



# Hadamard Products of Symbolic Powers and Hadamard Fat Grids

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**Abstract.** In this paper we address the question if, for points  $P, Q \in \mathbb{P}^2$ ,  $I(P)^m \star I(Q)^n = I(P \star Q)^{m+n-1}$  and we obtain different results according to the number of zero coordinates in  $P$  and  $Q$ . Successively, we use our results to define the so called Hadamard fat grids, which are the result of the Hadamard product of two sets of collinear points with given multiplicities. The most important invariants of Hadamard fat grids, as minimal resolution, Waldschmidt constant and resurgence, are then computed.

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## 1. Introduction

In the last few years, the Hadamard products of projective varieties have been widely studied from the point of view of Projective Geometry and Tropical Geometry. The main problem in this setting is the behaviour of the Hadamard product between varieties with many points with zero coordinates.

The paper [8], where Hadamard products of general linear spaces are studied, can be considered the first step in this direction. Successively, the second author, with Calussi, Fatabbi and Lorenzini, in [5] and [6], address the Hadamard products of linear varieties not necessarily in general position. Moreover, they show that the Hadamard product of two generic linear varieties  $V$  and  $W$  is projectively equivalent to a Segre embedding and that the Hilbert function of the Hadamard product  $V \star W$  of two varieties, with  $\dim(V), \dim(W) \leq 1$ , is the product of the Hilbert functions of the original varieties  $V$  and  $W$ .

An important result contained in [8] concerns the construction of star configurations of points, via Hadamard product. This construction found a generalization in [10] where the authors introduce a new construction using

the Hadamard product to present star configurations of codimension  $c$  of  $\mathbb{P}^n$  and which they called Hadamard star configurations. Successively, the first author and Calussi introduced a more general type of Hadamard star configuration; any star configuration constructed by their approach is called a weak Hadamard star configuration. In [1] they classify weak Hadamard star configurations and, in the case  $c = n$ , they investigate the existence of a (weak) Hadamard star configuration which is apolar to the generic homogeneous polynomial of degree  $d$ . The use of Hadamard products in this context permits a complete control both in the coordinates of the points forming the star configuration and both in the equations of the hyperplanes involved on it. This fact leads to the question if other interesting set of points can be constructed by Hadamard product. Recent results in this direction can be found in [3], where the second author along with Capresi and Carrucoli built Gorenstein sets of points in  $\mathbb{P}^3$ , with given  $h$ -vector, by creating, via Hadamard products, a stick figure of lines to which they apply the results of Migliore and Nagel [21], and also in [9] where the third author along with Carlini, Catalisano and Favacchio found a relation between star configurations where all the hyperplanes are osculating to the same rational normal curve (contact star configurations) and Hadamard products of linear varieties.

This paper addresses another question, proposed by the second author during the CMO Workshop “Ordinary and Symbolic Powers of Ideals” (May 14–19, 2017, Oaxaca, Mexico) stating

**Question 1.1.** Is it true that for  $P, Q$  points in  $\mathbb{P}^2$ ,  $I(P)^m \star I(Q)^n = I(P \star Q)^{m+n-1}$  ?

The answer of this question is given in Theorem 4.2. Successively this result is used in Section 5 to define the so called Hadamard fat grids, which are the result of the Hadamard product of two sets of collinear points with given multiplicities. Finally, we compute algebraic invariants of Hadamard fat grids, as minimal degree, minimal resolution, Waldschmidt constant and resurgence. We also point out that the results of Sect. 5 enlarge the known literature on the minimal graded resolution of sets of fat points in  $\mathbb{P}^2$  with all the same multiplicities supported on a complete intersection (see for instance [11, 18]).

## 2. Preliminary Results on Join and Hadamard Products of Ideals

Let  $S = \mathbb{K}[\mathbf{x}] = \mathbb{K}[x_0, \dots, x_N]$  be a polynomial ring over an algebraically closed field. Let  $\mathfrak{m} = \langle x_0, \dots, x_N \rangle$  be the homogeneous irrelevant ideal.

Let  $I_1, I_2, \dots, I_r$  be ideals in  $S$ . We introduce  $(N + 1)r$  new unknowns, grouped in  $r$  vectors  $\mathbf{y}_j = (y_{j0}, \dots, y_{jN})$ ,  $j = 1, 2, \dots, r$  and we consider the polynomial ring  $\mathbb{K}[\mathbf{x}, \mathbf{y}]$  in all  $(N + 1)r + N + 1$  variables.

Let  $I_j(\mathbf{y}_j)$  be the image of the ideal  $I_j$  in  $\mathbb{K}[\mathbf{x}, \mathbf{y}]$  under the map  $\mathbf{x} \mapsto \mathbf{y}_j$ . Then the *join*  $I_1 \star I_2 \star \dots \star I_r$ , as defined in [22], is the elimination ideal

$$(I_1(\mathbf{y}_1) + \dots + I_r(\mathbf{y}_r) + \langle x_i - y_{1i} - y_{2i} - \dots - y_{ri} \mid i = 0, \dots, N \rangle) \cap \mathbb{K}[\mathbf{x}]$$

and their Hadamard product  $I_1 \star I_2 \star \cdots \star I_r$ , as defined in [8], is the elimination ideal

$$(I_1(\mathbf{y}_1) + \cdots + I_r(\mathbf{y}_r) + \langle x_i - y_{1i}y_{2i} \cdots y_{ri} \mid i = 0, \dots, N \rangle) \bigcap \mathbb{K}[\mathbf{x}].$$

We define the  $r$ -th secant of an ideal  $I \subset \mathbb{K}[\mathbf{x}]$  to be the  $r$ -fold join  $I$  with itself:

$$I^{*r} := I * I * \cdots * I.$$

Similarly we define the  $r$ -th Hadamard power of an ideal  $I \subset \mathbb{K}[\mathbf{x}]$  to be the  $r$ -fold Hadamard product of  $I$  with itself:

$$I^{\star r} := I \star I \star \cdots \star I.$$

*Remark 2.1.* The Hadamard product of points in a projective space is a coordinate-wise product as in the case of the Hadamard product of matrices. Let  $p, q \in \mathbb{P}^N$  be two points with coordinates  $[p_0 : p_1 : \cdots : p_N]$  and  $[q_0 : q_1 : \cdots : q_N]$  respectively. If  $p_i q_i \neq 0$  for some  $i$ , the Hadamard product  $p \star q$  of  $p$  and  $q$ , is defined as  $p \star q = [p_0 q_0 : p_1 q_1 : \cdots : p_N q_N]$ . If  $p_i q_i = 0$  for all  $i = 0, \dots, n$  then we say  $p \star q$  is not defined. This definition extends to the Hadamard product of varieties in the following way. Let  $X$  and  $Y$  be two varieties in  $\mathbb{P}^N$ . Then the Hadamard product  $X \star Y$  is defined as

$$X \star Y = \overline{\{p \star q : p \in X, q \in Y, p \star q \text{ is defined}\}}.$$

Thus the defining ideal of the Hadamard product  $X \star Y$  of two varieties  $X$  and  $Y$ , that is, the ideal  $I(X \star Y)$ , equals the Hadamard product of the ideals  $I(X) \star I(Y)$  (see [8, Remark 2.6]).

**Definition 2.2.** If  $I$  is a homogeneous ideal of  $S$ , the  $m$ -th symbolic power of  $I$  is defined as

$$I^{(m)} = \bigcap_{p \in \text{Ass}(I)} (I^m S_p \cap S)$$

where  $\text{Ass}(I)$  denotes the set of the associated primes of  $I$ . If  $I$  is a radical ideal then  $I^{(m)} = \bigcap_{p \in \text{Ass}(I)} p^m$ .

In this paper we will always deal with ideals of fat points. Given distinct points  $p_1, \dots, p_s \in \mathbb{P}^N$  and non-negative integers  $m_i$  (not all 0), let  $Z = m_1 p_1 + \cdots + m_s p_s$  denote the scheme (called a set of fat points) defined by the ideal  $I_Z = \bigcap_{i=1}^s I(P_i)^{m_i}$ , where  $I(P_i)$  is the ideal generated by all homogeneous polynomials vanishing at  $P_i$ . When all the  $m_i$  are equal, we say that  $Z$  is a homogeneous set of fat points. For ideals of this type, the  $m$ -th symbolic power can be simply defined as  $I_Z^{(m)} = I_{mZ} = \bigcap_{i=1}^s I(P_i)^{m m_i}$ .

If  $I^m$  is the regular power of an ideal  $I$ , then there is clearly a containment  $I^m \subseteq I^{(m)}$  and indeed, for  $0 \neq I \subsetneq R$ ,  $I^r \subseteq I^{(m)}$  holds if and only if  $r \geq m$  [2, Lemma 8.1.4]. A much more difficult problem is to determine when there are containments of the form  $I^{(m)} \subseteq I^r$ . The results of [12] and [20] show that  $I^{(m)} \subseteq I^r$  holds whenever  $m \geq Nr$ , where  $N$  is the codimension of the ideal.

As a stepping stone, the second author and Harbourne [4] introduce an asymptotic quantity which we refer to as the *resurgence*, namely

$$\rho(I) = \sup\{m/r : I^{(m)} \not\subseteq I^r\}.$$

In particular, if  $m > \rho(I)r$ , then one is guaranteed that  $I^{(m)} \subseteq I^r$ . There are still, however, very few cases for which the actual value of  $\rho(I)$  is known, and they are almost all cases for which  $\rho(I) = 1$ . For example, by Macaulay’s unmixedness theorem it follows that  $\rho(I) = 1$  when  $I$  is a complete intersection (see Proposition 2.4). And if  $I$  is a monomial ideal, it is sometimes possible to compute  $\rho(I)$  directly. In this paper we will show that if  $I$  is the defining ideal of a Hadamard fat grid,  $\rho(I) = 1$  even if  $I$  is a not complete intersection (see Proposition 5.18 and Corollary 5.19).

We recall some results which will be useful for the rest of the paper.

**Proposition 2.3.** *Let  $I$  be a radical ideal in a polynomial ring over an algebraically closed field. Then  $I^{(t)} = I * m^t$ .*

*Proof.* See [23, Proposition 2.8]. □

**Proposition 2.4.** *Let  $I$  be a complete intersection ideal in a polynomial ring over an algebraically closed field. Then  $I^{(t)} = I^t$  for all  $t \geq 1$ .*

*Proof.* See [24, Lemma 5 and Theorem 2 of Appendix 6]. □

**Lemma 2.5.** *The Hadamard product distributes over intersections:*

$$\left( \bigcap_{l \in \mathcal{L}} J_l \right) * K = \bigcap_{l \in \mathcal{L}} (J_l * K).$$

*Proof.* The proof is analogous to [23, Lemma 2.6]. A polynomial  $f$  belongs to  $(\cap J_l) * K$  if and only if  $f(\mathbf{y}_1 \mathbf{y}_2) \in (\cap J_l)(\mathbf{y}_1) + K(\mathbf{y}_2)$  if and only if  $f(\mathbf{y}_1 \mathbf{y}_2) \in J_l(\mathbf{y}_1) + K(\mathbf{y}_2)$  for all  $l \in \mathcal{L}$  if and only if  $f \in \cap (J_l * K)$ . □

From the previous lemma we get the following

**Corollary 2.6.** *Let  $I, J$  be two ideals in  $\mathbb{K}[x_0, \dots, x_N]$  with primary decomposition respectively  $I = I_1 \cap I_2 \cap \dots \cap I_s$  and  $J = J_1 \cap J_2 \cap \dots \cap J_t$ , then*

$$I * J = \bigcap_{\substack{1 \leq i \leq s \\ 1 \leq j \leq t}} I_i * J_j. \tag{1}$$

*Remark 2.7.* The right-hand term in (1) is not in general a minimal primary decomposition of  $I * J$  since it can contain some redundant term. As a matter of fact consider, for example, the ideals in  $\mathbb{K}[x_0, x_1, x_2]$

$$\begin{aligned} I &= \langle 5x_0 - x_1 - x_2, 6x_1^2 - 13x_1x_2 + 6x_2^2 \rangle \\ J &= \langle 5x_0 - 4x_1 - 3x_2, 16x_1^2 - 26x_1x_2 + 9x_2^2 \rangle \end{aligned}$$

with primary decompositions

$$\begin{aligned} I &= I_1 \cap I_2 = \langle 3x_1 - 2x_2, x_0 - \frac{1}{5}x_1 - \frac{1}{5}x_2 \rangle \cap \langle 2x_1 - 3x_2, x_0 - \frac{1}{5}x_1 - \frac{1}{5}x_2 \rangle \\ J &= J_1 \cap J_2 = \langle 2x_1 - x_2, x_0 - \frac{4}{5}x_1 - \frac{3}{5}x_2 \rangle \cap \langle 8x_1 - 9x_2, x_0 - \frac{4}{5}x_1 - \frac{3}{5}x_2 \rangle. \end{aligned}$$

The primary decomposition of  $I \star J$  is

$$\langle 16x_1 - 27x_2, 4x_0 - 3x_2 \rangle \cap \langle 4x_1 - 3x_2, 2x_0 - x_2 \rangle \cap \langle 3x_1 - x_2, 3x_0 - x_2 \rangle$$

which does not have four ideals as expected. This is due to the fact that  $I_2 \star J_1 = I_1 \star J_2$ .

**Lemma 2.8.** *One has  $\mathfrak{m}^m \star \mathfrak{m}^n = \mathfrak{m}^{m+n-1}$ .*

*Proof.* Let  $\mathbf{u} = (u_0, \dots, u_n)$  an integer vector and denote by

$$\mathfrak{m}^{\mathbf{u}} = \langle x_i^{u_i} : u_i > 0 \rangle.$$

We decompose  $\mathfrak{m}^m = \cap_{\mathbf{u}} \mathfrak{m}^{\mathbf{u}}$  and  $\mathfrak{m}^n = \cap_{\mathbf{v}} \mathfrak{m}^{\mathbf{v}}$ . Then the proof follows from [22, Lemma 2.3] and [23, Lemma 2.6]. □

**Definition 2.9.** Let  $H_i \subset \mathbb{P}^n, i = 0, \dots, n$ , be the hyperplane  $x_i = 0$  and set

$$\Delta_i = \bigcup_{0 \leq j_1 < \dots < j_{n-i} \leq n} H_{j_1} \cap \dots \cap H_{j_{n-i}}.$$

In other words,  $\Delta_i$  is the  $i$ -dimensional variety of points having *at most*  $i + 1$  non-zero coordinates. Thus  $\Delta_0$  is the set of coordinates points and  $\Delta_{n-1}$  is the union of the coordinate hyperplanes. Note that elements of  $\Delta_i$  have *at least*  $n - i$  zero coordinates. We have the following chain of inclusions:

$$\Delta_0 = \{[1 : 0 : \dots : 0], \dots, [0 : \dots : 0 : 1]\} \subset \Delta_1 \subset \dots \subset \Delta_{n-1} \subset \Delta_n = \mathbb{P}^n. \tag{2}$$

Let  $R$  be the ring  $\mathbb{K}[x_0, x_1, \dots, x_n]$ . Given a vector of non-negative integers  $I = (i_0, \dots, i_n)$ , we denote by  $X^I$  the monomial  $x_0^{i_0} x_1^{i_1} \dots x_n^{i_n}$  and by  $|I| = i_0 + \dots + i_n$ . Similarly, if  $P$  is a point of  $\mathbb{P}^n$  with coordinates  $[p_0 : p_1 : \dots : p_n]$ , we denote by  $P^I$  the monomial  $X^I$  evaluated in  $P$ , that is  $p_0^{i_0} p_1^{i_1} \dots p_n^{i_n}$ . Moreover, if  $P$  is a point of  $\mathbb{P}^n \setminus \Delta_{n-1}$  with coordinates  $[p_0 : p_1 : \dots : p_n]$ , we denote by  $\frac{1}{P}$  the point with coordinates  $[\frac{1}{p_0} : \frac{1}{p_1} : \dots : \frac{1}{p_n}]$ .

**Definition 2.10.** Let  $f \in \mathbb{K}[x_0, x_1, \dots, x_n]$  be a homogenous polynomial, of degree  $d$ , of the form  $f = \sum_{|I|=d} a_I X^I$  and consider a point  $P \in \mathbb{P}^n \setminus \Delta_{n-1}$ . The Hadamard transformation of  $f$  by  $P$  is the polynomial

$$f^{\star P} = \sum_{|I|=d} \frac{a_I}{P^I} X^I. \tag{3}$$

**Theorem 2.11.** *Let  $I$  be an ideal in  $\mathbb{K}[x_0, \dots, x_n]$  and consider a point  $P \in \mathbb{P}^n \setminus \Delta_{n-1}$ . If  $f_1, \dots, f_s \in \mathbb{K}[x_0, \dots, x_n]$  is a generating set for  $I$ , that is  $I = \langle f_1, \dots, f_s \rangle$ , then  $f_1^{\star P}, \dots, f_s^{\star P}$  is a generating set for  $I(P) \star I$ .*

*Moreover, if  $f_1, \dots, f_s$  is a Gröbner basis for  $I$ , then  $f_1^{\star P}, \dots, f_s^{\star P}$  is a Gröbner basis for  $I(P) \star I$ .*

*Proof.* See [7, Theorem 3.5]. □

### 3. Preliminary Results on ACM Sets of Fat Points in $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we recall some known results and a standard technique used for sets of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  since they are the main tools to find a minimal graded free resolution of special sets of fat points in  $\mathbb{P}^2$  called Hadamard fat grids (see Definition 5.1 in Sect. 5). Indeed, we prove that a Hadamard fat grid inherits some properties from an arithmetically Cohen–Macaulay set of a fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , such as that its defining ideal is generated by product of linear forms. We should refer the reader to Chapters 4 and 6 in [17] for more details on arithmetically Cohen–Macaulay sets of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$ .

Let  $Z$  be a finite set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ , and let  $R/I_Z$  denote the associated  $\mathbb{N}^2$ -graded coordinate ring. When  $R/I_Z$  is Cohen–Macaulay, that is depth  $R/I_Z = \dim R/I_Z = 2$ , then  $Z$  is called an arithmetically Cohen–Macaulay (ACM for short) set of points.

*Remark 3.1.* For the ease of the reader, we now recall a standard argument that relates sets of (fat) points in  $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$  ( $n$ -times) and sets of (fat) points in  $\mathbb{P}^{2n-1}$  in the case  $n = 2$  (see for more details Section 3, Theorem 3.2, Corollaries 3.3 and 3.4 in [13] and, for instance, also [14–16]). We observe that in our case in  $\mathbb{P}^1$  a point is also a hyperplane, and this allows us to use hyperplane sections and related constructions for our study.

Let  $R = \mathbb{K}[x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}] = \mathbb{K}[\mathbb{P}^1 \times \mathbb{P}^1]$  be the coordinate ring for  $\mathbb{P}^1 \times \mathbb{P}^1$ , which we shall also view as the coordinate ring for  $\mathbb{P}^3$ . Let  $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$  be a finite set of fat points. Since  $I_Z$  defines both a set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  and a union of linear varieties (fat lines) in  $\mathbb{P}^3$ , by abuse of notation, we denote by  $Z$  also the subvariety of  $\mathbb{P}^3$  defined by this ideal.

In  $\mathbb{P}^1 \times \mathbb{P}^1$  a complete intersection of type  $(r, 0)$  and  $(0, s)$ , or simply an  $(r, s)$ -grid, can be viewed as 2 families of linear forms, and we denote by  $A_{1,i}$  the linear combinations of  $x_{1,0}$  and  $x_{1,1}$ , and by  $A_{2,i}$  the linear combinations of  $x_{2,0}$  and  $x_{2,1}$ . Set

$$\begin{aligned} |\{A_{1,i} \mid A_{1,i} \cap X \neq \emptyset\}| &= r \\ |\{A_{2,i} \mid A_{2,i} \cap X \neq \emptyset\}| &= s. \end{aligned}$$

Let  $T$  be the polynomial ring in  $r+s$  variables  $a_{1,1}, \dots, a_{1,r}, a_{2,1}, \dots, a_{2,s}$ . We form the monomial ideal in  $T$  given by the intersection of ideals of the form  $(a_{1,i}, a_{2,j})^{m_{ij}}$  corresponding to the components of  $X$ . This intersection defines a height 2 monomial ideal,  $J \subset T$ . Following Theorem 2 in [13], a set of fat points  $Z$  in  $\mathbb{P}^1 \times \mathbb{P}^1$  is ACM if and only if  $J \subset T$  is CM, where  $T$  is the ring previously defined. Since  $Z$  can be viewed as an ACM set of lines in  $\mathbb{P}^3$  (it is 1-dimensional) we can still find a suitable hyperplane section in order to get a set  $X$  of points in  $\mathbb{P}^2$  that shares the same Betti numbers as  $Z$ . In particular, with the above described method, we will show that a Hadamard fat grid in  $\mathbb{P}^2$  share the same graded Betti numbers as an ACM set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  (see Theorem 5.6).

In the sequel, it useful to consider in  $\mathbb{Z} \times \mathbb{Z}$ , resp. in  $\mathbb{N} \times \cdots \times \mathbb{N}$ , the partial (lexicographic) ordering induced by the usual one in  $\mathbb{Z}$ , resp. in  $\mathbb{N}$ ; we will denote it by “ $\leq$ ”.

Consider an  $(r, s)$ -grid, and let  $M = \{m_1, \dots, m_r\}$ ,  $N = \{n_1, \dots, n_s\}$  be two sets of non-negative integers with  $m_1 \leq m_2 \leq \dots \leq m_r$  and  $n_1 \leq n_2 \leq \dots \leq n_s$ . We have

$$\begin{aligned} \max_{i,j} \{m_i + n_j - 1\} &= m_r + n_s - 1, \\ \max_i \{m_i + n_j - 1\} &= m_r + n_j - 1, \\ \max_j \{m_i + n_j - 1\} &= m_i + n_s - 1. \end{aligned}$$

For each  $i = 1, \dots, r$  and  $j = 1, \dots, s$ , set

$$t_{ij}(h) = (m_i + n_j - 1 - h)_+ = \max\{0, m_i + n_j - 1 - h\} \text{ for } h \in \mathbb{N}^0.$$

For  $i = 1 \dots, r$ , let  $\mathcal{S}$  be the set of the  $s$ -tuples of type

$$\mathcal{S} = \{(t_{i1}(h), t_{i2}(h), \dots, t_{is}(h))\}. \tag{4}$$

The next Lemma 3.2 shows that the set  $\mathcal{S}$  of tuples associated to  $M$  and  $N$  is totally ordered. It will be useful to find the graded Betti numbers of a Hadamard fat grid (see Sect. 5).

We have

**Lemma 3.2.** *For  $i = 1 \dots, r$  and for  $h = 0 \dots, m_1 + n_s - 2$ , the set  $\mathcal{S}$  is a totally ordered set of tuples with usual lex ordering.*

*Proof.* Let  $r = 1$ . From construction, it is  $\max_j \{m_1 + n_j - 1\} = m_1 + n_s - 1$ . For  $h = 0 \dots, m_1 + n_s - 2$ , we have

$$\begin{aligned} \mathcal{S} &= \{(t_{11}(h), t_{12}(h), \dots, t_{1s}(h))\} \\ &= \{((m_1 + n_1 - 1 - h)_+, (m_1 + n_2 - 1 - h)_+, \dots, (m_1 + n_s - 1 - h)_+, )\} \\ &= \{(m_1 + n_1 - 1, m_1 + n_2 - 1, \dots, m_1 + n_s - 1), \\ &\quad (m_1 + n_1 - 2, m_1 + n_2 - 2, \dots, m_1 + n_s - 2), \\ &\quad \dots, (m_1 + n_1 - 1 - m_1 - n_s + 2, \dots, m_1 + n_s - 1 - m_1 - n_s + 2)\} \\ &= \{(m_1 + n_1 - 1, m_1 + n_2 - 1, \dots, m_1 + n_s - 1), \\ &\quad (m_1 + n_1 - 2, m_1 + n_2 - 2, \dots, m_1 + n_s - 2), \\ &\quad \dots, ((n_1 - n_s + 1)_+, (n_2 - n_s + 1)_+, \dots, (1))\}. \end{aligned}$$

Since  $n_1 \leq n_2 \leq \dots \leq n_s$ , it is  $m_1 + n_{j_1} - 1 \leq m_1 + n_{j_2} - 1$  for all  $j_1 \neq j_2$ , and since  $n_1 \leq n_s$  we have  $n_1 - n_s + 1 \leq 0$  if  $n_1 < n_s$  and  $n_1 - n_s + 1 = 1$  if  $n_1 = n_s$ . Hence  $\mathcal{S}$  is totally ordered for  $r = 1$ .

Suppose  $r > 1$  and the lemma true for all the sets  $\mathcal{S}'$  of  $s$ -tuples with  $r' < r$  and prove it for  $\mathcal{S}$ . Let  $1 \leq k \leq r$  an integer such that  $m_k + n_{j_1} - 1 \leq m_k + n_{j_2} - 1$  for all  $j_1 \neq j_2$ . From our construction, we have that  $k = r$ .

Define for  $i = 1 \dots, r$

$$\mathcal{S}' = \{(t'_{i1}(h), t'_{i2}(h), \dots, t'_{is}(h))\}$$

where

$$t'_{ij}(h) = \begin{cases} (m_i + n_j - 1 - h)_+ & \text{if } i \neq k \\ (m_k + n_j - 2 - h)_+ & \text{if } i = k. \end{cases}$$

By induction  $\mathcal{S}'$  is totally ordered. We observe that

$$\mathcal{S} = \mathcal{S}' \cup \{((m_k + n_1 - 1 - h)_+, (m_k + n_2 - 1 - h)_+, \dots, (m_k + n_s - 1 - h)_+)\}$$

and by construction and for all  $i = 1 \dots, r = k$ , we have

$$((m_i + n_1 - 1 - h)_+, \dots, (m_i + n_s - 1 - h)_+) \leq ((m_k + n_1 - 1 - h)_+, \dots, (m_k + n_s - 1 - h)_+).$$

Thus the set  $\mathcal{S}$  is totally ordered. □

For each integer  $0 \leq h \leq m_i + n_s - 2$ , with  $i = 1, \dots, r$  we define

$$a_{i,h} := \sum_{e=1}^s (m_i + n_e - 1 - h)_+ \quad \text{where } (n)_+ := \max\{n, 0\}.$$

Define

$$\tilde{\mathcal{A}}_Z := (a_{1,0}, \dots, a_{1,m_1+n_s-2}, a_{2,0}, \dots, a_{2,m_2+n_s-2}, \dots, a_{r,0}, \dots, a_{r,m_r+n_s-2}).$$

Finally, we define  $\mathcal{A}_Z$  to be the set of  $(m_1 + n_s - 1 + \dots + m_r + n_s - 1)$ -tuples, that is, the set of  $(\sum_{i=1}^r m_i + rn_s - r)$ -tuples that one gets by rearranging the elements of  $\tilde{\mathcal{A}}_Z$  in decreasing order.

We now recall how to compute the graded Betti numbers of  $I(Z)$  when  $Z$  is an ACM set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\mathcal{A}_Z = (\alpha_1, \dots, \alpha_m)$  be the tuple associated to  $Z$ . Define the following two sets from  $\mathcal{A}_Z$ :

$$C_Z := \{(m, 0), (0, \alpha_1)\} \cup \{(i - 1, \alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\}$$

$$V_Z := \{(m, \alpha_m)\} \cup \{(i - 1, \alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}$$

where we take  $\alpha_{-1} = 0$ . With this notation, we get that a minimal bigraded free resolution of an ACM set of points in  $\mathbb{P}^1 \times \mathbb{P}^1$  is given by

**Theorem 3.3.** [17, Theorem 6.27] *Suppose that  $Z$  is an ACM set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  with  $\mathcal{A}_Z = (\alpha_1, \dots, \alpha_m)$ . Let  $C_Z$  and  $V_Z$  be constructed from  $\mathcal{A}_Z$  as above. Then a bigraded minimal free resolution of  $I(Z)$  is given by*

$$0 \longrightarrow \bigoplus_{(v_1, v_2) \in V_Z} R(-v_1, -v_2) \longrightarrow \bigoplus_{(c_1, c_2) \in C_Z} R(-c_1, -c_2) \longrightarrow I(Z) \longrightarrow 0.$$

The following result will be one of the main tool that allows us to find that the Betti numbers of a given Hadamard fat grid in  $\mathbb{P}^2$  whose support is a complete intersection.

**Proposition 3.4.** *Let  $Z$  be a set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  whose support is a complete intersection of type  $(r, s)$  (or  $(r, s)$ -grid) and whose multiplicities are of type  $m_i + n_j - 1$  for  $i = 1, \dots, r$  and  $j = 1, \dots, s$ . Then  $Z$  is ACM.*

*Proof.* From Theorem 6.21 in [17] and Lemma 3.2, we get the conclusion. □



### 4. Hadamard Product of Symbolic Powers of Ideals of Points

We now focus our attention on the Hadamard product  $I(P)^m \star I(Q)^n$ , where  $P, Q \in \mathbb{P}^2$ .

Hence, we work on the polynomial ring  $S = \mathbb{K}[\mathbf{x}] = \mathbb{K}[x_0, x_1, x_2]$ , over an algebraically closed field and we still denote by  $\mathfrak{m} = \langle x_0, x_1, x_2 \rangle$  the irrelevant ideal. Since we will only consider two ideals at the time we slightly abuse our notation with respect to our original definition. More precisely we will use the extra variables  $\mathbf{y} = (y_0, y_1, y_2)$  and  $\mathbf{z} = (z_0, z_1, z_2)$  for the Hadamard product of the ideals  $I$  and  $J$ :

$$I \star J = (I(\mathbf{y}) + J(\mathbf{z}) + \langle x_i - y_i z_i \mid i = 0, 1, 2 \rangle) \bigcap \mathbb{K}[\mathbf{x}].$$

We set  $\mathcal{H} = (x_0 - y_0 z_0, x_1 - y_1 z_1, x_2 - y_2 z_2)$ .

We start with some preliminary results.

**Lemma 4.1.** *Let  $P = [p_0 : p_1 : p_2]$  be a point, then*

- (i) *if  $P \in \mathbb{P}^2 \setminus \Delta_1$ , then  $I(P) \star \mathfrak{m}^t = \mathfrak{m}^t$ ,*
- (ii) *if  $P \in \Delta_1$  then  $\mathfrak{m}^t \subseteq I(P) \star \mathfrak{m}^t$ .*

*Proof.* For i) we can use Theorem 2.11. Hence the generators of  $I(P) \star \mathfrak{m}^t$  are the Hadamard transformations, with respect to  $P$ , of the monomials in  $\mathfrak{m}^t$ . According to Definition 2.10, such transformations of the monomials in  $\mathfrak{m}^t$  are the same monomials scaled by a constant  $P^I$ , hence  $I(P) \star \mathfrak{m}^t = \mathfrak{m}^t$ .

For ii) we can assume, without loss of generality, that  $p_0 = 0$

$$\begin{aligned} I(P) \star \mathfrak{m}^t &= (I(P)(\mathbf{y}) + \mathfrak{m}(\mathbf{z})^n + \langle x_i - y_i z_i \mid i = 0, 1, 2 \rangle) \bigcap \mathbb{K}[\mathbf{x}] \\ &= \left( (y_0, p_2 y_1 - p_1 y_2) + (z_0, z_1, z_2)^t + \langle x_i - y_i z_i \mid i = 0, 1, 2 \rangle \right) \bigcap \mathbb{K}[\mathbf{x}] \\ &= (x_0) + (x_1, x_2)^t \end{aligned}$$

where the last equality follows from the fact that  $y_0$  annihilates  $y_0 z_0$ . Hence  $\mathfrak{m}^t \subseteq I(P) \star \mathfrak{m}^t$ . Similarly, if the point  $P$  has  $p_0 = p_1 = 0$  one has

$$\begin{aligned} I(P) \star \mathfrak{m}^t &= (I(P)(\mathbf{y}) + \mathfrak{m}(\mathbf{z})^n + \langle x_i - y_i z_i \mid i = 0, 1, 2 \rangle) \bigcap \mathbb{K}[\mathbf{x}] \\ &= \left( (y_0, y_1) + (z_0, z_1, z_2)^t + \langle x_i - y_i z_i \mid i = 0, 1, 2 \rangle \right) \bigcap \mathbb{K}[\mathbf{x}] \\ &= (x_0, x_1) + (x_2^t) \supset \mathfrak{m}^t. \end{aligned}$$

□

The following result gives a positive answers to Question 1.1 when the points  $P$  and  $Q$  have non-zero coordinates.

**Theorem 4.2.** *Let  $P$  and  $Q$  be two points in  $\mathbb{P}^2 \setminus \Delta_1$ . Then for  $m, n \geq 1$  one has  $I(P)^m \star I(Q)^n = I(P \star Q)^{m+n-1}$ .*

*Proof.* From Proposition 2.3 we know that

$$I(P)^m = I(P) \star \mathfrak{m}^m \text{ and } I(Q)^n = I(Q) \star \mathfrak{m}^n. \tag{5}$$

Thus,  $I(P)^m \star I(Q)^n = [I(P) \star \mathfrak{m}^m] \star [I(Q) \star \mathfrak{m}^n]$  can be computed by applying successively the definition of join of two ideals and of Hadamard products of ideals.

For this aim, we define

$$\begin{aligned} \mathcal{H}_x &= (x_0 - x'_0x''_0, x_1 - x'_1x''_1, x_2 - x'_2x''_2), \\ \mathcal{H}_y &= (x'_0 - y'_0 - y''_0, x'_1 - y'_1 - y''_1, x'_2 - y'_2 - y''_2), \\ \mathcal{H}_z &= (x''_0 - z'_0 - z''_0, x''_1 - z'_1 - z''_1, x''_2 - z'_2 - z''_2) \\ \hat{\mathcal{H}} &= (x_0 - (y'_0 + y''_0)(z'_0 + z''_0), x_1 - (y'_1 + y''_1)(z'_1 + z''_1), x_2 \\ &\quad - (y'_2 + y''_2)(z'_2 + z''_2)). \end{aligned}$$

Hence we have the following sequence of equalities

$$\begin{aligned} I(P)^m \star I(Q)^n &= [I(P) \star \mathfrak{m}^m] \star [I(Q) \star \mathfrak{m}^n] \\ &= ([I(P) \star \mathfrak{m}^m](\mathbf{x}') + [I(Q) \star \mathfrak{m}^n](\mathbf{x}'') + \mathcal{H}_x) \bigcap \mathbb{K}[\mathbf{x}] \\ &= \left( [I(P)(\mathbf{y}') + \mathfrak{m}^m(\mathbf{y}'') + \mathcal{H}_y] \bigcap \mathbb{K}[\mathbf{x}'] \right) \\ &\quad + \left( [I(Q)(\mathbf{z}') + \mathfrak{m}^n(\mathbf{z}'') + \mathcal{H}_z] \bigcap \mathbb{K}[\mathbf{x}''] \right) + \mathcal{H} \bigcap \mathbb{K}[\mathbf{x}] \\ &= \left( I(P)(\mathbf{y}') + \mathfrak{m}^m(\mathbf{y}'') + I(Q)(\mathbf{z}') + \mathfrak{m}^n(\mathbf{z}'') + \hat{\mathcal{H}} \right) \bigcap \mathbb{K}[\mathbf{x}] \\ &= \left( I(P)(\mathbf{y}') + I(Q)(\mathbf{z}') + \mathfrak{m}^m(\mathbf{y}'') + \mathfrak{m}^n(\mathbf{z}'') + \hat{\mathcal{H}} \right) \bigcap \mathbb{K}[\mathbf{x}]. \end{aligned}$$

Denote by  $\tilde{I}$  the ideal in the last equality in the previous formula, i.e.

$$\tilde{I} = I(P)(\mathbf{y}') + I(Q)(\mathbf{z}') + \mathfrak{m}^m(\mathbf{y}'') + \mathfrak{m}^n(\mathbf{z}'') + \hat{\mathcal{H}}.$$

Since the elements in  $\hat{\mathcal{H}}$  can be seen as

$$x_i = (y'_i + y''_i)(z'_i + z''_i) = y'_iz'_i + y'_iz''_i + z'_iy''_i + y''_iz''_i \tag{6}$$

we can make the following substitution of variables in the ideal  $\hat{\mathcal{H}}$

$$\begin{aligned} k_i^{(1)} &= y'_iz'_i, \\ k_i^{(2)} &= y'_iz''_i, \\ k_i^{(3)} &= z'_iy''_i, \\ k_i^{(4)} &= y''_iz''_i, \\ x_i &= k_i^{(1)} + k_i^{(2)} + k_i^{(3)} + k_i^{(4)} \end{aligned}$$

with  $i = 0, 1, 2$ .

By definition,  $k_i^{(1)} = y'_iz'_i$  are involved in  $I(P) \star I(Q) = I(P \star Q)$ . Note that  $k_i^{(2)} = y'_iz''_i$  are involved in  $I(P) \star \mathfrak{m}^n$ . By Lemma 4.1 (i), we know that  $I(P) \star \mathfrak{m}^n = \mathfrak{m}^n$ . Therefore we can write  $\tilde{k}_i^{(2)} = z''_i$ . Similarly  $k_i^{(3)} = z'_iy''_i$  are involved  $I(Q) \star \mathfrak{m}^m$ . Using the same lemma we have that  $I(Q) \star \mathfrak{m}^m = \mathfrak{m}^m$ , hence  $\tilde{k}_i^{(3)} = y''_i$ . Now we have

$$x_i = k_i^{(1)} + \tilde{k}_i^{(2)} + \tilde{k}_i^{(3)} + k_i^{(4)}. \tag{7}$$

Let  $f$  be a form of degree  $d$  in  $\tilde{I}$ . If  $\prod \left( \tilde{k}_i^{(2)} \right)^{r_i} \in \mathfrak{m}^n(\mathbf{z}'')$  then any term of  $f$  containing  $\prod \left( \tilde{k}_i^{(2)} \right)^{r_i}$  will be canceled. Hence, any term of  $f$  containing  $\prod \left( y_i''^{r_i} z_i''^{t_i} \right)$  will be canceled as well. Similarly, for any term of  $f$  containing

$\prod \left(\tilde{k}_i^{(3)}\right)^{t_i} \in \mathfrak{m}^m(\mathbf{y}'')$  it follows that any term of  $f$  containing  $\prod \left(y_i^{r_i} z_i^{t_i}\right)$  will be annihilated. Therefore, we conclude that any term of a form  $f$  of degree  $d$  in  $\tilde{I}$  containing  $\prod \left(k_i^{(4)}\right)^{d_i}$  is zero. Hence, we can cancel  $k_i^{(4)}$  from (7). We have that  $x_i = k_i^{(1)} + \tilde{k}_i^{(2)} + \tilde{k}_i^{(3)}$ . Let

$$\begin{aligned} \tilde{\mathcal{H}} &= \left(x_0 - k_0^{(1)} - \tilde{k}_0^{(2)} - \tilde{k}_0^{(3)}, x_1 - k_1^{(1)} - \tilde{k}_1^{(2)} - \tilde{k}_1^{(3)}, x_2 - k_2^{(1)} - \tilde{k}_2^{(2)} - \tilde{k}_2^{(3)}\right) \\ \tilde{\mathcal{H}}_{x'} &= \left(x'_0 - k_0^{(1)}, x'_1 - k_1^{(1)}, x'_2 - k_2^{(1)}\right) \\ \tilde{\mathcal{H}}_{x''} &= \left(x''_0 - \tilde{k}_0^{(2)} - \tilde{k}_0^{(3)}, x''_1 - \tilde{k}_1^{(2)} - \tilde{k}_1^{(3)}, x''_2 - \tilde{k}_2^{(2)} - \tilde{k}_2^{(3)}\right) \\ \tilde{\mathcal{H}}_x &= \left(x_0 - x''_0 - x'_0, x_1 - x''_1 - x'_1, x_2 - x''_2 - x'_2\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{I} \cap \mathbb{K}[\mathbf{x}] &= \left(I(P)(\mathbf{y}') + I(Q)(\mathbf{z}') + \mathfrak{m}^m(\mathbf{y}'') + \mathfrak{m}^n(\mathbf{z}'') + \tilde{\mathcal{H}}\right) \cap \mathbb{K}[\mathbf{x}] \\ &= \left(\left[I(P)(\mathbf{y}') + I(Q)(\mathbf{z}') + \tilde{\mathcal{H}}_{x'}\right] \cap \mathbb{K}[\mathbf{x}']\right) \\ &\quad + \left[\left(\mathfrak{m}^m(\mathbf{y}'') + \mathfrak{m}^n(\mathbf{z}'') + \tilde{\mathcal{H}}_{x''}\right) \cap \mathbb{K}[\mathbf{x}'']\right] + \tilde{\mathcal{H}} \cap \mathbb{K}[\mathbf{x}] \\ &= \left([I(P) \star I(Q)](\mathbf{x}') + [\mathfrak{m}^m \star \mathfrak{m}^n](\mathbf{x}'') + \tilde{\mathcal{H}}_x\right) \cap \mathbb{K}[\mathbf{x}] \\ &= [I(P) \star I(Q)] \star \mathfrak{m}^{m+n-1} \\ &= I(P \star Q) \star \mathfrak{m}^{m+n-1} \end{aligned}$$

and, by Proposition 2.3, the last ideal is equal to  $I(P \star Q)^{m+n-1}$  and the theorem is proved.  $\square$

*Remark 4.3.* If we remove the condition  $P, Q \in \mathbb{P}^2 \setminus \Delta_1$  in Theorem 4.2 we are able to prove only the following inclusion

$$I(P)^m \star I(Q)^n \supset I(P \star Q)^{m+n-1}.$$

To prove this inequality is enough to apply the proof of Theorem 4.2 using part ii) of Lemma 4.1.

We observe that if  $P \star Q$  is defined, and  $m = n = 1$  then it follows from the definition of Hadamard product that  $I(P) \star I(Q) = I(P \star Q)$ , and hence Question 1.1 has an affirmative answer. Theorem 4.2 shows that Question 1.1 has an affirmative answer, when we consider points which are not in the coordinates lines. When one of the point is taken in a coordinate line, the formula in Question 1.1 is no more valid, as stated in the following:

**Proposition 4.4.** *Let  $P$  and  $Q$  be two points in  $\mathbb{P}^2$ .*

- (a) *If  $P \in \mathbb{P}^2 \setminus \Delta_1$  and  $P \star Q$  is defined then for  $m = 1$  and  $n \geq 1$  Question 1.1 has an affirmative answer.*
- (b) *If  $P, Q \in \Delta_1 \setminus \Delta_0$  then Question 1.1 has no affirmative answer but for  $m = n = 1$ .*

*Proof.* (a) Without loss of generality let  $P = [p_0 : p_1 : p_2]$  with  $p_i \neq 0$  and assume that  $Q \in \Delta_0$  has exactly one non-zero coordinate, that is  $Q = [q_0 :$

$0 : 0]$ . Assume that  $m = 1$ . Since  $P \star Q = Q$ , therefore we only need to show that  $I(P) \star I(Q)^n = I(Q)^n$ . We have that

$$\begin{aligned} I(P) \star I(Q)^n &= (p_2x_1 - p_1x_2, p_2x_0 - p_0x_2) \star (x_1, x_2)^n \\ &= ((p_2y_1 - p_1y_2, p_2y_0 - p_2y_2) + (z_1, z_2)^n + \mathcal{H}) \bigcap \mathbb{K}[\mathbf{x}] \\ &= (x_1, x_2)^n = I(Q)^n. \end{aligned}$$

Let  $m > 1$ . Since  $x_1^n \notin I(Q)^{n+m-1}$  and we know that  $x_1^n \in I(P)^m \star I(Q)^n$  therefore  $I(Q)^{n+m-1} \neq I(P)^m \star I(Q)^n$ .

Note that in general we have

$$\begin{aligned} I(P)^m \star I(Q)^n &= (p_2x_1 - p_1x_2, p_2x_0 - p_0x_2)^m \star (x_1, x_2)^n \\ &= ((p_2y_1 - p_1y_2, p_2y_0 - p_2y_2)^m + (z_1, z_2)^n + \mathcal{H}) \bigcap \mathbb{K}[\mathbf{x}] \\ &= (x_1, x_2)^n = I(Q)^n = I(P \star Q)^n. \end{aligned}$$

Now let  $Q = [q_0 : q_1 : 0] \in \Delta_1 \setminus \Delta_0$  and  $m > 1$ . We have that

$$\begin{aligned} I(P)^m \star I(Q)^n &= (p_2x_1 - p_1x_2, p_2x_0 - p_0x_2)^m \star (q_1x_0 - q_0x_1, x_2)^n \\ &= ((p_2y_1 - p_1y_2, p_2y_0 - p_0y_2)^m + (q_1z_0 - q_0z_1, z_2)^n + \mathcal{H}) \bigcap \mathbb{K}[\mathbf{x}] \\ &= (x_2^n) + (\dots). \end{aligned}$$

Since  $x_2^n \notin I(P \star Q)^{m+n-1}$  therefore we have that  $I(P \star Q)^{m+n-1} \neq I(P)^m \star I(Q)^n$ .

If  $m = 1$  we have:

$$\begin{aligned} I(P) \star I(Q)^n &= (p_2x_1 - p_1x_2, p_2x_0 - p_0x_2) \star (q_1x_0 - q_0x_1, x_2)^n \\ &= ((p_2y_1 - p_1y_2, p_2y_0 - p_0y_2) + (q_1z_0 - q_0z_1, z_2)^n + \mathcal{H}) \bigcap \mathbb{K}[\mathbf{x}] \\ &= (x_2^n) + (x_2^{n-1}(p_1q_1x_0 - p_0q_0x_1)) + \dots + (p_1q_1x_0 - p_0q_0x_1)^n \\ &= (x_2, (p_1q_1x_0 - p_0q_0x_1))^n = I(P \star Q)^n. \end{aligned}$$

(b) Let  $m \leq n$ . Without loss of generality assume that  $P = [p_0 : 0 : p_2]$  and  $Q = [q_0 : q_1 : 0]$ . We have that  $I(P \star Q) = (x_1, x_2)$ .

$$\begin{aligned} I(P)^m \star I(Q)^n &= (p_2x_0 - p_0x_2, x_1)^m \star (q_1x_0 - q_0x_1, x_2)^n \\ &= ((p_2y_0 - p_0y_2, y_1)^m + (q_1z_0 - q_0z_1, z_2)^n + \mathcal{H}) \bigcap \mathbb{K}[\mathbf{x}] \\ &= (x_1^m, x_2^n) \not\subset (x_1, x_2)^{n+m-1} = I(P \star Q)^{n+m-1}. \end{aligned}$$

Since  $(x_1, x_2)^{n+m-1} \subset (x_1^m, x_2^n) \subset (x_1, x_2)^m$  therefore  $I(P \star Q)^{n+m-1} \subset I(P)^m \star I(Q)^n \subset I(P \star Q)^m$ .

Now again without loss of generality assume that  $P = [p_0 : p_1 : 0]$  and  $Q = [q_0 : q_1 : 0]$ . One can see that  $x_2^m \in I(P)^m \star I(Q)^n$ . Since  $x_2^m \notin I(P \star Q)^{m+n-1}$  therefore the equality fails. Similarly one can show that  $I(P \star Q)^{n+m-1} \subset I(P)^m \star I(Q)^n \subset I(P \star Q)^m$ .

□

### 5. Hadamard Fat Grids

In this section we introduce and study a particular set of fat points in  $\mathbb{P}^2$ , that we call *Hadamard fat grid*, whose support is a complete intersection. In particular, in Theorem 5.7 we describe a graded minimal free resolution of a Hadamard fat grid using the results from Sect. 3, enlarging the known literature on the minimal graded resolution of homogeneous sets of fat points in  $\mathbb{P}^2$  supported on a complete intersection. We also compute the Waldschmidt constant and the resurgence of the ideal defining a Hadamard fat grid (see Proposition 5.17 and Corollary 5.19).

Let  $P_M = \{P_1, \dots, P_r\}$  and  $Q_N = \{Q_1, \dots, Q_s\}$  be two sets of collinear points in  $\mathbb{P}^2 \setminus \Delta_1$  with assigned positive multiplicities, respectively,  $M = \{m_1, \dots, m_r\}$  and  $N = \{n_1, \dots, n_s\}$ . In terms of ideals we have

$$I(P_M) = I(P_1)^{m_1} \cap \dots \cap I(P_r)^{m_r} \text{ and } I(Q_N) = I(Q_1)^{n_1} \cap \dots \cap I(Q_s)^{n_s}.$$

**Definition 5.1.** Assume that  $P_i \star Q_j \neq P_k \star Q_l$  for all  $1 \leq i < k \leq r$  and  $1 \leq j < l \leq s$ . Then the set of fat points defined by  $I(P_M) \star I(Q_N)$ , is called a Hadamard fat grid and it is denoted by  $HFG(P_M, Q_N)$ .

According to Theorem 4.2 and Corollary 2.6, the ideal of  $HFG(P_M, Q_N)$  is

$$\bigcap_{i \in [r]} \bigcap_{j \in [s]} I(P_i \star Q_j)^{m_i+n_j-1}.$$

By Lemma 3.1 in [8], we know that the Hadamard product  $Z \star S$  of a collinear set  $Z$  by a point  $S \in \mathbb{P}^2 \setminus \Delta_1$  is still a collinear set lying on the line  $S \star L$ , where  $Z \subset L$ . Hence, if we denote by  $\ell_P$  and  $\ell_Q$  the lines in which the sets  $P_M$  and  $Q_N$  lie respectively, one has

$$P_i \star Q_j \in \ell_P \star Q_j \text{ for all } i = 1, \dots, r \text{ and for all } j = 1, \dots, s.$$

And similarly,

$$P_i \star Q_j \in P_i \star \ell_Q \text{ for all } j = 1, \dots, s \text{ and for all } i = 1, \dots, r.$$

This shows that  $HFG(P_M, Q_N)$  has the structure of a planar grid. Specifically, it is a set of fat points whose support is a complete intersection of type  $(r, s)$  in  $\mathbb{P}^2$ .

*Example 5.2.* Figure 1 shows a Hadamard fat grid for  $r = 4$  and  $s = 5$ ; Fig. 1a shows the geometric structure, with all lines involved in the grid, while in Fig. 1b we represent the multiplicities of each point of the grid.

*Remark 5.3.* From Fig. 1b we see that the multiplicities of the points in the grid have a specific behaviour. If we assume that  $m_i \leq m_{i+1}$ , for  $i = 1, \dots, r-1$  and  $n_j \leq n_{j+1}$  for each  $j = 1, \dots, s-1$ , then the multiplicities of  $P_i \star Q_j$  and  $P_i \star Q_{j+1}$  differ of  $n_{j+1} - n_j$  for all  $i = 1, \dots, r$ . Similarly the multiplicities of  $P_i \star Q_j$  and  $P_{i+1} \star Q_j$  differ of  $m_{i+1} - m_i$  for all  $j = 1, \dots, s$ . Hence the Hadamard fat grids are a subclass of all possible fat grids in the plane.

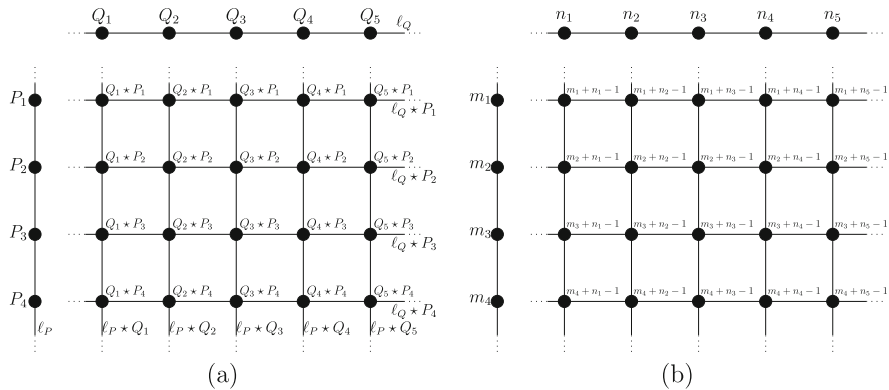


Figure 1.  $HFG(P_M, Q_N)$

From the rest of the paper we assume that  $s \geq r$  and the multiplicities are ordered in non-decreasing order, that is  $m_i \leq m_{i+1}$ , for  $i = 1, \dots, r - 1$  and  $n_j \leq n_{j+1}$  for each  $j = 1, \dots, s - 1$ . We denote by  $\mathcal{I}(P_M, Q_N)$  the ideal of  $HFG(P_M, Q_N)$ .

To simplify the notation, set  $m_{ij} = m_i + n_j - 1$  for  $i = 1, \dots, r$  and  $j = 1, \dots, s$ .

**Lemma 5.4.** *Let  $Z$  be a set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  whose support is a complete intersection of type  $(r, s)$  (or  $(r, s)$ -grid) and whose multiplicities  $m_{ij}$  are the same as a Hadamard Fat grid  $HFG(P_M, Q_N)$ . Then  $Z$  share the graded same Betti numbers as a set of fat points  $Y$  in  $\mathbb{P}^2$ .*

*Proof.* From Proposition 3.4,  $Z$  is an ACM set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  and from Remark 3.1, its ideal  $I_Z$  defines a set of fat lines in  $\mathbb{P}^3$ , and it still continues to be ACM and 1-dimensional. Hence, after a “proper hyperplane section” we get a set of fat points  $Y$  in  $\mathbb{P}^2$  that has the same graded Betti numbers as  $Z$ . That is, if  $\ell$  is a proper hyperplane section, we have

$$R/I_Z \cong \frac{R + (\ell)}{(\ell)} / \frac{I_Z + (\ell)}{(\ell)} \cong K[\mathbb{P}^2]/I_Y$$

where the former is ACM. □

**Lemma 5.5.** *Let  $X$  and  $X'$  be two sets of fat points in  $\mathbb{P}^2$  whose support is an  $(r, s)$ -grid and with the same multiplicities  $m_{ij}$  as a  $HFG(P_M, Q_N)$ . Then  $X$  and  $X'$  share the same graded Betti numbers.*

*Proof.* Following again the same method as Theorem 3.2 in [13], Remark 3.1 and Lemma 5.4, we construct the polynomial ring  $T$  in the  $r + s$  new variables and form a height 2 monomial ideal  $J$  in  $T$  given by the intersection of ideals of the form  $(a_{1,i}, a_{2,j})^{m_{ij}}$  corresponding to the components of  $X$ . Thus, starting from the scheme defined by the ideal  $J$ , after suitable sequences of proper hyperplane sections we can construct a set (fat) of lines in  $\mathbb{P}^3$  that corresponds to an ACM set of fat points  $Z$  in  $\mathbb{P}^1 \times \mathbb{P}^1$  and since, from Proposition 3.4,  $Z$  is ACM then, from Theorem 3.2 in [13],  $T/J$  is CM. Analogously, we

can construct a set of lines in  $\mathbb{P}^3$  that corresponds to an ACM set of fat points  $Z'$  in  $\mathbb{P}^1 \times \mathbb{P}^1$  and since  $Z'$  is ACM then  $T/J$  is CM. Again, starting from the scheme defined by the ideal  $J$ , after two other suitable sequences of hyperplane sections, we get two sets of fat points  $X$  and  $X'$  in  $\mathbb{P}^2$  that both share the same graded Betti numbers as  $Z$  and  $Z'$  ( and as  $J$ ).  $\square$

We show the main two results of this section.

**Theorem 5.6.** *Let  $X$  be a Hadamard fat grid  $HFG(P_M, Q_N)$  in  $\mathbb{P}^2$  and  $Z$  be an ACM set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  supported on an  $(r, s)$ -grid with the same multiplicities  $m_{ij}$  as the Hadamard fat grid  $X$ . Then  $X$  and  $Z$  share the same Betti numbers.*

*Proof.* From Lemma 5.4 we can construct a set of fat points  $Y$  an  $(r, s)$ -grid in  $\mathbb{P}^2$  that preserves the same multiplicities on  $m_{ij}$  and the same Betti numbers as  $Z$ . From Lemma 5.5,  $X$  and  $Y$  (and  $Z$ ) share the graded same Betti numbers.  $\square$

We now are able to compute a minimal free graded resolution of a Hadamard fat grid. We will use the results from previous Sect. 3, to prove the main result of this section.

**Theorem 5.7.** *Let  $X = HFG(P_M, Q_N)$  be a Hadamard fat grid in  $\mathbb{P}^2$ . Then a graded minimal free resolution of  $I(X)$  is given by*

$$0 \longrightarrow \bigoplus_{(v_1, v_2) \in V_X} R(-v_1 - v_2) \longrightarrow \bigoplus_{(c_1, c_2) \in C_X} R(-c_1 - c_2) \longrightarrow I(X) \longrightarrow 0.$$

*Proof.* We can construct  $\mathcal{A}_X = (\alpha_1, \dots, \alpha_m)$  associated to  $X$  using the method described as Sect. 3. Applying Theorems 3.3 and 5.6, we get the conclusion.  $\square$

In the sequel we adopt the following notation:

**Notation 5.8.** Let  $\mathcal{I}(P_M, Q_N)$  be the ideal of a Hadamard fat grid  $HFG(P_M, Q_N)$  where  $M = \{m_1, \dots, m_r\}$  and  $N = \{n_1, \dots, n_s\}$  with  $r \leq s$ ,  $m_i \leq m_{i+1}$ , for  $i = 1, \dots, r - 1$  and  $n_j \leq n_{j+1}$  for each  $j = 1, \dots, s - 1$ . Set  $a_i = m_{r-i+1} + n_s - 1$  and  $b_j = n_{s-j+1} - n_s$  for  $i = 1, \dots, r$ ,  $j = 1, \dots, s$ . Let  $H_i$  denotes the horizontal lines defining  $\ell_Q \star P_{r-i+1}$ , and  $V_j$  denotes the vertical lines defining  $\ell_P \star Q_{s-j+1}$ .

From Corollary 3.4 in [13] and Theorem 5.7, we have that

**Corollary 5.9.** *If  $X$  is a Hadamard fat grid  $X = HFG(P_M, Q_N)$  in  $\mathbb{P}^2$ , then its homogeneous ideal is minimally generated by products of linear forms of type  $H_i$  and  $V_j$ .*

**Theorem 5.10.** *A minimal set of generators of the ideal  $\mathcal{I}(P_M, Q_N)$  consists of  $m_r + n_s$  generators of types  $H_1^{a_1-k} \dots H_r^{a_r-k} \cdot V_1^{b_1+k} \dots V_s^{b_s+k}$  for  $k =$*

$0, \dots, m_r + n_s - 1$  where we adopt the convention that  $H_i^{a_i - k} = 1$  if  $a_i - k \leq 0$  and  $V_j^{b_j + k} = 1$  if  $b_j + k \leq 0$ . That is, a minimal set of generators is of type

$$\begin{aligned}
 & H_1^{m_r + n_s - 1} H_2^{m_{r-1} + n_s - 1} \dots H_r^{m_1 + n_s - 1} \cdot V_1^0 V_2^{n_{s-1} - n_s} \dots V_{s-1}^{n_2 - n_s} V_s^{n_1 - n_s}, \\
 & H_1^{m_r + n_s - 2} H_2^{m_{r-1} + n_s - 2} \dots H_r^{m_1 + n_s - 2} \cdot V_1^1 V_2^{n_{s-1} - n_s + 1} \dots V_{s-1}^{n_2 - n_s + 1} V_s^{n_1 - n_s + 1}, \\
 & \vdots \\
 & H_1^0 H_2^{m_{r-1} - m_r} \dots H_r^{m_1 - m_r} \cdot V_1^{m_r + n_s - 1} \dots V_{s-1}^{n_2 + m_r - 1} V_s^{n_1 + m_r - 1}.
 \end{aligned} \tag{8}$$

*Proof.* We apply Theorems 3.3, 5.6, 5.7 and Corollary 5.9. □

We will show the above results with an example.

*Example 5.11.* Let  $X = HFG(P_M, Q_N)$  be a Hadamard fat grid where  $M = (2, 3, 3)$  and  $N = (2, 3, 4, 4)$  as in Fig. 2.

In this case,

$$\begin{aligned}
 A_X &= (21, 21, 17, 17, 17, 13, 13, 13, 9, 9, 9, 5, 5, 5, 2, 2, 2), \\
 V_X &= \{(2, 21), (5, 17), (8, 13), (11, 9), (14, 5), (17, 2)\}
 \end{aligned}$$

and

$$C_X = \{(0, 21), (2, 17), (5, 13), (8, 9), (11, 5), (14, 2), (17, 0)\}.$$

Then, using Theorem 5.7, a graded minimal free resolution of  $I(X)$  is given by

$$\begin{aligned}
 0 &\longrightarrow R(-23) \oplus R(-22) \oplus R(-21) \oplus R(-20) \oplus R^2(-19) \longrightarrow \\
 &R(-21) \oplus R(-19) \oplus R(-18) \oplus R^2(-17) \oplus R^2(-16) \longrightarrow I(X) \longrightarrow 0
 \end{aligned}$$

And using Theorem 5.10, a minimal set of seven generators is given by

$$\begin{aligned}
 & H_1^6 H_2^6 H_3^5 \\
 & H_1^5 H_2^5 H_3^4 \cdot V_1^1 V_2^1 \\
 & H_1^4 H_2^4 H_3^3 \cdot V_1^2 V_2^2 V_3^1 \\
 & H_1^3 H_2^3 H_3^2 \cdot V_1^3 V_2^3 V_3^2 V_4^1 \\
 & H_1^2 H_2^2 H_3^1 \cdot V_1^4 V_2^4 V_3^3 V_4^2 \\
 & H_1^1 H_2^1 \cdot V_1^5 V_2^5 V_3^4 V_4^3 \\
 & V_1^6 V_2^6 V_3^5 V_4^4.
 \end{aligned} \tag{9}$$

We are able to compute the Waldschmidt constant and the resurgence of a Hadamard fat grid  $\mathcal{I}(P_M, Q_N)$ . We recall the following

**Definition 5.12.** For a given homogeneous ideal  $0 \neq I \subsetneq \mathbb{K}[x_0, \dots, x_N]$ , its Waldschmidt constant  $\hat{\alpha}(I)$  is defined as

$$\hat{\alpha}(I) = \lim_{m \rightarrow \infty} \alpha(I^{(m)})/m$$

where  $\alpha(I)$  is the least degree  $d$  such that  $I_d \neq (0)$ .



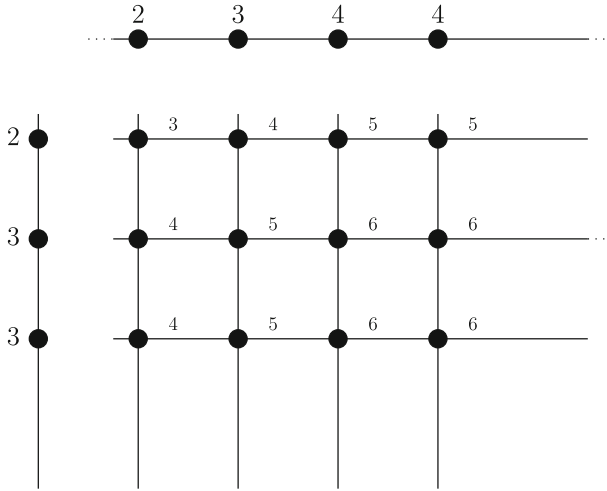


Figure 2.  $X = HFG(P_M, Q_N)$ , with  $M = (2, 3, 3)$  and  $N = (2, 3, 4, 4)$

Because of the subadditivity of  $\alpha$ , this limit exists (see Lemma 2.3.1 of [7]). Moreover,  $\hat{\alpha}(I) > 0$  (see Lemma 2.3.2 of [7]).

We need some preliminary result.

**Proposition 5.13.** *Let  $X = HFG(P_M, Q_N)$  be a Hadamard fat grid. The minimal degree  $\alpha(\mathcal{I}(P_M, Q_N))$  of a generator of the ideal  $\mathcal{I}(P_M, Q_N)$  is given by*

$$\alpha(\mathcal{I}(P_M, Q_N)) = \sum_{i=1}^r m_i + \sum_{i=1}^r n_{s-i+1} - r.$$

*Proof.* By Theorem 5.10, we know that the degrees of the minimal generators of the ideal  $\mathcal{I}(P_M, Q_N)$  are of type

$$\left\{ \sum_{i=1}^r \deg H_{r-i+1}^{m_i+n_s-t} + \sum_{j=1}^s \deg V_{s-j+1}^{n_j-n_s+t-1} \right\}_{t=1}^{m_r+n_s},$$

where  $H_i^{a_i-k} = 1$  if  $a_i - k \leq 0$  and  $V_j^{b_j+k} = 1$  if  $b_j + k \leq 0$  for  $k = 0, \dots, m_r + n_s - 1$ . It means that we only consider non-negative degrees.

Since  $m_1 \leq m_2 \leq \dots \leq m_r$ , therefore

$$\deg H_r^{m_1+n_s-t} \leq \deg H_{r-1}^{m_2+n_s-t} \leq \dots \leq \deg H_1^{m_r+n_s-t}.$$

On the other hand,  $\deg H_r^{m_1+n_s-t} > 0$  for  $t = 1, \dots, m_1 + n_s - 1$ . It follows that

$$0 < \deg H_r^{m_1+n_s-t} \leq \deg H_{r-1}^{m_2+n_s-t} \leq \dots \leq \deg H_1^{m_r+n_s-t} \quad \forall t = 1, \dots, m_1 + n_s - 1. \tag{10}$$

We can have a similar argument for the other summation as follows.

Since  $n_s \geq n_{s-1} \geq \dots \geq n_{s-r+1} \geq \dots \geq n_1$ , therefore

$$\deg V_s^{n_1-n_s+t-1} \leq \dots \leq \deg V_r^{n_{s-r+1}-n_s+t-1} \leq \dots \leq \deg V_2^{n_{s-1}-n_s+t-1}$$

$$\leq \deg V_1^{n_s - n_s + t - 1}.$$

One can also observe that  $\deg V_1^{n_s - n_s + t - 1} = 0$  for  $t = 1$ . Hence for  $t = 1$  we have,

$$\deg V_s^{n_1 - n_s} \leq \dots \leq \deg V_r^{n_s - r + 1 - n_s} \leq \dots \leq \deg V_2^{n_{s-1} - n_s} \leq \deg V_1^{n_s - n_s} = 0.$$

For  $t = 1, \dots, m_r + n_s$ , let  $G = \{g_1, g_2, \dots, g_t\}$  be a set of minimal generators of  $\mathcal{I}(P_M, Q_N)$ . We claim that  $g_t \geq g_{n_s - n_{s-r+1}}$  for  $1 \leq t \leq n_s - n_{s-r+1}$ .

Set  $\alpha = \sum_{i=1}^r m_i + (\sum_{i=1}^r n_{s-i+1}) - r$ .

We observe that

$$\begin{aligned} \alpha &= \sum_{i=1}^r (m_i + n_s - n_s + n_{s-r+1}) + \sum_{j=1}^s (n_j - n_s + n_s - n_{s-r+1} - 1) \\ &= \sum_{i=1}^r (m_i + n_{s-r+1}) + \sum_{j=1}^s (n_j - n_{s-r+1} - 1) \\ &= \sum_{i=1}^r m_i + r n_{s-r+1} \\ &\quad + (n_1 - n_{s-r+1} - 1) + (n_2 - n_{s-r+1} - 1) + \dots \\ &\quad + (n_{s-r} - n_{s-r+1} - 1) + (n_{s-r+1} - n_{s-r+1} - 1) \\ &\quad + (n_{s-r+2} - n_{s-r+1} - 1) + \dots + (n_s - n_{s-r+1} - 1). \end{aligned}$$

Since

$$n_s \geq n_{s-1} \geq \dots \geq n_{s-r+1} \geq n_{s-r} \geq \dots \geq n_1, \tag{11}$$

we have

$$\begin{aligned} \alpha &= \sum_{i=1}^r m_i + r n_{s-r+1} \\ &\quad + (n_{s-r+1} - n_{s-r+1} - 1) + (n_{s-r+2} - n_{s-r+1} - 1) + \dots + (n_s - n_{s-r+1} - 1) \\ &= \sum_{i=1}^r m_i + r n_{s-r+1} + \sum_{i=1}^r n_{s-i+1} - r n_{s-r+1} - r \\ &= \sum_{i=1}^r m_i + \sum_{i=1}^r n_{s-i+1} - r. \end{aligned}$$

We also note that from (11), we have

$$\left( \sum_{i=r+1}^s n_{s-i+1} - n_s \right) \leq 0. \tag{12}$$

Then, for  $1 \leq t \leq n_s - n_{s-r+1}$  and using (12), we have

$$\begin{aligned} g_t \geq \alpha &\iff \sum_{i=1}^r (m_i + n_s - t) + \sum_{j=1}^s (n_j - n_s + t - 1) \\ &\geq \sum_{i=1}^r m_i + \left( \sum_{i=1}^r n_{s-i+1} \right) - r \end{aligned}$$

$$\begin{aligned}
 &\iff \left( \sum_{j=1}^s n_j \right) + rn_s - rt - sn_s + s(t-1) \geq \left( \sum_{i=1}^r n_{s-i+1} \right) - r \\
 &\iff \left( \sum_{i=1}^r n_{s-i+1} \right) + \left( \sum_{i=r+1}^s n_{s-i+1} \right) + rn_s - rt - sn_s + s(t-1) \\
 &\geq \left( \sum_{i=1}^r n_{s-i+1} \right) - r \\
 &\iff \left( \sum_{i=r+1}^s n_{s-i+1} \right) - (s-r)n_s + s(t-1) + r - rt \geq 0 \\
 &\iff \left( \sum_{i=r+1}^s n_{s-i+1} - n_s \right) + s(t-1) + r - rt \geq 0 \\
 &\iff t(s-r) + r - s \geq 0 \\
 &\iff t \geq 1.
 \end{aligned}$$

This proves that  $\alpha$  is the minimum value among all the degrees of the minimal generators of a Hadamard fat grid. □

*Example 5.14.* Using again the values of Example 5.11 we get

$$\alpha(\mathcal{I}(P_M, Q_N)) = 2 + 3 + 3 + 4 + 4 + 3 - 3 = 16$$

as expected from its minimal resolution. In this case we have two generators of minimal degree, namely  $H_1^5 H_2^5 H_3^4 \cdot V_1^1 V_2^1$  and  $H_1^4 H_2^4 H_3^3 \cdot V_1^2 V_2^2 V_3^1$ .

*Remark 5.15.* Note that since  $m_1 \leq m_2 \leq \dots \leq m_r$  and  $n_1 \leq n_2 \leq \dots \leq n_s$  therefore the maximum degree of generators of the ideal  $\mathcal{I}(P_M, Q_N)$  is as the following:

$$\beta(\mathcal{I}(P_M, Q_N)) = \max \left\{ \sum_{j=1}^s (m_r + n_j - 1), \sum_{i=1}^r (n_s + m_i - 1) \right\}.$$

We need a preliminary lemma to study the symbolic powers  $\mathcal{I}(P_M, Q_N)^{(t)}$ .

**Lemma 5.16.** *The  $t$ -th symbolic power of  $\mathcal{I}(P_M, Q_N)$  is a Hadamard fat grid.*

*Proof.* We know that:

$$\mathcal{I}(P_M, Q_N) = \bigcap_{i=1}^r \bigcap_{j=1}^s I(P_i \star Q_j)^{m_i+n_j-1}.$$

Hence

$$\begin{aligned}
 \mathcal{I}(P_M, Q_N)^{(t)} &= \bigcap_{i=1}^r \bigcap_{j=1}^s I(P_i \star Q_j)^{t(m_i+n_j-1)} \\
 &= \bigcap_{i=1}^r \bigcap_{j=1}^s I(P_i \star Q_j)^{((tm_i-(t-1))+tn_j)-1}.
 \end{aligned}$$

Therefore, the sets  $P_{M'} \star Q_{N'}$  with multiplicities  $M' = \{tm_1 - (t-1), \dots, tm_r - (t-1)\}$  and  $N' = \{tn_1, \dots, tn_s\}$  is the Hadamard fat grid whose ideal is  $\mathcal{I}(P_M, Q_N)^{(t)}$ . □

We are now able to prove the following

**Proposition 5.17.** *The Waldschmidt constant  $\hat{\alpha}(\mathcal{I}(P_M, Q_N))$  of the Hadamard fat grid  $HFG(P_M, Q_N)$  is equal to the least degree of a minimal set of generators of its defining ideal, i.e.  $\hat{\alpha}(\mathcal{I}(P_M, Q_N)) = \alpha(\mathcal{I}(P_M, Q_N))$ .*

*Proof.* Since the  $t$ -symbolic power of the ideal of a Hadamard fat grid  $HFG(P_M, Q_N)$  defines a new Hadamard fat grid  $HFG(P_{M'}, Q_{N'})$ , one has

$$\alpha(\mathcal{I}(P_M, Q_N)^{(t)}) = \alpha(\mathcal{I}(P_{M'}, Q_{N'}))$$

where  $M'$  and  $N'$  are defined as in Lemma 5.16. Hence we can apply Proposition 5.13 to compute  $\alpha(\mathcal{I}(P_{M'}, Q_{N'}))$  obtaining

$$\begin{aligned} \alpha(\mathcal{I}(P_M, Q_N)^{(t)}) &= \sum_{i=1}^r (tm_i - (t-1)) + t \sum_{i=1}^r n_{s-i+1} - r \\ &= t \sum_{i=1}^r m_i - r(t-1) + t \sum_{i=1}^r n_{s-i+1} - r \\ &= t \sum_{i=1}^r m_i + t \sum_{i=1}^r n_{s-i+1} - rt \\ &= t \left( \sum_{i=1}^r m_i + \sum_{i=1}^r n_{s-i+1} - r \right) \\ &= t\alpha(\mathcal{I}(P_M, Q_N)). \end{aligned}$$

Therefore,

$$\hat{\alpha}(\mathcal{I}(P_M, Q_N)) = \lim_{t \rightarrow \infty} \frac{\alpha(\mathcal{I}(P_M, Q_N)^{(t)})}{t} = \alpha(\mathcal{I}(P_M, Q_N)).$$

□

We pass now to compute the resurgence  $\rho(I) = \sup\{m/r : I^{(m)} \not\subseteq I^r\}$  of  $I = \mathcal{I}(P_M, Q_N)$ .

It is important to notice that even though  $\mathcal{I}(P_M, Q_N)$  defines a set of fat points in  $\mathbb{P}^2$  that is not a complete intersection, we will show that  $\mathcal{I}(P_M, Q_N)^t = \mathcal{I}(P_M, Q_N)^{(t)}$  and hence that its resurgence is equal to 1.

**Proposition 5.18.** *Let  $\mathcal{I}(P_M, Q_N)$  be the ideal of a Hadamard fat grid, then  $\mathcal{I}(P_M, Q_N)^t = \mathcal{I}(P_M, Q_N)^{(t)}$  for all  $t \geq 1$ .*

*Proof.* By Theorem 5.10, a minimal set of generators of the ideal  $\mathcal{I}(P_M, Q_N)$  consists of  $m_r + n_s$  generators of types

$$g_k = H_1^{a_1-k} \dots H_r^{a_r-k} \cdot V_1^{b_1+k} \dots V_s^{b_s+k}, \tag{13}$$

where

$$a_i = m_{r-i+1} + n_s - 1, \quad (i = 1, \dots, r), \quad b_j = n_{s-j+1} - n_s, \quad (j = 1, \dots, s),$$

$$k = 0, 1, \dots, m_r + n_s - 1.$$

The ordinary power  $\mathcal{I}(P_M, Q_N)^t$  is generated by all possible products of  $t$  generators of  $\mathcal{I}(P_M, Q_N)$ :

$$g_0^{t_0} g_1^{t_1} \dots g_{m_r+n_s-1}^{t_{m_r+n_s-1}} \tag{14}$$

where  $\sum t_i = t$ . Substituting (13) in (14) we get that the generators of  $I^t$  are of the form

$$H_1^{ta_1-k} \dots H_r^{ta_r-k} \cdot V_1^{tb_1+k} \dots V_s^{tb_s+k},$$

where  $k = 0, \dots, t(m_r + n_s - 1)$ .

By Lemma 5.16, we know that  $\mathcal{I}(P_M, Q_N)^{(t)}$  is the ideal of a Hadamard fat grid given by the sets  $P_{M'} \star Q_{N'}$  with multiplicities  $M' = \{tm_1 - (t - 1), \dots, tm_r - (t - 1)\}$  and  $N' = \{tn_1, \dots, tn_s\}$ . Hence, again by Theorem 5.10,  $\mathcal{I}(P_M, Q_N)^{(t)}$  is generated by

$$\overline{g_k} = H_1^{\overline{a_1-k}} \dots H_r^{\overline{a_r-k}} \cdot V_1^{\overline{b_1+k}} \dots V_s^{\overline{b_s+k}}, \tag{15}$$

where

$$\begin{aligned} \overline{a_i} &= tm_{r-i+1} - t + 1 - tn_s - 1 = t(m_{r-i+1} + n_s - 1) = ta_i, & (i = 1, \dots, r), \\ \overline{b_j} &= tn_{s-j+1} - tn_s = t(n_{s-j+1} - n_s) = tb_j, & (j = 1, \dots, s), \end{aligned}$$

and

$$\overline{k} = 0, 1, \dots, tm_r - t + 1 + tn_s - 1.$$

Since  $tm_r - t + 1 + tn_s - 1 = t(m_r + n_s - 1)$ , every generator in  $\mathcal{I}(P_M, Q_N)^{(t)}$  is also a generator in  $\mathcal{I}(P_M, Q_N)^t$ , giving  $\mathcal{I}(P_M, Q_N)^{(t)} \subset \mathcal{I}(P_M, Q_N)^t$ . The other inclusion is obvious by definition of symbolic powers, hence we get  $\mathcal{I}(P_M, Q_N)^t = \mathcal{I}(P_M, Q_N)^{(t)}$ .  $\square$

Using the definition of resurgence we get the following

**Corollary 5.19.** *Let  $\mathcal{I}(P_M, Q_N)$  be the ideal of a Hadamard fat grid, then  $\rho(\mathcal{I}(P_M, Q_N)) = 1$ .*

*Remark 5.20.* Corollary 5.19 can also be recovered by the recent Theorem 2.3 in [19] where the authors provide various examples and questions about computing the resurgence of homogeneous ideals.

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## Declarations

**Conflict of interest** The authors declare no competing interests.

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