alpha x^2(t)+\alpha_2y^2(t)+z^2(t)\right) \\ &+\left(\int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\right)^2 \\ &+d_2^2(1+\alpha+\alpha_1)\int_0^t |H_2(t-s_2)|y^2(s_2)\mathrm{d}s_2 \\ &+H_2(0)\int_0^t\int_t^\infty |H_2(u-s_2)|\mathrm{d}u\, v_2^2(y(s_2))\mathrm{d}s_2. \end{aligned} \tag{4.3}$$

By the Schwarz inequality [8], we have

$$\begin{aligned} \left(\int_0^t H_2(t-s_2)v_2(x(s_2))\mathrm{d}s_2\right)^2 &= \left(\int_0^t |H_2(t-s_2)|^{\frac{1}{2}}|H_2(t-s_2)|^{\frac{1}{2}}v_2(x(s_2))\mathrm{d}s_2\right)^2 \\ &\leq \int_0^t |H_2(t-s_2)|\mathrm{d}s_2\int_0^t |H_2(t-s)2)|v_2^2(x(s_2))\mathrm{d}s_2. \end{aligned}$$

Applying the conditions of Theorem 2.3, we obtain

$$\begin{aligned} V_2 \leq &(1+\alpha+\alpha_2+L)\Big(\alpha x^2(t)+\alpha_2 y^2(t)+z^2(t)\Big) \\ &+\int_0^t |H_2(t-s_2)|\mathrm{d}s_2 \int_0^t |H_2(t-s_2)|v_2^2(x(s_2))\mathrm{d}s_2 \\ &+ d_2^2(1+\alpha+\alpha_1)\int_0^t |H_2(t-s_2)|y^2(s_2)\mathrm{d}s_2 \\ &+ \theta_2 d_2^2 \int_0^t \int_t^\infty |H_2(u-s_2)|\mathrm{d}u y^2(s_2)\mathrm{d}s_2. \end{aligned}$$

It follows that

$$\begin{aligned} V_2 \ \leq &\ (1+\alpha+\alpha_2+L)\Big(\alpha x^2+\alpha_2 y^2+z^2\Big) \\ &+W\Bigg[\int_0^t \Big\{d_2(1+\alpha+\alpha_1)|H_2(t-s_2)|+d_2^2\theta_2\int_{t-s_2}^\infty H_2(u)\mathrm{d}u\Big\}y^2(s_2)\mathrm{d}s_2\Bigg]. \end{aligned}$$

If we let

$$d_2(1+\alpha+\alpha_1)|H_2(t-s_2)|+d_2^2\theta_2\int_{t-s_2}^\infty H_2(u)\mathrm{d}u=\Phi(t-s_2),$$

then, we get

$$\begin{aligned} V_2 \ \leq &\ (1+\alpha+\alpha_2+L)\Big(\alpha x^2+\alpha_2 y^2+z^2\Big) \\ &+W\Big(\int_0^t \Phi(t-s_2)y^2(s_2)\mathrm{d}s_2\Big). \end{aligned}$$

Since $1+\alpha+\alpha_2+L>0$, then we have a positive constant γ_2, such that

$$\begin{aligned} V_2 \leq &\gamma_2\left(\alpha x^2+\alpha_2 y^2+z^2\right) \\ &+W\Big(\int_0^t \Phi(t-s_2)\big(x^2(s_2)+y^2(s_2)+z^2(s_2)\big)\Big). \end{aligned} \tag{4.4}$$

Now, (4.2) becomes

$$\begin{aligned} V_2 &\geq \left(z+\alpha x+\int_0^y f_2(\xi)\mathrm{d}\xi-\int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\right)^2 \\ &\geq \left(|z+\alpha x+\int_0^y f_2(\xi)\mathrm{d}\xi|-\int_0^t |H_2(t-s_2)v_2(y(s_2))|\,\mathrm{d}s_2\right)^2. \end{aligned}$$

By (ii), we have $\int_0^\infty |H_2(u)|\mathrm{d}u=L<1$ and by the assumption (i) of Theorem 2.3, we conclude that

$$V_2 \ \geq \Big(|z|+\alpha|x|+\alpha_1|y|-d_2|y|\Big)^2. \tag{4.5}$$

Differentiating the LKF $V_2(t, x_t, y_t, z_t)$ with respect to t

$$\begin{aligned}\frac{\mathrm{d}V_2}{\mathrm{d}t} =& 2\Big(z + \alpha x + \int_0^y f_2(\xi)\mathrm{d}\xi - \int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\Big)\\ &\times\Big(\dot{z} + \alpha\dot{x} + f_2(y)z - \frac{d}{dt}\int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\Big)\\ &+ H_2(0)\frac{d}{dt}\int_0^t\int_t^\infty |H_2(u-s_2)|\mathrm{d}u\, v_2^2(y(s_2))\mathrm{d}s_2.\end{aligned}$$

From Leibnitz rule [23] Pg. 17 and the identity [6] Pg. 41, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\int_t^\infty |H_2(u-s_2)|\mathrm{d}u\, v_2^2(y(s_2))\mathrm{d}s_2 =& \int_t^\infty |H_2(u-t)|\mathrm{d}u\, v_2^2(y(t))\\ &- \int_0^t |H_2(t-s_2)|v_2^2(y(s_2))\mathrm{d}s_2.\end{aligned}$$

By using the equivalent system (4.1), we obtain

$$\begin{aligned}\frac{\mathrm{d}V_2}{\mathrm{d}t} =& 2\Big(z + \alpha x + \int_0^y f_2(\xi)\mathrm{d}\xi - \int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\Big)\\ &\times\Big(- H(0)v(y) + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\\ &- \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t H_2(t-s_2)v_2(y(s_2))\mathrm{d}s_2\Big) + H_2(0)\int_t^\infty |H_2(u-t)|\mathrm{d}u\, v_2^2(y(t))\\ &- H_2(0)\int_0^t |H_2(t-s_2)|v_2^2(y(s_2))\mathrm{d}s_2.\end{aligned}$$

From condition (i), we get

$$\begin{aligned}\frac{\mathrm{d}V_2}{\mathrm{d}t} \leq& - 2H_2(0)d_2yz - 2\alpha H_2(0)d_2xy - 2H_2(0)d_2\alpha_1 y^2\\ &+ H_2(0)\Big(\int_0^t |H_2(t-s_2)|\mathrm{d}s_2 + \int_t^\infty |H_2(u-t)|\mathrm{d}u\Big)d_2^2y^2.\end{aligned}$$

It follows from condition (iv) and the inequality $|mn| \leq \frac{1}{2}(m^2+n^2)$ that

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} \leq - \theta_1\Big\{\big(d_2 + \alpha d_2 + 2\alpha_1 d_2 - \theta_3 d_2^2\big)y^2 + \alpha d_2 x^2 + d_2 z^2\Big\}.$$

Thus, one can conclude for a positive constant $D_2 > 0$ that

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} \leq -D_2\left(x^2 + y^2 + z^2\right), \tag{4.6}$$

where $D_2 = \theta_1 \min\{d_2 + \alpha d_2 + 2\alpha_1 d_2 - \theta_3 d_2^2, \alpha d_2, d_2\}$. From the results (4.4), (4.5) and (4.6), we note that all assumptions of Theorem 2.1 are satisfied, then the zero solution of (1.2) is UAS.

Thus, the proof of Theorem 2.3 is now complete.

5. Illustrative Examples

Example 5.1. Consider the following VIDE with delay

$$\ddot{x} + (10x^{\frac{1}{2}} - 5\sin(x))\dot{x} + \int_0^t e^{t-s_1-1}\big(6\sin^2(x(s_1)) + 5\sin^3(x(s_1))\big)\mathrm{d}s_1 = 0. \tag{5.1}$$

Note that

$$f_1(x) = 10x^{\frac{1}{2}} - 5\sin(x), \;\; f_1(0) = 0.$$

So, we find

$$10 \le 10x^{\frac{1}{2}} - 5\sin(x) \le 46, \;\text{ so }\; a_1 = 46 \;\text{ and }\; a_2 = 10,$$

and

$$f_1'(x) = 5x^{-\frac{1}{2}} - 5\cos x, \;\; |f_1'(x)| \le 10 = c_1.$$

Figure 1, shows the behaviour of $f_1(x)$ and $f_1'(x)$ on the interval $t \in [2, 20]$ and $t \in [0, 90]$, respectively.

Moreover, we have

$$v_1(x) = 6\sin^2 x + 5\sin^3 x, \;\text{ so, }\; 1 \le v_1(x) \le 5 \;\text{ then, we get }\; b_1 = 5, \; b_2 = 1,$$

and

$$v_1'(x) = 12\sin x\cos x + 15\sin^2 x\cos x, \;\;\text{ therefore }\;\; |v_1'(x)| \le 12 = c_2.$$

Figure 2, illustrates the behaviour of $v_1(x)$ and $v_1'(x)$ through the interval $t \in [0, 90]$.

Also, we have

$$\int_0^\infty |H_1(u)|\mathrm{d}u = \frac{1}{e} = L,$$

and

$$L_1 = \int_t^\infty e^{t-s_1-1}\mathrm{d}s_1 + \int_0^t e^{t-s_1-1}\mathrm{d}s_1 = \frac{1}{e}.$$

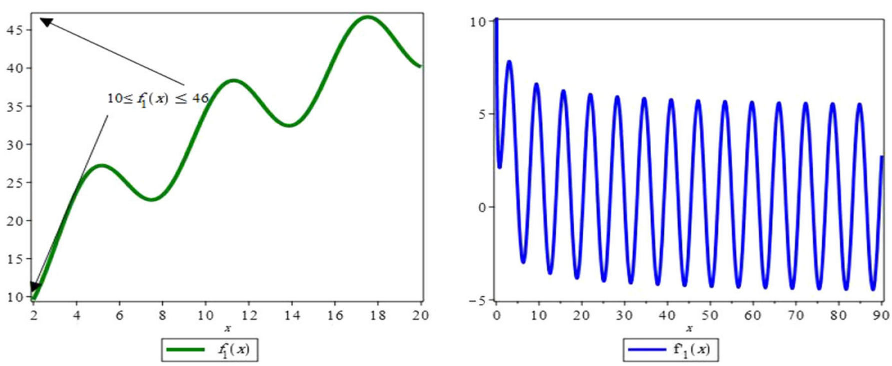

Figure 1. Trajectories of the functions $f_1(x)$ and $f_1'(x)$

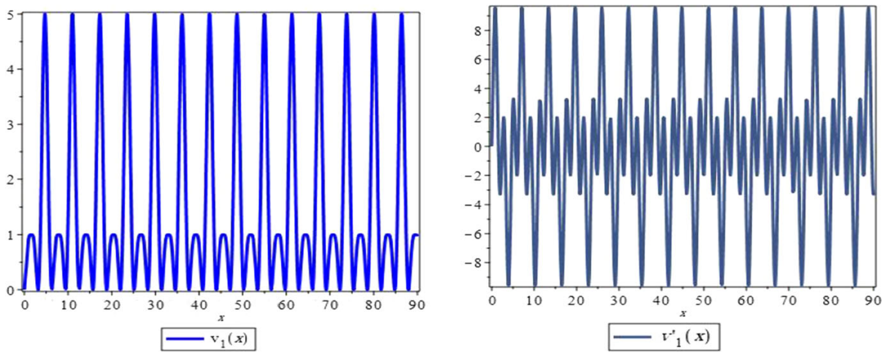

Figure 2. Trajectories of the functions $v_1(x)$ and $v_1'(x)$

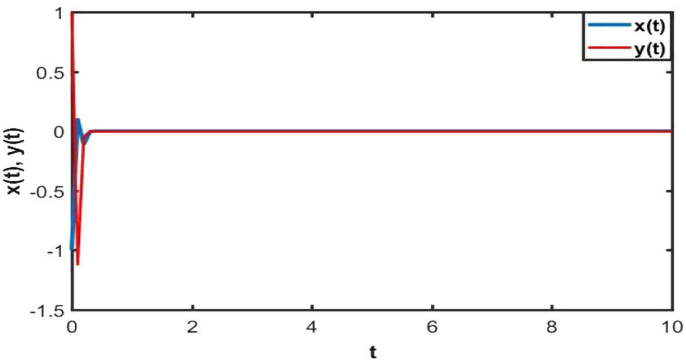

Figure 3. Trajectories of the solutions for Example 5.1

Then, we get

$$H_1(0) = \frac{1}{e}, \quad \frac{1}{2e} \le H_1(0) \le \frac{2}{e},$$

and

$$\beta_1 b_2 c_1 + 2\beta_1 L_1 c_2^2 = 21.3 \text{ and } c_1 b_1 L = 18.4.$$

So, it is clear that

$$\beta_1 b_2 c_1 + 2\beta_1 L_1 c_2^2 > c_1 b_1 L, \quad 2a_2 = 20 > c_1 b_1 L.$$

We can see that the behaviour of the solutions $(x(t), y(t))$ with the initial values $(x_0 = 0, y_0 = 1)$ for (5.1) by Fig. 3.

Thus, all the hypotheses of Theorem 2.2 are satisfied.
Then, the zero solution of (5.1) is UAS.

Example 5.2. Consider the following VIDE with delay

$$\dddot{x} + 3\sin(\dot{x})\ddot{x} + 8\dot{x} + \int_0^t 2e^{t-s_2-1}(2\sin^2 s_2)\mathrm{d}s_2 = 0. \tag{5.2}$$

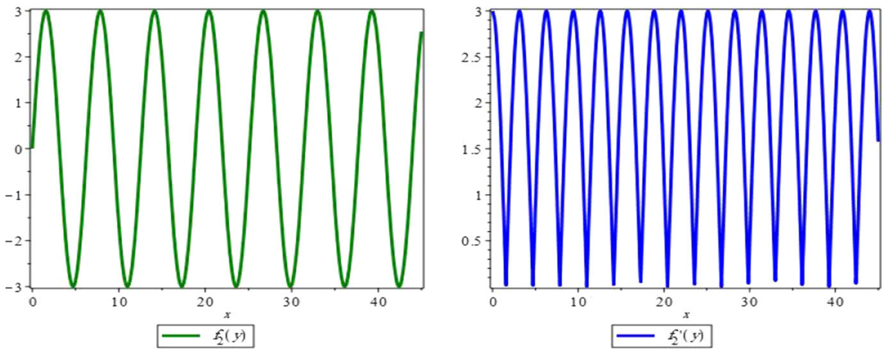

Figure 4. Trajectories of the functions $f_2(y)$ and $f_2'(y)$

It follows that

$$f_2(y) = 3\sin y, \text{ then } \; f_2(0) = 0.$$

So, we get

$$1 \leq f_2(y) \leq 3, \text{ then } \; \alpha_1 = 1, \; \alpha_2 = 3,$$

and

$$f_2'(y) = 3\cos y, \text{ then } \; |3\cos y| \leq 3 = d_1.$$

Figure 4, shows the behaviour of $f_2(x)$ and $f_2'(x)$ on the interval $t \in [0, 50]$.

Moreover

$$v_2(y) = 2\sin^2 y, \; \text{ then } \; v_2(0) = 0.$$

So, we get

$$0 \leq v_2(y) \leq 2, \; \text{ then } \; \alpha_3 = 0, \; \alpha_4 = 2,$$

and

$$v_2'(y) = 4\sin y \cos y, \text{ then } \; |4\sin y \cos y| \leq 2 = d_2.$$

Figure 5, illustrates the path of $v_2(x)$ and $v_2'(x)$ on the interval $t \in [0, 50]$.
Also, we have

$$\int_0^{\infty} |H_2(u)| du = \frac{1}{e} = L,$$

and

$$\int_t^{\infty} 2e^{t-s_2-1} \mathrm{d}s_2 + \int_0^t 2e^{t-s_2-1} \mathrm{d}s_1 = \frac{2}{e} = \theta_3.$$

Also, we have

$$H_2(0) = \frac{2}{e}, \quad \frac{e}{4} \leq H(0) \leq \frac{1}{e}.$$

and

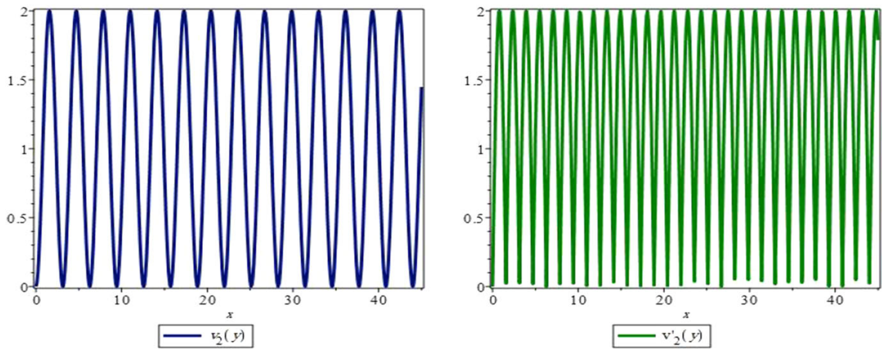

Figure 5. Trajectories of the functions $v_2(y)$ and $v_2'(y)$

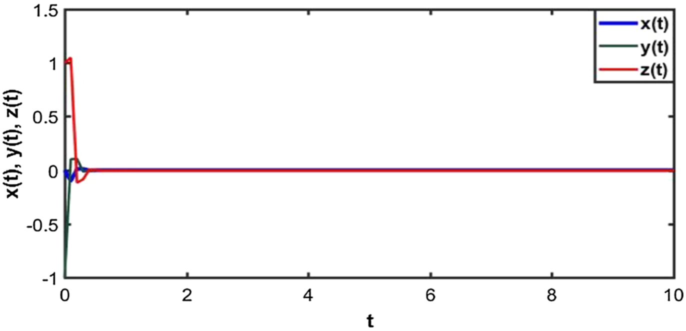

Figure 6. Path of the solutions for Example 5.2

$$\int_t^{\infty} 2e^{t-s_2-1}\mathrm{d}s_2 + \int_0^t 2e^{t-s_2-1}\mathrm{d}s_2 = \frac{2}{e} < \frac{\theta_1\theta_3}{\theta_2}.$$

So, it is clear that

$$1 + \alpha + 2\alpha_1 > d_2\theta_3,$$

Figure 6, shows the behaviour of the solutions $(x(t), y(t), z(t))$ with the initial values $(x_0 = 0, y_0 = 1, z_0 = 1)$ for (5.2).

Thus, all the hypotheses of Theorem 2.3 are verified.
Then, the zero solution of (5.2) is UAS.

6. Conclusion

This work emphasizes the stability of solutions to certain nonlinear second-order and third-order VIDE with delay.

By employing Lyapunov's second method, a suitable LKF was constructed and used to establish the sufficient conditions of Theorems 2.2 and 2.3.

Two numerical examples were given and all functions were drawn to prove the sufficient conditions of Theorems 2.2 and 2.3, and also orbits of the numerical solutions were drawn with assigned initial conditions to demonstrate the effectiveness of the obtained results.

The results obtained in this paper extend many existing and exciting results on nonlinear VIDE.

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Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

References

[1] Appleby, J.A.D., Reynold, D.W.: On the non-exponential convergence of asymptotically stable solutions of linear scalar volterra integro-differential equation. J. Integr. Equ. Appl. **14**(2), 109–118 (2022)

[2] Berezansky, L., Braverman, E., Akça, H.: On oscillation of a linear delay integro-differential equation. Dyn. Syst. Appl. **8**(2), 219–234 (1999)

[3] Berezansky, L., Domoshnitsky, A.: On stability of a second order integro-differential equation. Nonlinear Dyn. Syst. Theory **19**(1), 117–123 (2019)

[4] Berezansky, L., Braverman, E.: On exponential stability of linear delay equations with oscillatory coefficients and kernels. Differ. Integr. Equ. **35**, 559–580 (2022)

[5] Bohner, M., Tunç, O., Tunç, C.: Qualitative analysis of caputo fractional integro-differential equations with constant delays. Comput. Appl. Math. **40**(214), 1–17 (2021)

[6] Burton, T.A.: Volterra Integral and Differential Equations. Academic Press, New York (1983)

[7] Burton, T.A.: Stabitity and Periodic Solutions of Ordinary and Functional Differential Equations. Academic Press, Cambridge (1985)

[8] Clason, C.: Introduction to Functional Analysis. Springer Nature, Switzerland (2019)

[9] El Hajji, M.: Boundedness and asymptotic stability of nonlinear Volterra integro-differential equations using Lyapunov functional. J. King Saud Univ. Sci. **31**, 1516–1521 (2019)

[10] Graef, J.R., Tunç, C.: Continuability and boundedness of multi-delay functional integro-differential equations of the second-order. RACSAM **109**, 169–173 (2015)

[11] Raffoul, Y., Rai, H.: Uniform stability in nonlinear infinite delay Volterra integro-differential equations using Lyapunov functionals. Nonauton. Dyn. Syst. **3**, 14–23 (2016)

[12] Rahman, M.: Integral Equations and Their Applications. WIT Press, Boston (2007)

[13] Rama-Mohana-Rao, M., Srinivas, P.: Asymptotic behavior of solutions of Volterra integro-differential equations. Proc. Am. Math. Soc. **94**(1), 55–60 (1985)

[14] Tunç, C.: New stability and boundedness results to Volterra integro-differential equations with delay. J. Egypt. Math. Soc. **24**, 210–213 (2016)

[15] Tunç, C.: A note on the qualitative behaviors of non-linear Volterra integro-differential equation. J. Egypt. Math. Soc. **24**(2), 187–192 (2016)

[16] Tunç, C., Tunç, O.: A note on the qualitative analysis of Volterra integro-differential equations. J. Taibah Univ. Sci. **13**(1), 490–496 (2019)

[17] Tunç, C., Tunç, O.: On the stability, integrability and boundedness analyses of systems of integro-diferential equations with time-delay retardation. RACSAM **115**, 895 (2021)

[18] Tunç, C., Tunç, O.: On the fundamental analyses of solutions to nonlinear integro-differential equations of the second-order. Mathematics **10**, 1–18 (2022)

[19] Tunç, O., Atan, S., Tunç, C., Yao, J.C.: Qualitative analyses of integro-fractional differential equations with caputo derivatives and retardations via the lyapunov-razumikhin method. Axioms **10**(58), 1–19 (2021)

[20] Tunç, C., Tunç, O., Yao, J.C.: On the new qualitative results in integro-differential equations with caputo fractional derivative and multiple kernels and delays. J. Nonlinear Convex Anal. **23**(11), 2577–2591 (2022)

[21] Tunç, O., Tunç, C., Yao, J., Wen, C.: New fundamental results on the continuous and discrete integro-differential equations. Mathematics **10**, 852 (2022)

[22] Tunç, O., Tunç, C., Yao, J., Wen, C.: On the qualitative analyses solutions of new mathematical models of integro-differential equations with infinite delay. Math. Meth. Appl. Sci. **2023**, 1–17 (2023)

[23] Wazwaz, A.M.: Linear and Nonlinear Integral Equations. Methods and Applications. Higher Education Press, Springer, Beijing, Berlin (2011)

[24] Yoshizawa, T.: Stability Theory by Lyapunov's Second Method. The Mathematical Society of Japan (1966)

[25] Zhang, B.: Construction of Liapunov functionals for linear Volterra integro-differential equations and stability of delay systems. Elect. J. Qual. Theory Differ. Equ. **30**, 1–17 (2000)

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