# Correction to: Equilibrium of Surfaces in a Vertical Force Field 

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In the previous paper [2, Section 3.2], we describe the family of titled $\left[\varphi, \vec{e}_{3}\right]$ catenary cylinders as surfaces obtained from a $\left[\varphi, \vec{e}_{3}\right]$-catenary cylinder $\Sigma$, by rotation of angle $\theta \in] 0, \pi / 2[$ about the $x$-axis and dilation by scale factor $1 / \cos \theta$. The authors state that the resulting surface, $\widetilde{\Sigma}$, is always a $\left[\varphi, \vec{e}_{3}\right]$ minimal surface, but this is only true when the starting $\left[\varphi, \vec{e}_{3}\right]$-catenary cylinder is a grim reaper translating soliton, which follows directly from the relationship between their mean curvatures. In fact, following the same notation as in [2, Section 3.2], $\widetilde{H}=\cos \theta H$, and $\widetilde{\Sigma}$ will be a $\left[\varphi, \vec{e}_{3}\right]$-minimal surface if and only if

$$
\dot{\varphi}(u)=\dot{\varphi}(u+y \sin \theta) \quad \text { for any } y \in \mathbb{R}
$$

that is, if and only if $\Sigma$ is a grim reaper translating soliton. In this case, $\widetilde{\psi}$ will be a tilted grim reaper cylinder.

The following result updates the classification of complete flat $\left[\varphi, \vec{e}_{3}\right]$ minimal surfaces in $\mathbb{R}^{3}$ [2, Theorem 3.7]:

Theorem 1. Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a diffeomorphism and $\Sigma$ be a complete flat $\left[\varphi, \vec{e}_{3}\right]$-minimal surface in $\mathbb{R}^{3}$. Then, one of the following statements holds

- $\Sigma$ is a vertical plane.
- $\Sigma$ is a grim reaper cylinder (maybe tilted).
- $\Sigma$ is a $\left[\varphi, \vec{e}_{3}\right]$-catenary cylinder.

Proof. From basic differential geometry, $\Sigma=\gamma \times \Pi^{\perp}$ is a ruled surface and its Gauss map is constant along the rules, where $\gamma$ is a complete regular curve in a plane $\Pi \subset \mathbb{R}^{3}$. Thus, $\Sigma$ can be parametrized by $\psi(s, t)=\gamma(s)+t \vec{v}$, with $\gamma(s)$ a complete regular curve contained in a plane $\Pi$ and $\vec{v}$ a unit vector

[^0]orthogonal to $\Pi$. We may assume that $(s, t) \in \mathbb{R}^{2}$ and $\left|\gamma^{\prime}\right|=1$, then, the Gauss map $N$ of $\psi$ and its mean curvature $H$ are given by
$$
N(s, t)=\gamma^{\prime}(s) \wedge \vec{v}, \quad H(s, t)=\kappa_{\gamma}(s),
$$
where $\kappa_{\gamma}$ is the curvature of $\gamma$. Hence, $\Sigma$ is $\left[\varphi, \vec{e}_{3}\right]$-minimal if and only if the following relation holds
$$
\kappa_{\gamma}(s)=-\dot{\varphi}\left(\left\langle\gamma(s)+t \vec{v}, \vec{e}_{3}\right\rangle\right)\left\langle\gamma^{\prime}(s) \wedge \vec{v}, \vec{e}_{3}\right\rangle .
$$

Differentiating with respect to $t$ in the above expression, we obtain that

$$
0=\ddot{\varphi}\left(\left\langle\gamma(s)+t \vec{v}, \vec{e}_{3}\right\rangle\right)\left\langle\vec{v}, \vec{e}_{3}\right\rangle\left\langle\gamma^{\prime}(s) \wedge \vec{v}, \vec{e}_{3}\right\rangle .
$$

If $\left\langle\vec{v}, \vec{e}_{3}\right\rangle=0$, arguing as in [2, Theorem 3.7], for any horizontal rule $\mathcal{L}$ of $\Sigma$ there exists a $\left[\varphi, \vec{e}_{3}\right]$-catenary cylinder $\mathcal{C}$ containing $\mathcal{L}$ and tangent to $\Sigma$ along $\mathcal{L}$. Thus, from standard theory of uniqueness of solution for an ODE, up to horizontal translation, $\Sigma$ must coincide with $\mathcal{C}$.
If $\left\langle\vec{v}, \vec{e}_{3}\right\rangle \neq 0$,

$$
0=\ddot{\varphi}\left(\left\langle\gamma(s)+t \vec{v}, \vec{e}_{3}\right\rangle\right)\left\langle\gamma^{\prime}(s) \wedge \vec{v}, \vec{e}_{3}\right\rangle
$$

and we can assume that $\ddot{\varphi}$ does not vanish everywhere otherwise, from [1], $\varphi$ is a linear function and $\Sigma$ is either a vertical plane or a grim reaper cylinder (maybe tilted). Thus, we have that $\gamma^{\prime} \wedge \vec{v}$ is orthogonal to $\vec{e}_{3}$ and $\Sigma$ must be a vertical plane.

Now, the Corollary 3.8 in [2] updates to
Corollary 2. Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be a increasing diffeomorphism with $\dddot{\varphi} \leq 0$ and $\Sigma$ be a complete locally convex $\left[\varphi, \vec{e}_{3}\right]$-minimal surface in $\mathbb{R}^{3}$. If the Gauss curvature vanishes anywhere, then one of the following statements holds

- $\Sigma$ is a vertical plane.
- $\Sigma$ is a grim reaper cylinder (maybe tilted).
- $\Sigma$ is a $\left[\varphi, \vec{e}_{3}\right]$-catenary cylinder.

Remark 3. Although it does not affect any of the results shown throughout the paper, the Eqs. (6) and (7) in [2, Lemma 2.1] must be change to

$$
\begin{align*}
& \nabla^{2} H=-\eta \nabla^{2} \dot{\varphi}-(\nabla \mathcal{A})(\nabla \varphi, \cdot, \cdot)-H \mathcal{A}^{[2]}-\mathcal{B}  \tag{6}\\
& \Delta \mathcal{A}+(\nabla \mathcal{A})(\nabla \varphi, \cdot, \cdot)+\eta \nabla^{2} \dot{\varphi}+|\mathcal{A}|^{2} \mathcal{A}+\mathcal{B}=0 \tag{7}
\end{align*}
$$

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## References

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