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Correction to: Equilibrium of Surfaces in a Vertical Force Field

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In the previous paper [2, Section 3.2], we describe the family of *titled* $[\varphi, \vec{e}_3]$-*catenary cylinders* as surfaces obtained from a $[\varphi, \vec{e}_3]$-catenary cylinder Σ, by rotation of angle $\theta \in]0, \pi/2[$ about the x-axis and dilation by scale factor $1/\cos\theta$. The authors state that the resulting surface, $\widetilde{\Sigma}$, is always a $[\varphi, \vec{e}_3]$-minimal surface, but this is only true when the starting $[\varphi, \vec{e}_3]$-catenary cylinder is a grim reaper translating soliton, which follows directly from the relationship between their mean curvatures. In fact, following the same notation as in [2, Section 3.2], $\widetilde{H} = \cos\theta\, H$, and $\widetilde{\Sigma}$ will be a $[\varphi, \vec{e}_3]$-minimal surface if and only if

$$\dot{\varphi}(u) = \dot{\varphi}(u + y\ \sin\theta) \quad \text{for any } y \in \mathbb{R},$$

that is, if and only if Σ is a grim reaper translating soliton. In this case, $\widetilde{\psi}$ will be a *tilted grim reaper cylinder*.

The following result updates the classification of complete flat $[\varphi, \vec{e}_3]$-minimal surfaces in $\mathbb{R}^3$ [2, Theorem 3.7]:

Theorem 1. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism and Σ be a complete flat $[\varphi, \vec{e}_3]$-minimal surface in $\mathbb{R}^3$. Then, one of the following statements holds*

- *Σ is a vertical plane.*
- *Σ is a grim reaper cylinder (maybe tilted).*
- *Σ is a $[\varphi, \vec{e}_3]$-catenary cylinder.*

Proof. From basic differential geometry, $\Sigma = \gamma \times \Pi^{\perp}$ is a ruled surface and its Gauss map is constant along the rules, where γ is a complete regular curve in a plane $\Pi \subset \mathbb{R}^3$. Thus, Σ can be parametrized by $\psi(s,t) = \gamma(s) + t\vec{v}$, with $\gamma(s)$ a complete regular curve contained in a plane Π and $\vec{v}$ a unit vector

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orthogonal to Π. We may assume that $(s,t)\in\mathbb{R}^2$ and $|\gamma'|=1$, then, the Gauss map N of ψ and its mean curvature H are given by

$$N(s,t)=\gamma'(s)\wedge\vec{v},\quad H(s,t)=\kappa_\gamma(s),$$

where κ_γ is the curvature of γ. Hence, Σ is $[\varphi,\vec{e}_3]$-minimal if and only if the following relation holds

$$\kappa_\gamma(s)=-\dot{\varphi}(\langle\gamma(s)+t\vec{v},\vec{e}_3\rangle)\langle\gamma'(s)\wedge\vec{v},\vec{e}_3\rangle.$$

Differentiating with respect to t in the above expression, we obtain that

$$0=\ddot{\varphi}(\langle\gamma(s)+t\vec{v},\vec{e}_3\rangle)\langle\vec{v},\vec{e}_3\rangle\langle\gamma'(s)\wedge\vec{v},\vec{e}_3\rangle.$$

If $\langle\vec{v},\vec{e}_3\rangle=0$, arguing as in [2, Theorem 3.7], for any horizontal rule $\mathcal{L}$ of Σ there exists a $[\varphi,\vec{e}_3]$-catenary cylinder $\mathcal{C}$ containing $\mathcal{L}$ and tangent to Σ along $\mathcal{L}$. Thus, from standard theory of uniqueness of solution for an ODE, up to horizontal translation, Σ must coincide with $\mathcal{C}$.
If $\langle\vec{v},\vec{e}_3\rangle\neq0$,

$$0=\ddot{\varphi}(\langle\gamma(s)+t\vec{v},\vec{e}_3\rangle)\langle\gamma'(s)\wedge\vec{v},\vec{e}_3\rangle$$

and we can assume that $\ddot{\varphi}$ does not vanish everywhere otherwise, from [1], φ is a linear function and Σ is either a vertical plane or a grim reaper cylinder (maybe tilted). Thus, we have that $\gamma'\wedge\vec{v}$ is orthogonal to $\vec{e}_3$ and Σ must be a vertical plane. □

Now, the Corollary 3.8 in [2] updates to

Corollary 2. *Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a increasing diffeomorphism with $\dddot{\varphi}\leq0$ and Σ be a complete locally convex $[\varphi,\vec{e}_3]$-minimal surface in $\mathbb{R}^3$. If the Gauss curvature vanishes anywhere, then one of the following statements holds*

- *Σ is a vertical plane.*
- *Σ is a grim reaper cylinder (maybe tilted).*
- *Σ is a $[\varphi,\vec{e}_3]$-catenary cylinder.*

Remark 3. Although it does not affect any of the results shown throughout the paper, the Eqs. (6) and (7) in [2, Lemma 2.1] must be change to

$$\nabla^2H=-\eta\nabla^2\dot{\varphi}-(\nabla\mathcal{A})(\nabla\varphi,\,\cdot,\,\cdot\,)-H\mathcal{A}^{[2]}-\mathcal{B}\tag{6}$$
$$\Delta\mathcal{A}+(\nabla\mathcal{A})(\nabla\varphi,\,\cdot,\,\cdot\,)+\eta\nabla^2\dot{\varphi}+|\mathcal{A}|^2\mathcal{A}+\mathcal{B}=0,\tag{7}$$

References

[1] Martín, F., Savas-Halilaj, A., Smoczyk, K.: On the topology of translating solitons of the mean curvature flow. Calc. Var. **54**, 2853–2882 (2015)

[2] Martínez, A., Martínez-Triviño, A.L.: Equilibrium of surfaces in a vertical force field. Mediterr. J. Math. **19**, 3 (2022). https://doi.org/10.1007/s00009-021-01877-4

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