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Convolution Integral Operators in Variable Bounded Variation Spaces

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Abstract. Working in the frame of variable bounded variation spaces in the sense of Wiener, introduced by Castillo, Merentes, and Rafeiro, we prove convergence in variable variation by means of the classical convolution integral operators. In the proposed approach, a crucial step is the convergence of the variable modulus of smoothness for absolutely continuous functions. Several preliminary properties of the variable $p(\cdot)$-variation are also presented.

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1. Introduction

The study of variable exponent Lebesgue spaces has been a challenging topic in the last 30 years. Such spaces, introduced by Orlicz [24] and then developed by Nakano [22,23], are a generalization of the classical Lebesgue spaces: the basic idea is that the constant exponent p of the L^p-spaces is replaced by a variable function $p(\cdot)$. Such spaces share, with the classical L^p-spaces, several properties, but nevertheless they also present some significant differences. Among them, for example, the variable exponent spaces are not invariant under translation. The study of such spaces had a wide development for their intrinsic interest, and also for the important applications that they have in partial differential equations, calculus of variations, harmonic analysis, as well as in several applied problems such as, for example, digital image processing (see, e.g., [11,15,28]) or the study of electrorheological fluids (see, e.g., [25,26]).

Following the idea of variable spaces, in [14], Castillo, Merentes, and Rafeiro introduced the variable bounded variation spaces in the sense of Wiener ($BV^{p(\cdot)}$), a generalization of the spaces of bounded p-variation [27], that generalized in turn the classical BV-space in the sense of Jordan. We recall that the space of bounded p-variation in the sense of Wiener is defined as the space of functions for which the p-variation is finite, that is

$$V^p[f] := \sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p < +\infty,$$

where the supremum is taken over all the possible increasing sequences $t_0 < t_1 < \cdots < t_n$ in $\mathbb{R}$. Taking $p = 1$, the above space reduces to the classical BV-space in sense of Jordan. The idea of Castillo, Merentes, and Rafeiro was to replace p by a variable function $p(\cdot)$ with suitable properties, defining therefore the $BV^{p(\cdot)}$-spaces, namely the variable bounded variation spaces in the sense of Wiener, that are the setting of the present paper. We recall that a variable exponent version of the Riesz variation was introduced and studied in [12,13], while we refer to [7] for an extensive treatment about classical and non-classical BV-spaces.

Our main goal will be to obtain a convergence result for the classical convolution integral operators with respect to convergence in variable variation in the sense of Wiener, recalling that convergence in variation is the natural notion of convergence in BV-spaces. Convergence results for the convolution integral operators within BV-spaces were obtained using several notions of variation, besides the classical Jordan variation (see, e.g., [8,9]), such as the φ-variation in the sense of Musielak–Orlicz [9,21], the Riesz φ-variation [1], or, in the multidimensional setting, the Tonelli variation [8]. About variable spaces, in [16], there are results about pointwise and norm convergence for convolution operators in the variable Lebesgue spaces $L^{p(\cdot)}$.

As mentioned before, if variable spaces share several properties with classical Lebesgue spaces, there are also significant differences and some important properties do not hold any more. As an example, it is not true that the translation operator applied to a function belonging to a variable Lebesgue space belongs to the same space, as it holds in L^p-spaces [16], and of course the same happens in $BV^{p(\cdot)}$-spaces (see Example 3). Another delicate point is about the additivity of the variation on intervals: analogously to what happens in the case of Musielak–Orlicz φ-variation (see [21, 1.17 and 1.18]), the classical additivity property on intervals is replaced by suitable inequalities (see Proposition 3.2). These facts make the problem of convergence in variable variation much more delicate with respect to working with the classical variation.

The paper is organized as follows. After a preliminary section in which we state the main notations and preliminaries, we present some properties of the variable variation in the sense of Wiener that will be useful in the following (Sect. 3). Then, in Sect. 4, we present the main results: starting from an estimate in variable variation for the convolution operators, we prove a result of convergence for the modulus of smoothness, that is naturally reformulated in the context of $BV^{p(\cdot)}$-spaces: to do this, several preliminary results are necessary to provide a kind of approximation by means of step-type functions (Proposition 4.3, Theorem 4.4). As a consequence, we obtain the convergence in variable variation by means of convolution operators. For all these results, it is crucial the assumption of $p(\cdot)$-absolute continuity on the function, that is, the obvious reformulation of absolute continuity in the context of variable bounded variation spaces. This is absolutely natural, since, also in case of the classical Jordan variation, convergence in variation can be obtained just in the subspace of BV of the absolutely continuous functions. A similar situation occurs also working with other concepts of variation (see, e.g., [21] for the Musielak–Orlicz φ-variation, [6] for the multidimensional φ-variation in the sense of Tonelli) and with other kind of operators (see, e.g., [2,10] for Mellin integral operators or [3–5] for sampling-type discrete operators).

2. Notations and Preliminaries

We first recall the definition of variable bounded variation spaces in the sense of Wiener, adapting to the case of functions defined on the whole real line that one given in [14] for functions defined on an interval $[a, b]$.

Definition 2.1. An admissible function is a function $p : \mathbb{R} \to [1, +\infty)$, such that $p_+ := \sup_{x\in\mathbb{R}} p(x) < +\infty$.

We will also use the notation $p_- := \inf_{x\in\mathbb{R}} p(x)$: notice that, obviously, $p_- \geq 1$. From now on, $p(\cdot)$ will denote an admissible function.

Definition 2.2. Let $f : \mathbb{R} \to \mathbb{R}$. The $p(\cdot)$-variation in the Wiener's sense of f is defined as

$$V^{p(\cdot)}[f] := \sup_{\Pi^*} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^{p(x_{i-1})},$$

where Π^* is a *tagged sequence*, i.e., an increasing sequence $t_0 < t_1 < \cdots < t_n$, together with a finite sequence of numbers $x_0, x_1, \ldots, x_{n-1}$ subject to the condition $t_i \leq x_i \leq t_{i+1}$, $\forall i = 0, \ldots, n-1$.

Definition 2.3. By

$$BV^{p(\cdot)}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : \ V^{p(\cdot)}[\lambda f] < +\infty, \text{ for some } \lambda > 0\},$$

we denote the space of functions of bounded $p(\cdot)$-variation on $\mathbb{R}$.

The definition in case of functions defined on an interval $[a, b] \subset \mathbb{R}$ (or on a halfline) is given in a similar way [14].

Definition 2.4. Let $f : [a, b] \to \mathbb{R}$. The $p(\cdot)$-variation in the Wiener's sense of f is defined as

$$V^{p(\cdot)}[f, [a, b]] := \sup_{\Pi^*} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^{p(x_{i-1})},$$

where Π^* is a *tagged partition*, i.e., a partition $t_0 = a < t_1 < \cdots < t_n = b$ of $[a, b]$ together with a finite sequence of numbers $x_0, x_1, \ldots, x_{n-1}$ subject to the condition $t_i \leq x_i \leq t_{i+1}$, $\forall i = 0, \ldots, n-1$.

Definition 2.5. By

$$BV^{p(\cdot)}([a,b]) := \{f : [a,b] \to \mathbb{R} :\ V^{p(\cdot)}[\lambda f,[a,b]] < +\infty,\ \text{for some}\ \ \lambda > 0\},$$

we denote the space of functions of bounded $p(\cdot)$-variation in the Wiener's sense on $[a,b]$.

In [14], $BV^{p(\cdot)}([a,b])$ is actually defined by means of the norm

$$\|f\|_{BV^{p(\cdot)}([a,b])} := \inf\left\{\lambda > 0 :\ V^{p(\cdot)}\left[\frac{f}{\lambda},[a,b]\right] \le 1\right\},$$

i.e., as the space of functions $f : [a,b]\to\mathbb{R}$ for which $\|f\|_{BV^{p(\cdot)}([a,b])}<+\infty$. The reason is that, regarding such definitions in the theory of modular spaces, it can be proved (see [14]) that $\|f\|_{BV^{p(\cdot)}([a,b])}$ is a Luxemburg norm; therefore, $BV^{p(\cdot)}([a,b])$ is a Banach space. Instead of the norm, we choose to use $V^{p(\cdot)}[\lambda f]$, that turns out to be a pseudomodular[1] (see [14]), so that $BV^{p(\cdot)}(\mathbb{R})$ is a modular space (see, e.g., [20,21]).

The natural convergence is therefore the so-called "modular convergence".

Definition 2.6. A family of functions $(f_w)_{w>0} \subset BV^{p(\cdot)}([a,b])$ is convergent in variation (modular convergent) to $f \in BV^{p(\cdot)}([a,b])$ if there exists $\lambda > 0$, such that

$$V^{p(\cdot)}[\lambda(f_w - f),[a,b]] \to 0, \quad \text{as } w \to +\infty.$$

Besides modular convergence, that is the notion of convergence that we will use in the present paper, the norm $\|f\|_{BV^{p(\cdot)}([a,b])}$ induces the usual norm convergence. We recall that norm convergence (i.e., $\|f_w - f\|_{BV^{p(\cdot)}([a,b])} \to 0$ as $w \to +\infty$) is equivalent to

$$V^{p(\cdot)}[\lambda(f_w - f),[a,b]] \to 0, \quad \text{as } w \to +\infty, \quad \text{for every } \lambda > 0.$$

In general, norm convergence is stronger than modular convergence: in case of $p_+ < +\infty$, as assumed here, it can be proved that actually they are equivalent (see also [16]).

Proposition 2.7. *Given a family of functions $(f_w)_w \subset BV^{p(\cdot)}([a,b])$, then $(f_w)_w$ converges in variation to $f \in BV^{p(\cdot)}([a,b])$ if and only if $(f_w)_w$ converges in norm to f.*

Proof. Being obvious that convergence in norm implies convergence in variation, we prove the converse. Let $\bar{\lambda} > 0$ be such that $V^{p(\cdot)}[\bar{\lambda}(f_w - f),[a,b]] \to 0$, as $w \to +\infty$. Let us fix any $\lambda > 0$ and assume w.l.g. that $\lambda > \bar{\lambda}$, since in the

[1]We recall that, if X is a vector space on K ($K = \mathbb{C}$ or $K = \mathbb{R}$), then a convex, left-continuous function $\rho : X \to [0,\infty)$ is called a convex pseudomodular on X if, for every x and y in X

(i) $\rho(0) = 0$,
(ii) $\rho(\alpha x) = \rho(x)$, for all $\alpha \in K$, such that $|\alpha| = 1$,
(iii) $\rho(\alpha x + (1-\alpha)y) \le \alpha\rho(x) + (1-\alpha)\rho(y)$ for all $\alpha \in [0,1]$.

If ρ is a pseudomodular on X, then the set $X_\rho = \{x \in X :\ \lim_{\lambda\to 0^+} \rho(\lambda x) = 0\}$ is a modular space.

other case, the proof is obvious. For a fixed $\epsilon > 0$, let $\bar{w} > 0$ be such that $V^{p(\cdot)}[\bar{\lambda}(f_w - f), [a,b]] < \epsilon \left(\frac{\bar{\lambda}}{\lambda}\right)^{p_+}$, for every $w \geq \bar{w}$. Then

$$V^{p(\cdot)}[\lambda(f_w - f), [a,b]] = V^{p(\cdot)}\left[\frac{\lambda}{\bar{\lambda}}\bar{\lambda}(f_w - f), [a,b]\right]$$
$$\leq \left(\frac{\lambda}{\bar{\lambda}}\right)^{p_+} V^{p(\cdot)}[\bar{\lambda}(f_w - f), [a,b]] < \epsilon,$$

for every $w \geq \bar{w}$. □

We recall that it is easy to prove that (see [14]), if $p(\cdot) \leq q(\cdot)$, then

$$V^{p(\cdot)}[\lambda f] \geq V^{q(\cdot)}[\lambda f], \tag{2.1}$$

and therefore, $BV^{p(\cdot)}(\mathbb{R}) \subset BV^{q(\cdot)}(\mathbb{R})$.

We will study approximation properties in $BV^{p(\cdot)}(\mathbb{R})$ of the classical convolution integral operators defined as

$$(T_w f)(s) = \int_{\mathbb{R}} K_w(t) f(s-t)\,\mathrm{d}t, \;\; w > 0, s \in \mathbb{R},$$

for $f : \mathbb{R} \to \mathbb{R}$ of bounded $p(\cdot)$-variation, where $(K_w)_{w>0}$ is a family of kernel functions that satisfy the usual assumptions of approximate identities, that is

$(K1)$ $K_w \in L^1(\mathbb{R})$, $\|K_w\|_1 \leq A$, for some constant $A > 0$ and for every $w > 0$ and $\int_{\mathbb{R}} K_w(t)\,\mathrm{d}t = 1$, for every $w > 0$;

$(K2)$ for any fixed $\delta > 0$, $\int_{|t|>\delta} |K_w(t)|\,\mathrm{d}t \to 0$, as $w \to +\infty$.

To get the main convergence result, it is necessary to introduce the concept of *variable absolute continuity*, adapted from [14] to the case of functions defined on the whole real space.

Definition 2.8. A function $f \in BV^{p(\cdot)}(\mathbb{R})$ is absolutely $p(\cdot)$-continuous if

$$\lim_{\delta \to 0^+} \sup_{\Pi_\delta} \sum_{i=1}^{n} |\lambda[f(t_i) - f(t_{i-1})]|^{p(x_{i-1})} = 0,$$

for some $\lambda > 0$, where Π_δ is a tagged sequence with mesh not greater than δ ($\max_{1 \leq i \leq n} |t_i - t_{i-1}| \leq \delta$). By $AC^{p(\cdot)}(\mathbb{R})$, we will denote the space of all the absolutely $p(\cdot)$-continuous functions $f : \mathbb{R} \to \mathbb{R}$.

It is immediate to see that, in the particular case $p(\cdot) = 1$, the above definition reduces (for $\lambda = 1$) to the classical absolute continuity, expressed in terms of convergence of the modulus of continuity, i.e., $\lim_{\delta \to 0^+} \omega_\delta^1(f) = 0$. In general, denoted by $AC(\mathbb{R})$, the space of *absolutely continuous functions on* $\mathbb{R}$, i.e., the BV-functions on $\mathbb{R}$ for which $\lim_{\delta \to 0^+} \sup_{S_\delta} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = 0$, where the supremum is taken over all the increasing sequences S_δ on $\mathbb{R}$ with mesh not greater than δ, we have that $AC(\mathbb{R}) \subset AC^{p(\cdot)}(\mathbb{R})$. Indeed, first of all recall that $p(\cdot) \geq 1$ implies $BV(\mathbb{R}) \subset BV^{p(\cdot)}(\mathbb{R})$. Moreover, if $f \in AC(\mathbb{R})$, in correspondence to $0 < \epsilon < 1$ (w.l.g.), there exists $\bar{\delta} > 0$, such that, for every $0 < \delta < \bar{\delta}$, $\sup_{S_\delta} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \epsilon < 1$, and so, in particular, $|f(t_i)-f(t_{i-1})| \le \sum_{i=1}^n |f(t_i)-f(t_{i-1})| < 1$, for every sequence $t_0 < t_1 < \cdots < t_n$ with mesh not greater than δ. Now, if Π_δ is a tagged sequence with mesh not greater than δ

$$|f(t_i)-f(t_{i-1})|^{p(x_{i-1})} \le |f(t_i)-f(t_{i-1})|,$$

since $p(\cdot) \ge 1$, and so

$$\sup_{\Pi_\delta} \sum_{i=1}^n |f(t_i)-f(t_{i-1})|^{p(x_{i-1})} \le \sup_{S_\delta} \sum_{i=1}^n |f(t_i)-f(t_{i-1})| < \epsilon.$$

3. Some Properties of the $p(\cdot)$-Variation

In this section, we will present some general results about the $p(\cdot)$-variation that will be useful to study the problem of convergence in variation by means of convolution integral operators. For other basic properties of the $p(\cdot)$-variation, we refer to [17–19].

Proposition 3.1. *If $f \in BV^{p(\cdot)}(\mathbb{R})$, there exists $\lambda > 0$, such that*

(a) $V^{p(\cdot)}[\lambda f] \le 1$;

(b) *for every increasing sequence $t_0 < t_1 < \cdots < t_n$, $\lambda |f(t_i)-f(t_{i-1})| \le 1$, for every $i = 1, \ldots, n$.*

Proof. To prove (a), let $\mu > 0$ be such that $V^{p(\cdot)}[\mu f] < +\infty$; if $V^{p(\cdot)}[\mu f] \le 1$, there is nothing to prove. If $V^{p(\cdot)}[\mu f] > 1$, then $\left\{V^{p(\cdot)}[\mu f]\right\}^{p(\cdot)} \ge V^{p(\cdot)}[\mu f]$, since $p(\cdot) \ge 1$, and so, if $t_0 < t_1 < \cdots < t_n$, $x_0, \ldots, x_{n-1}$ is a tagged sequence

$$\sum_{i=1}^n \left[\frac{\mu}{V^{p(\cdot)}[\mu f]}|f(t_i)-f(t_{i-1})|\right]^{p(x_{i-1})} \le \frac{\sum_{i=1}^n [\mu|f(t_i)-f(t_{i-1})|]^{p(x_{i-1})}}{V^{p(\cdot)}[\mu f]} \le 1;$$

therefore, passing to the supremum over all the tagged sequences in $\mathbb{R}$, we conclude that

$$V^{p(\cdot)}[\lambda f] \le 1$$

for $\lambda = \frac{\mu}{V^{p(\cdot)}[\mu f]}$. About (b), it is sufficient to notice that by (a)

$$[\lambda|f(t_i)-f(t_{i-1})|]^{p(\cdot)} \le V^{p(\cdot)}[\lambda f] \le 1,$$

and hence

$$\lambda|f(t_i)-f(t_{i-1})| \le 1,$$

for every $i = 1, \ldots, n$. □

The following proposition is a generalization to the $p(\cdot)$-variation of the classical additivity of the variation on intervals.

Proposition 3.2. *If $f \in BV^{p(\cdot)}([a,b])$ and $a < c < b$, then, for some $\lambda > 0$*

(a) $V^{p(\cdot)}[\lambda f, [a,c]] + V^{p(\cdot)}[\lambda f, [c,b]] \le V^{p(\cdot)}[\lambda f, [a,b]]$;

(b) $V^{p(\cdot)p_+/p_-}[\lambda f, [a,b]] \le 2^{p_+^2/p_- - 1}\left\{V^{p(\cdot)}[\lambda f, [a,c]] + V^{p(\cdot)}[\lambda f, [c,b]]\right\}$.

Proof. Let $\lambda > 0$ be given by Proposition 3.1. For (a), it is sufficient to notice that, if $t_0 = a < t_1 < \cdots < t_m = c$, $x_0, \ldots, x_{m-1}$ is a tagged partition of $[a, c]$ and $t_0^{'} = c < t_1^{'} < \cdots < t_k^{'} = b$, $x_0^{'}, \ldots, x_{k-1}^{'}$ is a tagged partition of $[c, b]$; obviously, the union $t_0 = a < \cdots < t_k^{'} = b$, $x_0, \ldots, x_{k-1}^{'}$ is a tagged partition of $[a, b]$, and hence

$$\sum_{j=1}^{m}[\lambda|f(t_j) - f(t_{j-1})|]^{p(x_{j-1})} + \sum_{j=1}^{k}[\lambda|f(t_j^{'}) - f(t_{j-1}^{'})|]^{p(x_{j-1}^{'})} \leq V^{p(\cdot)}[\lambda f, [a, b]].$$

Therefore, passing to the supremum over all the tagged partitions of $[a, c]$ and $[c, b]$

$$V^{p(\cdot)}[\lambda f, [a, c]] + V^{p(\cdot)}[\lambda f, [c, b]] \leq V^{p(\cdot)}[\lambda f, [a, b]].$$

To prove (b), let us consider a tagged partition of $[a, b]$ $\tau_0 = a < \cdots < \tau_m = b$, $x_0, \ldots, x_{m-1}$. There will be some interval, say $[\tau_{j-1}, \tau_j]$, that contains c. By the convexity of the power function $u^{p(\cdot)p_+/p_-}$, $u \geq 0$, there holds

$$\begin{aligned}
&[\lambda|f(\tau_j) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \\
&\quad = [\lambda|f(\tau_j) - f(c) + f(c) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \\
&\quad \leq \frac{1}{2}\Big\{[2\lambda|f(\tau_j) - f(c)|]^{p(x_{j-1})p_+/p_-} + [2\lambda|f(c) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-}\Big\} \\
&\quad \leq 2^{p_+^2/p_- -1}\Big\{[\lambda|f(\tau_j) - f(c)|]^{p(x_{j-1})p_+/p_-} + [\lambda|f(c) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-}\Big\}.
\end{aligned}$$

Now if, for example, $x_{j-1} \in [\tau_{j-1}, c[$, taking into account that $\lambda|f(\tau_j) - f(c)| \leq 1$ and $\lambda|f(c) - f(\tau_{j-1})| \leq 1$, then for some $\bar{x}_c \in [c, \tau_j]$, we have

$$\begin{aligned}
&[\lambda|f(\tau_j) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \\
&\quad \leq 2^{p_+^2/p_- -1}\Big\{[\lambda|f(\tau_j) - f(c)|]^{p(\bar{x}_c)} + [\lambda|f(c) - f(\tau_{j-1})|]^{p(x_{j-1})}\Big\},
\end{aligned}$$

since $p(x_{j-1})/p_- \geq 1$ and $p_+ \geq p(\bar{x}_c)$. Taking into account that $\lambda|f(\tau_i) - f(\tau_{i-1})| \leq 1$, for every i, and that $p(\cdot)p_+/p_- \geq p(\cdot)$, this implies that

$$\begin{aligned}
&\sum_{i=1}^{m}[\lambda|f(\tau_i) - f(\tau_{i-1})|]^{p(x_{j-1})p_+/p_-} \\
&\quad = \sum_{i\neq j}[\lambda|f(\tau_i) - f(\tau_{i-1})|]^{p(x_{i-1})p_+/p_-} + [\lambda|f(\tau_j) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \\
&\quad \leq 2^{p_+^2/p_- -1}\sum_{i\neq j}[\lambda|f(\tau_i) - f(\tau_{i-1})|]^{p(x_{i-1})} \\
&\qquad + 2^{p_+^2/p_- -1}\Big\{[\lambda|f(\tau_j) - f(c)|]^{p(\bar{x}_c)} + [\lambda|f(c) - f(\tau_{j-1})|]^{p(x_{j-1})}\Big\} \\
&\quad \leq 2^{p_+^2/p_- -1}\Big\{V^{p(\cdot)}[\lambda f, [a, c]] + V^{p(\cdot)}[\lambda f, [c, b]]\Big\},
\end{aligned}$$

and the thesis follows passing to the supremum over all the tagged partitions of $[a, b]$. □

The following Proposition is a generalization of the previous result in case of functions that vanish on a partition of $[a, b]$.

Proposition 3.3. *Let $f \in BV^{p(\cdot)}([a,b])$ and let $t_0 = a < t_1 < \cdots < t_n = b$ be a partition of $[a,b]$. Then, for some $\lambda > 0$*

(a) $\sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]] \leq V^{p(\cdot)}[\lambda f, [a,b]]$;

(b) $V^{p(\cdot)}[\lambda f, [a,b]] \leq n^{p_+^2/p_- -1} \sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]]$;

(c) *if, in addition, $f(t_i) = 0$ for every $i = 0, \ldots, n$,* $V^{p(\cdot)p_+/p_-}[\lambda f, [a,b]] \leq 2^{p_+^2/p_- -1} \sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]]$.

Proof. Let $\lambda > 0$ be given by Proposition 3.1. Part (a) follows with analogous reasonings to (a) of Proposition 3.2.

To prove (b), let us consider a tagged partition of $[a,b]$ $\tau_0 = a < \cdots < \tau_m = b$, $x_0, \ldots, x_{m-1}$. There will be some intervals $[\tau_{j-1}, \tau_j]$ that contain some t_i, say $\tau_{j-1} \leq t_i < \cdots < t_{i+\nu_j} \leq \tau_j$: for such intervals, there holds, by the convexity of the power function $u^{p(\cdot)p_+/p_-}$, $u \geq 0$

$$\begin{aligned}
&[\lambda|f(\tau_j) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \\
&\quad \leq \frac{1}{\nu_j + 1}\Big\{[(\nu_j+1)\lambda|f(\tau_{j-1}) - f(t_i)|]^{p(x_{j-1})p_+/p_-} \\
&\qquad + [(\nu_j+1)\lambda|f(t_i) - f(t_{i+1})|]^{p(x_{j-1})p_+/p_-} \\
&\qquad + \cdots + [(\nu_j+1)\lambda|f(t_{i+\nu_j}) - f(\tau_j)|]^{p(x_{j-1})p_+/p_-}\Big\} \qquad (3.1)\\
&\quad \leq (\nu_j+1)^{p_+^2/p_- -1}\Big\{[\lambda|f(\tau_{j-1}) - f(t_i)|]^{p(x_{j-1})p_+/p_-} \\
&\qquad + [\lambda|f(t_i) - f(t_{i+1})|]^{p(x_{j-1})p_+/p_-} \\
&\qquad + \cdots + [\lambda|f(t_{i+\nu_j}) - f(\tau_j)|]^{p(x_{j-1})p_+/p_-}\Big\}.
\end{aligned}$$

Now, taking into account that $\lambda|f(\tau_{j-1}) - f(t_i)| \leq 1$, $\lambda|f(t_i) - f(t_{i+1})| \leq 1, \ldots, \lambda|f(t_{i+\nu_j}) - f(\tau_j)| \leq 1$, then

$$\begin{aligned}
&[\lambda|f(\tau_j) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \leq (\nu_j+1)^{p_+^2/p_- -1}\Big\{[\lambda|f(\tau_{j-1}) - f(t_i)|]^{p(\bar{x}_{i-1})} \\
&\quad + [\lambda|f(t_{i+1}) - f(t_i)|]^{p(\bar{x}_i)} + \cdots + [\lambda|f(t_{i+\nu_j}) - f(\tau_j)|]^{p(\bar{x}_{i+\nu_j})}\Big\}
\end{aligned}$$

for some $\bar{x}_{i-1} \in [\tau_{j-1}, t_i]$, $\bar{x}_{k-1} \in [t_{k-1}, t_k]$, $k = i+1, \ldots, i+\nu_j$, $\bar{x}_{i+\nu_j} \in [t_{i_j+\nu}, \tau_j]$, since $p(x_{j-1})/p_- \geq 1$ for every $j = 1, \ldots, n$, and $p_+ \geq p(\cdot)$. This implies that

$$\begin{aligned}
&[\lambda|f(\tau_j) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \\
&\quad \leq (\nu_j+1)^{p_+^2/p_- -1}\Big\{V^{p(\cdot)}[\lambda f, [\tau_{j-1}, t_i]] + \cdots + V^{p(\cdot)}[\lambda f, [t_{i+\nu_j}, \tau_j]]\Big\};
\end{aligned}$$

hence, summing over $j = 1, \ldots, m$, applying (a), and taking into account that $\nu_j \leq n-1$, for every j

$$\sum_{j=1}^{m} [\lambda|f(\tau_j) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \leq n^{p_+^2/p_- -1} \sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]].$$

Therefore, the inequality follows passing to the supremum over all the tagged partitions of $[a,b]$.

To prove (c), one can proceed as in the previous case, for a tagged partition of $[a,b]$ $\tau_0 = a < \cdots < \tau_m = b$, $x_0, \ldots, x_{m-1}$. Then, for the intervals $[\tau_{j-1}, \tau_j]$ that contain some t_i, say $\tau_{j-1} \le t_i < \cdots < t_{i+\nu_j} \le \tau_j$, the estimate (3.1) can be replaced by the following:

$$\begin{aligned}
&[\lambda|f(\tau_j) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \\
&\quad\le \frac{1}{2}\left\{[2\lambda|f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} + [2\lambda|f(\tau_j)|]^{p(x_{j-1})p_+/p_-}\right\} \\
&\quad\le 2^{p_+^2/p_- -1}\left\{[\lambda|f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} + [\lambda|f(\tau_j)|]^{p(x_{j-1})p_+/p_-}\right\} \\
&\quad= 2^{p_+^2/p_- -1}\left\{[\lambda|f(\tau_{j-1}) - f(t_i)|]^{p(x_{j-1})p_+/p_-}\right. \\
&\qquad+ [\lambda|f(t_i) - f(t_{i+1})|]^{p(x_{j-1})p_+/p_-} \\
&\qquad\left.+ \cdots + [\lambda|f(t_{i+\nu_i}) - f(\tau_j)|]^{p(x_{j-1})p_+/p_-}\right\}.
\end{aligned}$$

Now, with similar reasonings as before, it is possible to conclude that

$$\sum_{j=1}^{m}[\lambda|f(\tau_j) - f(\tau_{j-1})|]^{p(x_{j-1})p_+/p_-} \le 2^{p_+^2/p_- -1}\sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]],$$

and the inequality follows passing to the supremum over all the tagged partitions of $[a,b]$.

□

As an immediate consequence of the previous Proposition, we have the following:

Corollary 3.4. *If $f \in BV^{p(\cdot)}(\mathbb{R})$ and $t_0 < t_1 < \cdots < t_n$ is an increasing sequence in $\mathbb{R}$, such that $f(t_i) = 0$ for every $i = 0, \ldots, n$, then, for some $\lambda > 0$*

(a) $V^{p(\cdot)}[\lambda f, (-\infty, t_0]] + \sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]] + V^{p(\cdot)}[\lambda f, [t_n, +\infty)] \le V^{p(\cdot)}[\lambda f]$;

(b) $V^{p(\cdot)p_+/p_-}[\lambda f] \le 2^{p_+^2/p_- -1}\Big\{V^{p(\cdot)}[\lambda f, (-\infty, t_0]] + \sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]] + V^{p(\cdot)}[\lambda f, [t_n, +\infty)]\Big\}$.

Another classical result for the variation that can be extended to the $p(\cdot)$-variation is the subadditivity with respect to functions:

Proposition 3.5. *If $f_1, \ldots, f_m \in BV^{p(\cdot)}(\mathbb{R})$, $m \in \mathbb{N}$, then $f_1 + \cdots + f_m \in BV^{p(\cdot)}(\mathbb{R})$ and, for some $\lambda > 0$*

$$V^{p(\cdot)}[\lambda(f_1 + \cdots + f_m)] \le m^{p_+ -1}\Big\{V^{p(\cdot)}[\lambda f_1] + \cdots + V^{p(\cdot)}[\lambda f_m]\Big\}.$$

Proof. Let $\lambda > 0$ be such that $V^{p(\cdot)}[\lambda f_i] < +\infty$, for every $i = 1, \ldots, m$, and let $t_0 < t_1 < \cdots < t_n$, $x_0 < \cdots < x_{n-1}$ be a tagged sequence. Then, by the monotonicity and the convexity of the power function $u^{p(\cdot)}$, $u \ge 0$

$$\sum_{i=1}^{n} |\lambda(f_1 + \cdots + f_m)(t_i) - \lambda(f_1 + \cdots + f_m)(t_{i-1})|^{p(x_{i-1})}$$

$$\leq \sum_{i=1}^{n} [\lambda|f_1(t_i) - f_1(t_{i-1})| + \cdots + \lambda|f_m(t_i) - f_m(t_{i-1})|]^{p(x_{i-1})}$$
$$\leq \sum_{i=1}^{n} \left\{ \frac{1}{m} \left[m\lambda|f_1(t_i) - f_1(t_{i-1})| \right]^{p(x_{i-1})} + \cdots \right.$$
$$\left. + \frac{1}{m} \left[m\lambda|f_m(t_i) - f_m(t_{i-1})| \right]^{p(x_{i-1})} \right\}$$
$$\leq m^{p_+ - 1} \left\{ V^{p(\cdot)}[\lambda f_1] + \cdots + V^{p(\cdot)}[\lambda f_m] \right\},$$

and the thesis follows passing to the supremum over all the tagged sequences in $\mathbb{R}$. □

Finally, we prove a relation between the $p(\cdot)$-variations of a function and its shifted version $\tau_t f(u) := f(u - t)$, $t, u \in \mathbb{R}$, that will be fundamental to work with the convolution integral operators.

Proposition 3.6. *For every* $t \in \mathbb{R}$

$$V^{p(\cdot)}[\tau_t f] = V^{\tau_{-t} p(\cdot)}[f].$$

Therefore, $\tau_t f \in BV^{p(\cdot)}(\mathbb{R})$, $t \in \mathbb{R}$, *if and only if* $f \in BV^{\tau_{-t} p(\cdot)}(\mathbb{R})$.

Proof. Let $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$ be a fixed tagged sequence: then, for $t \in \mathbb{R}$, $s_0 - t < s_1 - t < \cdots < s_n - t$, $x_0 - t < \cdots < x_{n-1} - t$ is again a tagged sequence. Therefore

$$\sum_{i=1}^{n} |f(s_i - t) - f(s_{i-1} - t)|^{p(x_{i-1})} = \sum_{i=1}^{n} |f(s_i - t) - f(s_{i-1} - t)|^{p((x_{i-1} - t) + t)}$$
$$\leq V^{p(\cdot + t)}[f],$$

and so, passing to the supremum over all the tagged sequences in $\mathbb{R}$

$$V^{p(\cdot)}[\tau_t f] \leq V^{\tau_{-t} p(\cdot)}[f].$$

On the other side, if $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$ is a tagged sequence, so is $s_0 + t < s_1 + t < \cdots < s_n + t$, $x_0 + t < \cdots < x_{n-1} + t$, and therefore

$$\sum_{i=1}^{n} |f(s_i) - f(s_{i-1})|^{p(x_{i-1} + t)} = \sum_{i=1}^{n} |f((s_i + t) - t) - f((s_{i-1} + t) - t)|^{p(x_{i-1} + t)}$$
$$\leq V^{p(\cdot)}[\tau_t f].$$

Again passing to the supremum over all the tagged sequences, we have

$$V^{\tau_{-t} p(\cdot)}[f] \leq V^{p(\cdot)}[\tau_t f],$$

and the result is proved. □

Example. The previous Proposition suggests an important difference between the variable variation and the classical notion of variation: the space $BV^{p(\cdot)}(\mathbb{R})$

is not invariant under translation, as the classical BV-spaces. Indeed, let us consider for example the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x, & x = \frac{1}{k},\ k = 1, 2, \dots, \\ 0, & \text{otherwise}, \end{cases}$$

and the admissible function

$$p(x) = \begin{cases} 2, & x \in [0, 1], \\ 1, & \text{otherwise}. \end{cases}$$

Then, $f \in BV^{p(\cdot)}(\mathbb{R})$, since

$$V^{p(\cdot)}[f] = 2 \sum_{k=1}^{+\infty} \left| f\left(\frac{1}{k}\right) \right|^2 = 2 \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty.$$

Nevertheless, if we consider $h \in \mathbb{R}$, such that $|h| > 1$, then by Proposition 3.6, for every $\lambda > 0$

$$V^{p(\cdot)}[\lambda \tau_h f] = V^{p(\cdot - h)}[\lambda f] = \sup_{\Pi^*} \sum_{i=1}^{n} |\lambda f(t_i) - \lambda f(t_{i-1})|^{p(x_i - h)} = 2 \sum_{k=1}^{+\infty} \frac{\lambda}{k} = +\infty,$$

taking into account that, for every $x_i \in [0, 1]$, $x_i - h \notin [0, 1]$. Therefore, $\tau_h f \notin BV^{p(\cdot)}(\mathbb{R})$.

4. Convergence Results by Means of Convolution Operators

We first point out that, if $f \in BV^{p(\cdot)}(\mathbb{R})$, the operators $T_w f$ are well defined. Indeed, it is immediate to see that $f \in BV^{p(\cdot)}(\mathbb{R})$ implies that f is bounded, and therefore, for some $M > 0$

$$|(T_w f)(s)| \leq \int_{\mathbb{R}} |K_w(t)||f(s - t)| \, \mathrm{d}t \leq M \|K_w\|_1 \leq MA,$$

for every $s \in \mathbb{R}$, $w > 0$, by $(K1)$.

The first result will be an estimate in variable variation for $T_w f$, $w > 0$.

Theorem 4.1. *If $f \in BV^{p(\cdot)}(\mathbb{R})$ and $(K1)$ is satisfied, then there exists $\mu > 0$, such that*

$$V^{p_+/p_- p(\cdot)}[\mu T_w f] \leq V^{p(\cdot)}[\lambda f], \tag{4.1}$$

where $\lambda > 0$ is such that $V^{p(\cdot)}[\lambda f] < +\infty$. As a consequence, T_w maps $BV^{p(\cdot)}(\mathbb{R})$ in $BV^{p_+/p_- p(\cdot)}(\mathbb{R})$, for every $w > 0$.

Proof. Let $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$ be a tagged sequence in $\mathbb{R}$. Then, for $\mu > 0$, we have

$$\begin{aligned} S_w &:= \sum_{i=1}^{n} |\mu[(T_w f)(s_i) - (T_w f)(s_{i-1})]|^{p_+/p_- p(x_{i-1})} \\ &= \sum_{i=1}^{n} \left| \mu \int_{\mathbb{R}} K_w(t) f(s_i - t) \, \mathrm{d}t - \mu \int_{\mathbb{R}} K_w(t) f(s_{i-1} - t) \, \mathrm{d}t \right|^{p_+/p_- p(x_{i-1})}. \end{aligned}$$

Now, by the convexity of the function $u^{p_+/p_-p(\cdot)}$, $u \geq 0$, by $(K1)$ and Jensen's inequality

$$\begin{aligned} S_w &\leq A^{-1} \int_{\mathbb{R}} |K_w(t)| \sum_{i=1}^{n} [\mu A|f(s_i - t) - f(s_{i-1} - t)|]^{p_+/p_-p(x_{i-1})} \, \mathrm{d}t \\ &\leq A^{-1} \int_{\mathbb{R}} |K_w(t)| V^{p_+/p_-p(\cdot)}[\mu A \tau_t f] \, \mathrm{d}t, \end{aligned}$$

and so, by Proposition 3.6

$$S_w \leq A^{-1} \int_{\mathbb{R}} |K_w(t)| V^{p_+/p_-p(\cdot+t)}[\mu A f] \, \mathrm{d}t.$$

Since, for every $t \in \mathbb{R}$, $p_+/p_-p(\cdot + t) \geq p(\cdot)/p_-p(\cdot + t) \geq p(\cdot)$, then by (2.1) and $(K1)$

$$S_w \leq A^{-1} \int_{\mathbb{R}} |K_w(t)| V^{p(\cdot)}[\mu A f] \, \mathrm{d}t \leq V^{p(\cdot)}[\mu A f].$$

Therefore, if $0 < \mu < \frac{\lambda}{A}$, passing to the supremum over all the possible tagged sequences in $\mathbb{R}$, we conclude that

$$V^{p_+/p_-p(\cdot)}[\mu T_w f] \leq V^{p(\cdot)}[\lambda f].$$

□

Remark 4.2. In the particular case of $p(\cdot) = 1$, the variable bounded variation reduces to the classical Jordan variation and the estimate of Theorem 4.1 becomes

$$V[T_w f] \leq V[f],$$

that is, the variation-diminishing property for the classical convolution integral operators (see, e.g., [8]). In the case $p(\cdot) \equiv p$, $p > 1$, the variable bounded variation coincides with the Wiener p-variation and the estimate of Theorem 4.1 reduces to the variation-diminishing property for the Wiener p-variation (see, e.g., [21] for the generalization of such result in the case of the Musielak–Orlicz φ-variation). We point out that, in all these cases, with "classical" notions of variation, the convolution integral operators map the space of functions of bounded variation (in the sense of Jordan, Wiener...) in itself, while here they map $BV^{p(\cdot)}(\mathbb{R})$ into $BV^{p_+/p_-p(\cdot)}(\mathbb{R})$. Actually, this is natural, in the setting of variable exponent spaces: for instance, similar phenomena occur working in variable Lebesgue spaces, that are not translation invariant (see, e.g., [16]).

To prove the main convergence theorem, a crucial step is a result about the convergence of the modulus of smoothness of an absolutely continuous function: the modulus of smoothness, in the context of $BV^{p(\cdot)}$-spaces, for $f : \mathbb{R} \to \mathbb{R}$, is defined as

$$\omega^{p(\cdot)}(f, \delta) := \sup_{|t| \leq \delta} V^{p(\cdot)}[\tau_t f - f], \quad \delta > 0.$$

We will now prove some preliminary results. The first proposition guarantees the possibility to approximate a $BV^{p(\cdot)}$-function by means of a step function.

Proposition 4.3. *If $f \in AC^{p(\cdot)}(\mathbb{R})$, there exists $\lambda > 0$, such that, for every $\epsilon > 0$, there exist $a, b \in \mathbb{R}$ and $\delta > 0$, such that, if $t_0 = a < t_1 < \cdots < t_n = b$ is a partition of $[a, b]$ with $t_i - t_{i-1} < \delta$ for every $i = 1, \ldots, n$, then*

(a) $V^{p(\cdot)}[\lambda f, (-\infty, a]] < \epsilon$ *and* $V^{p(\cdot)}[\lambda f, [b, +\infty)] < \epsilon$;
(b) $\sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]] < \epsilon$;
(c) *the step functions $\nu_1, \nu_2 : \mathbb{R} \to \mathbb{R}$ defined as*

$$\nu_1(t) := \begin{cases} f(a), & t < a, \\ f(t_{i-1}), & t_{i-1} \le t < t_i, \\ f(b), & t \ge b, \end{cases}$$

$$\nu_2(t) := \begin{cases} f(a), & t \le a, \\ f(t_i), & t_{i-1} < t \le t_i, \\ f(b), & t > b, \end{cases}$$

are such that $V^{p(\cdot)p_+/p_-}[\lambda(f - \nu_1)] < \epsilon$ *and* $V^{p(\cdot)p_+/p_-}[\lambda(f - \nu_2)] < \epsilon$.

Proof. About (a), it is sufficient to recall that $f \in AC^{p(\cdot)}(\mathbb{R})$ implies in particular $f \in BV^{p(\cdot)}(\mathbb{R})$, and hence, there exists $\lambda > 0$, such that $V^{p(\cdot)}[\lambda f] < +\infty$. Since $V^{p(\cdot)}[\lambda f] = \lim_{n\to+\infty} V^{p(\cdot)}[\lambda f, [-x_n, x_n]]$ where $(x_n)_n$ is an increasing sequence in $\mathbb{R}$, then by Corollary 3.4

$$V^{p(\cdot)}[\lambda f, (-\infty, -x_n]] + V^{p(\cdot)}[\lambda f, [x_n, +\infty)] \le V^{p(\cdot)}[\lambda f] - V^{p(\cdot)}[\lambda f, [-x_n, x_n]] \to 0,$$

as $n \to +\infty$. Therefore

$$\lim_{n\to+\infty} V^{p(\cdot)}[\lambda f, (-\infty, -x_n]] = \lim_{n\to+\infty} V^{p(\cdot)}[\lambda f, [x_n, +\infty)] = 0,$$

that implies (a).

To prove (b), it is sufficient to notice that, by the $p(\cdot)$-absolute continuity of f, for some $\lambda > 0$, in correspondence to $\epsilon > 0$, there exists $\delta > 0$, such that $\sum_{i=1}^{n} |\lambda[f(t_i) - f(t_{i-1})]|^{p(x_{i-1})} < \epsilon$, for every tagged sequence $t_0 < t_1 < \cdots < t_n$, such that $t_i - t_{i-1} < \delta$, for every i. Therefore, if one considers a tagged sequence $\tau_0^i < \cdots < \tau_{m_i}^i$, $y_0^i, \ldots, y_{m_i-1}^i$ in each interval $[t_{i-1}, t_i]$, there holds

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} [\lambda[f(\tau_j^i) - f(\tau_{j-1}^i)]|^{p(y_{j-1}^i)} < \epsilon,$$

and the thesis follows passing to the supremum over all the possible tagged sequences in $[t_{i-1}, t_i]$.

Let us now prove (c). In correspondence to $\epsilon > 0$, let $\lambda > 0$, $a, b \in \mathbb{R}$ and $\delta > 0$ be given by (a) and (b), so that, if $t_0 = a < t_1 < \cdots < t_n = b$ is a partition of $[a, b]$ with $t_i - t_{i-1} < \delta$ for every $i = 1, \ldots, n$, then

(i) $V^{p(\cdot)}[\lambda f, (-\infty, a]] < \frac{\epsilon}{2^{p_+^2/p_- + 1}} (< \epsilon)$ and $V^{p(\cdot)}[\lambda f, [b, +\infty)] < \frac{\epsilon}{2^{p_+^2/p_- + 1}} (< \epsilon)$;
(ii) $\sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]] < \frac{\epsilon}{2^{p_+^2/p_- + p_+}} (< \epsilon)$.

By Corollary 3.4 (taking into account that $(f - \nu_k)(t_i) = 0$ for every $i = 0, \ldots, n$, $k = 1, 2$)

$$\begin{aligned}
V^{p(\cdot)p_+/p_-}[\lambda(f-\nu_k)] &\le 2^{p_+^2/p_- - 1}\Big\{V^{p(\cdot)}[\lambda(f-\nu_k), (-\infty, a]] \\
&\quad + \sum_{i=1}^{n} V^{p(\cdot)}[\lambda(f-\nu_k), [t_{i-1}, t_i]] \\
&\quad + V^{p(\cdot)}[\lambda(f-\nu_k), [b, +\infty)]\Big\} \\
&= 2^{p_+^2/p_- - 1}\Big\{V^{p(\cdot)}[\lambda f, (-\infty, a]] \\
&\quad + \sum_{i=1}^{n} V^{p(\cdot)}[\lambda(f-\nu_k), [t_{i-1}, t_i]] \\
&\quad + V^{p(\cdot)}[\lambda f, [b, +\infty)]\Big\} \\
&< 2^{p_+^2/p_- - 1}\left\{\frac{\epsilon}{2^{p_+^2/p_-}} + \sum_{i=1}^{n} V^{p(\cdot)}[\lambda(f-\nu_k), [t_{i-1}, t_i]]\right\}.
\end{aligned}$$

By Proposition 3.5, and since obviously $V^{p(\cdot)}[\lambda\nu_k, [t_{i-1}, t_i]] \le V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]]$, we have that

$$\begin{aligned}
V^{p(\cdot)}[\lambda(f-\nu_k), [t_{i-1}, t_i]] &\le 2^{p_+ - 1}\{V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]] + V^{p(\cdot)}[\lambda\nu_k, [t_{i-1}, t_i]]\} \\
&\le 2^{p_+} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]],
\end{aligned}$$

for every $i = 1, \ldots, n$. Therefore, by (ii)

$$V^{p(\cdot)p_+/p_-}[\lambda(f-\nu_k)] < 2^{p_+^2/p_- - 1}\left\{\frac{\epsilon}{2^{p_+^2/p_-}} + 2^{p_+}\frac{\epsilon}{2^{p_+^2/p_- + p_+}}\right\} = \epsilon,$$

$k = 1, 2$. □

Theorem 4.4. *If $f \in AC^{p(\cdot)}(\mathbb{R})$, then there exists $\lambda > 0$, such that*

$$\lim_{t\to 0^-} V^{p(\cdot)p_+/p_-}[\lambda(\tau_t\nu_1 - \nu_1)] = 0, \qquad \lim_{t\to 0^+} V^{p(\cdot)p_+/p_-}[\lambda(\tau_t\nu_2 - \nu_2)] = 0,$$

where ν_1 and ν_2 are defined as in Proposition 4.3.

Proof. Let us fix $\epsilon > 0$; by Proposition 4.3, there exist $\lambda, \delta > 0$, $a, b \in \mathbb{R}$ and two step functions $\nu_1 : \mathbb{R} \to \mathbb{R}$, $\nu_2 : \mathbb{R} \to \mathbb{R}$, such that

$$\sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]] < \frac{\epsilon}{2^{p_+^2/p_- + p_+ - 1}}, \tag{4.2}$$

where $t_0 = a < t_1 < \cdots < t_n = b$ is a partition of $[a, b]$, such that $t_i - t_{i-1} < \delta$, for every $i = 1, \ldots, n$, and $V^{p(\cdot)p_+/p_-}[\lambda(f - \nu_k)] < \epsilon$, $k = 1, 2$.

Let now $0 < \beta < \min_{i=1,\ldots,n}\{t_i - t_{i-1}\}$. If $-\beta < t < 0$, then $t_{i-1} < t_{i-1} - t < t_i$ and $\tau_t\nu_1(t_{i-1}) = \nu_1(t_{i-1} - t) = \nu_1(t_{i-1}) = f(t_{i-1})$: therefore, $V^{p(\cdot)}[\lambda\tau_t\nu_1, [t_{i-1}, t_i]] = V^{p(\cdot)}[\lambda\nu_1, [t_{i-1}, t_i]]$. Then, using (*c*) of Proposition 3.3, Proposition 3.5, and Eq. (4.2), we obtain

$$V^{p(\cdot)p_+/p_-}[\lambda(\tau_t\nu_1 - \nu_1), [a, b]] \le 2^{p_+^2/p_- - 1}\sum_{i=1}^{n} V^{p(\cdot)}[\lambda(\tau_t\nu_1 - \nu_1), [t_{i-1}, t_i]]$$

$$\leq 2^{p_+^2/p_- + p_+ -2} \sum_{i=1}^{n} \Big\{ V^{p(\cdot)}[\lambda \tau_t \nu_1, [t_{i-1}, t_i]]$$
$$+ V^{p(\cdot)}[\lambda \nu_1, [t_{i-1}, t_i]] \Big\}$$
$$= 2^{p_+^2/p_- + p_+ -1} \sum_{i=1}^{n} V^{p(\cdot)}[\lambda \nu_1, [t_{i-1}, t_i]]$$
$$\leq 2^{p_+^2/p_- + p_+ -1} \sum_{i=1}^{n} V^{p(\cdot)}[\lambda f, [t_{i-1}, t_i]] < \epsilon.$$

Moreover, taking into account that $\nu_1(u) = \tau_t \nu_1(u) = f(a)$, for every $u \leq a$, and $\nu(u) = \tau_t \nu(u) = f(b)$, for every $u \geq b$

$$V^{p(\cdot)p_+/p_-}[\lambda(\tau_t \nu_1 - \nu_1)] = V^{p(\cdot)p_+/p_-}[\lambda(\tau_t \nu_1 - \nu_1), [a, b]] < \epsilon.$$

The proof of the other limit relation for ν_2 follows with analogous reasonings. □

We will now prove a result of convergence for the modulus of smoothness in case of $AC^{p(\cdot)}$-functions.

Theorem 4.5. *If $f \in AC^{p(\cdot)}(\mathbb{R})$, then for some $\lambda > 0$, there holds*

$$\lim_{\delta \to 0^+} \omega^{p(\cdot)p_+^2/p_-^2}(\lambda f, \delta).$$

Proof. We will prove that

$$\lim_{t \to 0} V^{p(\cdot)p_+^2/p_-^2}[\lambda(\tau_t f - f)] = 0,$$

that is equivalent to the thesis. For a fixed $\epsilon > 0$, by Proposition 4.3, there exist $\lambda, \delta > 0$, $a, b \in \mathbb{R}$ and two step functions $\nu_1 : \mathbb{R} \to \mathbb{R}$ and $\nu_2 : \mathbb{R} \to \mathbb{R}$ (associated with a partition of $[a, b]$ with mesh not greater than δ), such that

$$V^{p(\cdot)p_+/p_-}[\lambda(f - \nu_k)] < \frac{\epsilon}{3^{p_+}}, \quad k = 1, 2. \tag{4.3}$$

If $t < 0$, by Proposition 3.5

$$V^{p(\cdot)p_+^2/p_-^2}[\lambda(\tau_t f - f)] \leq 3^{p_+ -1} \Big\{ V^{p(\cdot)p_+^2/p_-^2}[\lambda(\tau_t f - \tau_t \nu_1)]$$
$$+ V^{p(\cdot)p_+^2/p_-^2}[\lambda(\tau_t \nu_1 - \nu_1)] + V^{p(\cdot)p_+^2/p_-^2}[\lambda(\nu_1 - f)] \Big\}.$$

By Eqs. (2.1) and (4.3), there holds

$$V^{p(\cdot)p_+^2/p_-^2}[\lambda(\nu_1 - f)] \leq V^{p(\cdot)p_+/p_-}[\lambda(\nu_1 - f)] < \frac{\epsilon}{3^{p_+}},$$

and by Theorem 4.4

$$V^{p(\cdot)p_+^2/p_-^2}[\lambda(\tau_t \nu_1 - \nu_1)] \leq V^{p(\cdot)p_+/p_-}[\lambda(\tau_t \nu_1 - \nu_1)] < \frac{\epsilon}{3^{p_+}},$$

for sufficiently small $t < 0$. Finally, by Proposition 3.6 and (2.1)

$$V^{p(\cdot)p_+^2/p_-^2}[\lambda(\tau_t f - \tau_t \nu_1)] = V^{p(\cdot + t)p_+^2/p_-^2}[\lambda(f - \nu_1)] \leq V^{p(\cdot)p_+/p_-}[\lambda(f - \nu_1)].$$

Therefore, we conclude that for $t < 0$ small enough

$$V^{p(\cdot)p_+^2/p_-^2}[\lambda(\tau_t f - f)] \leq 3^{p_+ - 1}\frac{3\epsilon}{3^{p_+}} = \epsilon.$$

Now, replacing ν_1 by ν_2, it is possible to prove that, for $t > 0$ sufficiently small

$$V^{p(\cdot)p_+^2/p_-^2}[\lambda(\tau_t f - f)] \leq \epsilon,$$

and the proof is complete. □

We are now ready to prove the main result about convergence in $p(\cdot)$-variation.

Theorem 4.6. *Let $f \in AC^{p(\cdot)}(\mathbb{R})$. If $(K1)$ and $(K2)$ are satisfied, then*

$$\lim_{w \to +\infty} V^{p_+^2/p_-^2 p(\cdot)}[\lambda(T_w f - f)] = 0,$$

for some $\lambda > 0$.

Proof. For a fixed tagged sequence $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$ and $\lambda > 0$, there holds

$$\begin{aligned} S_w &:= \sum_{i=1}^{n} |\lambda[(T_w f - f)(s_i) - (T_w f - f)(s_{i-1})]|^{p_+^2/p_-^2 p(x_{i-1})} \\ &= \sum_{i=1}^{n} \Bigg|\lambda \int_{\mathbb{R}} K_w(t)[f(s_i - t) - f(s_i)]\,\mathrm{d}t \\ &\quad -\lambda \int_{\mathbb{R}} K_w(t)[f(s_{i-1} - t) - f(s_{i-1})]\,\mathrm{d}t\Bigg|^{p_+^2/p_-^2 p(x_{i-1})} \\ &\leq \sum_{i=1}^{n} \Bigg[\lambda \int_{\mathbb{R}} |K_w(t)||[f(s_i - t) - f(s_i)] \\ &\quad - [f(s_{i-1} - t) - f(s_{i-1})]|\,\mathrm{d}t\Bigg]^{p_+^2/p_-^2 p(x_{i-1})}, \end{aligned}$$

taking into account of $(K1)$. Now, by Jensen's inequality and $(K1)$

$$\begin{aligned} S_w &\leq A^{-1} \int_{\mathbb{R}} |K_w(t)| \sum_{i=1}^{n} \Bigg|\lambda A\Big\{[f(s_i - t) - f(s_i)] \\ &\quad - [f(s_{i-1} - t) - f(s_{i-1})]\Big\}\Bigg|^{p_+^2/p_-^2 p(x_{i-1})}\,\mathrm{d}t \\ &= A^{-1}\left\{\int_{|t|\leq\delta} + \int_{|t|>\delta}\right\} |K_w(t)| \sum_{i=1}^{n} \Bigg|\lambda A\Big\{[f(s_i - t) - f(s_i)] \\ &\quad - [f(s_{i-1} - t) - f(s_{i-1})]\Big\}\Bigg|^{p_+^2/p_-^2 p(x_{i-1})}\,\mathrm{d}t \\ &:= I_1^\delta + I_2^\delta, \end{aligned}$$

for every $\delta > 0$. Let us fix $\epsilon > 0$. By Theorem 4.5 there exist $\bar{\lambda}, \bar{\delta} > 0$ such that $\omega^{p_+^2/p_-^2 p(\cdot)}(\bar{\lambda} f, \bar{\delta}) < \frac{\epsilon}{2}$, and so, in correspondence of $\bar{\delta}$, for $0 < \lambda < \bar{\lambda} A^{-1}$

$$I_1^{\bar{\delta}} \leq A^{-1} \int_{|t| \leq \bar{\delta}} |K_w(t)| V^{p_+^2/p_-^2 p(\cdot)}[\lambda A(\tau_t f - f)] \, \mathrm{d}t$$
$$\leq \omega^{p_+^2/p_-^2 p(\cdot)}(\lambda A f, \bar{\delta}) A^{-1} \int_{|t| \leq \bar{\delta}} |K_w(t)| \, \mathrm{d}t < \frac{\epsilon}{2},$$

by $(K1)$.

About I_2^{δ}, by Proposition 3.5 and Eq. (2.1)

$$I_2^{\delta} \leq A^{-1} 2^{p_+ - 1} \int_{|t| > \delta} |K_w(t)| \left\{ V^{p_+^2/p_-^2 p(\cdot)}[\lambda A \tau_t f] + V^{p_+^2/p_-^2 p(\cdot)}[\lambda A f] \right\} \mathrm{d}t$$
$$\leq A^{-1} 2^{p_+} V^{p_+/p_- p(\cdot)}[\lambda A f] \int_{|t| > \delta} |K_w(t)| \, \mathrm{d}t$$
$$\leq A^{-1} 2^{p_+} V^{p(\cdot)}[\mu f] \int_{|t| > \delta} |K_w(t)| \, \mathrm{d}t,$$

for $0 < \lambda < \mu A^{-1}$, where μ is such that $V^{p(\cdot)}[\mu f] < +\infty$.[2] By assumption $(K2)$, there exists $\tilde{w} > 0$ such that $\int_{|t| > \bar{\delta}} |K_w(t)| \, \mathrm{d}t < \frac{\epsilon}{A^{-1} 2^{p_+ + 1} V^{p(\cdot)}[\mu f]}$, for every $w \geq \tilde{w}$, and so

$$I_2^{\bar{\delta}} < \frac{\epsilon}{2}.$$

Therefore, if $0 < \lambda < \min\{\bar{\lambda} A^{-1}, \mu A^{-1}\}$

$$S_w \leq I_1^{\bar{\delta}} + I_2^{\bar{\delta}} < \epsilon,$$

for sufficiently large $w > 0$, and the thesis follows passing to the supremum over all the possible tagged sequences in $\mathbb{R}$. □

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Declarations

Conflict of Interest The first author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), of RITA (Research ITalian network on Approximation) and of the UMI group "Teoria dell'Approssimazione e Applicazioni" and is partially supported by the "Department of Mathematics and Computer Science" of the University of Perugia, by the "Fondo Ricerca di Base" 2019 and 2020 of the University of Perugia, by 2018 (B.I.M.) and 2019 (M.I.R.A.) Projects funded by the Fondazione Cassa di Risparmio

[2] W.l.g. $V^{p(\cdot)}[\mu f] \neq 0$, since otherwise the result is trivial.

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