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Newton-Like Components in the Chebyshev–Halley Family of Degree n Polynomials

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Abstract. We study the Chebyshev–Halley methods applied to the family of polynomials $f_{n,c}(z) = z^n + c$, for $n \geq 2$ and $c \in \mathbb{C}^*$. We prove the existence of parameters such that the immediate basins of attraction corresponding to the roots of unity are infinitely connected. We also prove that, for $n \geq 2$, the corresponding dynamical plane contains a connected component of the Julia set, which is a quasiconformal deformation of the Julia set of the map obtained by applying Newton's method to $f_{n,-1}$.

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1. Introduction

Numerical methods have been extensively used to give accurate approximations of the solutions of systems of nonlinear equations. Those equations or systems of equations correspond to a wide source of scientific models from biology to engineering and from economics to social sciences, and so their solutions are the cornerstone of applied mathematics. One of the most studied families of numerical methods are the so called root-finding algorithms; that is, iterative methods which asymptotically converge to the zeros (or some of the zeros) of the non linear equation, say $g(z) = 0$. Although g could in general describe an arbitrary high dimensional problem, in this paper we focus on the one dimensional case, i.e. $g : \mathbb{C} \to \mathbb{C}$.

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The universal and most studied root-finding algorithm is known as *Newton's method*. If g is holomorphic, we generate a sequence $\{z_n\}_{n\geq 0}$ of approximations of a root of g, using Newton's method, defined as follows

$$z_{n+1} = z_n - \frac{g(z_n)}{g'(z_n)}, \quad z_0 \in \mathbb{C}.$$

It is well known that if $z_0 \in \mathbb{C}$ is chosen close enough to one of the solutions of the equation $g(z) = 0$, say α, then the sequence $\{z_n = g^n(z_0)\}_{n\geq 0}$ has the limit α when n tends to ∞. Moreover the speed of (local) convergence is generically quadratic (see, for instance, [2]). It was Cayley (see [10]) the first to consider Newton's method as a (holomorphic) dynamical system, that is studying the convergence of these sequences for all possible seeds $x_0 \in \mathbb{C}$ at once, under the assumption that g was a degree 2 or 3 polynomial. This was known as *Cayley's problem*.

Many authors have studied alternative iterative methods having, for instance, a better local speed of convergence. Two of the best known root-finding algorithms of order of convergence 3 are *Chebyshev's method* and *Halley's method* (see [2]). They are included in the *Chebyshev–Halley family* of root-finding algorithms, which was introduced in [11] (see also [1]), and is defined as follows. Let g be a holomorphic map. Then

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\,\frac{L_g(z_n)}{1 - \alpha L_g(z_n)}\right)\frac{g(z_n)}{g'(z_n)}, \tag{1}$$

where $\alpha \in [0, 1]$ and $L_g(z) = \frac{g(z)g''(z)}{(g'(z))^2}$. Notice that in a real setting, it suffices for g to be a doubly differentiable function such that $g''(x)$ is continuous.

For $\alpha = 0$, we have Chebyshev's method and for $\alpha = \frac{1}{2}$ Halley's method. As α tends to ∞, the Chebyshev–Halley algorithms tend to Newton's method. The main goal of the paper is to show that the unbounded connected component of the Julia set of the Chebyshev–Halley maps applied to $z^n - c$ (for large enough α) is homeomorphic to the Julia set of the map obtained by applying Newton's method to $z^n - 1$.

Next, we give a brief introduction of complex dynamics for holomorphic maps defined over the Riemann sphere, that is, rational maps. For a more detailed description of the topic, see, for instance, [4] and [15]. Let $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. A point $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if $Q(z_0) = z_0$. The *multiplier* λ of the fixed point $z_0 \in \mathbb{C}$ is $\lambda{:=}Q'(z_0)$ (for $z_0 = \infty$, the multiplier is computed by conjugating with a map which sends $z_0 = \infty$ to a finite point). If $|\lambda| < 1$, then z_0 is an *attracting fixed point*. If $\lambda = 0$, we say that z_0 is a *superattracting fixed point*. The *basin of attraction* of an attracting fixed point z_0 is the set $A_Q(z_0) = \{z \in \hat{\mathbb{C}} | \lim_{n\to\infty} Q^n(z) = z_0\}$. We denote by $A^*_Q(z_0)$ the connected component of the basin of attraction which contains z_0, also known as the *immediate basin of attraction* of z_0. Any immediate basin of attraction contains at least one critical point, that is, a point $c \in \hat{\mathbb{C}}$ such that $Q'(c) = 0$.

The Fatou set $F(Q)$ is defined as the set of points of normality. A point $z \in \hat{\mathbb{C}}$ is said to be normal if the family $\{Q^n\}_{n\geq 1}$ is normal for some neighborhood U of z. A connected component of the Fatou set is called a

Fatou component. The complement of the Fatou set is called the Julia set, denoted by $J(Q)$. Both Fatou and Julia sets are dynamically invariant. The behaviour of Q on its corresponding Julia set $J(Q)$ is chaotic. Moreover, the Julia set $J(Q)$ is non-empty.

Root-finding algorithms are a natural topic for complex dynamics. In particular, maps obtained by applying Newton's method to polynomials are a much studied topic (see [3,13,16,18]). It is proven in [16] that the Julia set of such maps is connected, so all Fatou components are simply connected.

Previously, Campos, Canela, and Vindel have studied the Chebyshev–Halley family applied to $f_{n,c}(z) = z^n + c$, $c \in \mathbb{C}^*$ (see [6,7]). The maps obtained by applying the Chebyshev–Halley family to $f_{n,c}$ are all conjugated to the map obtained by applying the Chebyshev–Halley family to $f_n(z){:=}f_{n,-1}(z) = z^n - 1$ (see Lemma 2.1). By applying the Chebyshev–Halley method to $f_n(z) = z^n - 1$ we obtain the map:

$$O_{n,\alpha}(z) = \frac{(1-2\alpha)(n-1) + (2-4\alpha-4n+6\alpha n-2\alpha n^2)z^n + (n-1)(1-2\alpha-2n+2\alpha n)z^{2n}}{2nz^{n-1}(\alpha(1-n) + (-\alpha-n+\alpha n)z^n)}. \tag{2}$$

The map $O_{n,\alpha}$ has degree $2n$ and it has $4n-2$ critical points, counting multiplicity. The point $z = 0$ is a critical point of multiplicity $n-2$, which is mapped to the fixed point $z = \infty$. The n-th roots of unity are superattracting fixed points of local degree 3, and therefore, they have multiplicity 2 as critical points. This leaves n free critical points. They are given by

$$c_{n,\alpha,\xi} = \xi \left(\frac{\alpha(n-1)^2(2\alpha-1)}{n(2n-1) - \alpha(4n-1)(n-1) + 2\alpha^2(n-1)^2} \right)^{\frac{1}{n}}, \tag{3}$$

where $\xi^n = 1$. This family is symmetric with respect to rotation by the nth root of unity (see Lemma 2.2). This symmetry ties the orbits of the n free critical points, so the family $O_{n,\alpha}$ has only one degree of freedom (see Fig. 2).

In [7], the authors studied in detail the topology of the immediate basins of attraction of the fixed points of $O_{n,\alpha}(z)$ given by the nth root of unity, that is, the zeros of $f_n(z)$. In what follows we refer to these basins as

$$A_{n,\alpha}(\xi){:=}A_{O_{n,\alpha}}(\xi) \, \left[A^*_{n,\alpha}(\xi){:=}A^*_{O_{n,\alpha}}(\xi) \right],$$

where $\xi^n = 1$. For one particular case, the immediate basins of attraction are infinitely connected (see Fig. 1). We study the Julia set of $O_{n,\alpha}$ for this particular case and relate it to the Julia set of the map obtained by applying Newton's method to f_n. We realise this using a quasiconformal surgery construction, which erases the holes in the immediate basins of attraction. The construction is a simpler case of one in [14]. However, realising the surgery is still needed, as we prove the uniqueness of the resulted quasiconformal map, to show that the quasirational map presents the necessary symmetries and is precisely N_{f_n}.

Theorem A. *Fix $n \geq 2$ and assume that $A^*_{n,\alpha}(1)$ is infinitely connected for some $\alpha \in \mathbb{C}$. Then there exists an invariant Julia component Π (which contains $z = \infty$) which is a quasiconformal copy of the Julia set of N_{f_n}, where*

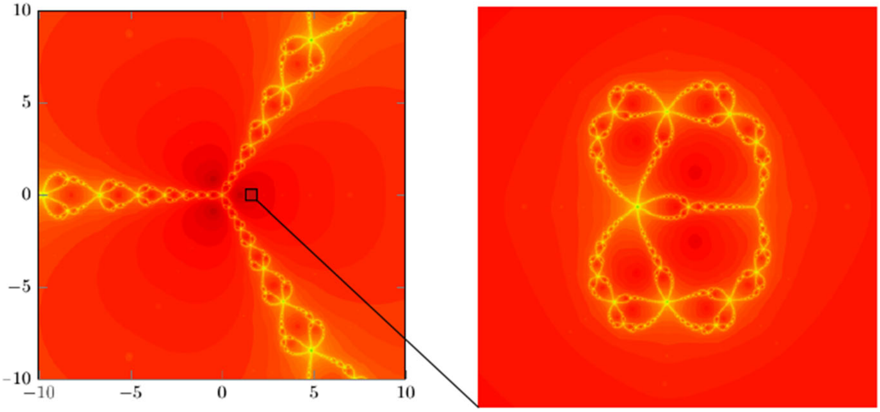


Figure 1. Left figure illustrates the dynamical plane of $O_{n,\alpha}$ for $n = 3$ and $\alpha = 10$. In the right figure (which shows $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \in [1.620; 1.623]$ and $\mathrm{Im}(z) \in [-0.0015; 0.0015]$ in the same dynamical plane), we can see a component of the Julia set which lies in $A^*(1)$

N_{f_n} is the map obtained by applying Newton's method to the polynomial $f_n(z) = z^n - 1$.

We finish the paper by proving that there exist parameters such that the hypothesis of Theorem A holds. We split the proof of Theorem B in two cases, $n = 2$ and $n \geq 3$. For the case $n = 2$, much work was previously done in [6], and the map is conjugate to a Blaschke product. For the case $n \geq 3$, the map is not conjugate to a Blaschke product. We provide a map conjugate to $O_{n,\alpha}$, for which we prove, using various properties and computational arguments, that the immediate basin of attraction of $z = \infty$ is infinitely connected. Numerical computations confirm to us the existence of such hyperbolic components (see Fig. 2).

Theorem B. *Let $n \geq 2$. Then there exists $\alpha_0 > 0$ large enough such that for $\alpha > \alpha_0$, $\alpha \in \mathbb{R}$, $A^*_{n,\alpha}(1)$ is infinitely connected. Moreover, for $n = 2$, the statement is true for any $\alpha \in \mathbb{C}$ such that $|\alpha| > \alpha_0$.*

The paper is organised as follows. In Sect. 2 we briefly introduce the tools later used in the paper. In Sect. 3 we prove Theorem A. In Sect. 4 we prove Theorem B.

2. Preliminaries

In this section we present the main tools that we use along the paper. Before, we introduce some notation. Let $U \subset \mathbb{C}$ be a multiply connected open set. We denote by $\mathrm{Fill}(U)$ the minimal simply connected open set which contains U but not $z = \infty$. Let $\gamma \in \mathbb{C}$ be a Jordan curve. We denote by $\mathrm{Ext}\,(\gamma)$ and $\mathrm{Int}\,(\gamma)$ the connected components of $\hat{\mathbb{C}} \setminus \gamma$ that contain $z = \infty$ and do not

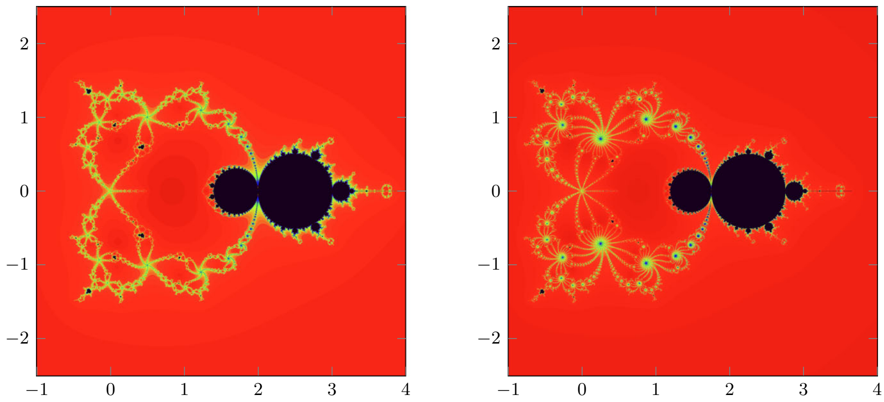


Figure 2. Left figure illustrates the parameter plane of $O_{n,\alpha}$ for $n = 3$. In the right figure we can see the parameter plane of $O_{n,\alpha}$ for $n = 5$

contain $z = \infty$, respectively. We denote the circle centered at the origin and of radius $c > 0$ by $\mathbb{S}_c$. Finally, if $U \subset \mathbb{C}$ we denote by $\overline{U}$ its closure.

Let $O_{n,\alpha,c}$ be the map obtained by applying the Chebyshev–Halley method with parameter α to the map $f_{n,c} = z^n + c$. The following lemma, indicated but not proven in [6], states that for any $c \in \mathbb{C}^*$, the map $O_{n,\alpha,c}$ is conjugated to $O_{n,\alpha,-1} = O_{n,\alpha}$. We give the proof for the sake of completeness.

Lemma 2.1. *Let $c \in \mathbb{C}^*$, $c = re^{i2\pi k}$, where $r > 0$ and $k \in [0, 1]$. Let $u = \sqrt[n]{r}e^{i\frac{(2k+1)\pi}{n}}$ and $\eta_c(z) = uz$, $\eta_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Then $O_{n,\alpha,c}$ and $O_{n,\alpha,-1}$ are conjugated by the map η_c, i.e. $O_{n,\alpha,c} \circ \eta_c(z) = \eta_c \circ O_{n,\alpha,-1}(z)$.*

Proof. First, we need to compute $O_{n,\alpha,c}$. We use the Chebyshev–Halley family of methods definition (see 1), which can be rewritten as

$$O_{n,\alpha,c}(z) = \frac{2zf'_{n,c}(z) - 2f_{n,c}(z) - f_{n,c}(z)L_{f_{n,c}}(z) + 2\alpha L_{f_{n,c}}(z)(f_{n,c}(z) - zf'_{n,c}(z))}{2f'_{n,c}(z)(1 - \alpha L_{f_{n,c}}(z))},$$

where $f_{n,c}(z) = z^n + c$, $f'_{n,c}(z) = nz^{n-1}$, $L_{f_{n,c}}(z) = \frac{(n-1)(z^n+c)}{nz^n}$. This gives us the expression of $O_{n,\alpha,c}$:

$$\begin{aligned} &O_{n,\alpha,c}(z) \\ &\quad = \frac{c^2(1-2\alpha)(n-1) - c(2-4\alpha-4n+6\alpha n-2\alpha n^2)z^n + (n-1)(1-2\alpha-2n+2\alpha n)z^{2n}}{2nz^{n-1}[-c\alpha(1-n) + (-\alpha-n+\alpha n)z^n]}. \end{aligned}$$

Now we prove the conjugation. Observe that $u^n = -c$, and the map η_c maps roots of $f_{n,1}$ to roots of $f_{n,c}$ (therefore, it also maps the superattracting fixed points of $O_{n,\alpha}$ to the superattracting fixed points of $O_{n,\alpha,c}$). Then

$$\begin{aligned} &O_{n,\alpha,c}\left(\eta_c(z)\right) \\ &\quad = \frac{c^2(1-2\alpha)(n-1) + c^2(2-4\alpha-4n+6\alpha n-2\alpha n^2)z^n + c^2(n-1)(1-2\alpha-2n+2\alpha n)z^{2n}}{u^{n-1}2nz^{n-1}[-c\alpha(1-n) - c(-\alpha-n+\alpha n)z^n]} \\ &= u\frac{(1-2\alpha)(n-1) + (2-4\alpha-4n+6\alpha n-2\alpha n^2)z^n + (n-1)(1-2\alpha-2n+2\alpha n)z^{2n}}{2nz^{n-1}[\alpha(1-n) + (-\alpha-n+\alpha n)z^n]} \\ &= \eta_c\left(O_{n,\alpha}(z)\right). \end{aligned}$$

$\square$

The next lemma shows that the map $O_{n,\alpha}$ is symmetric with respect to rotation by an nth root of unity.

Lemma 2.2. [6, Lemma 6.2] *Let $n \in \mathbb{N}$ and let ξ be an nth root of unity, i.e. $\xi^n = 1$. Then $I_\xi(z){:=}\xi z$ conjugates $O_{n,\alpha}$ with itself, i.e. $O_{n,\alpha} \circ I_\xi(z) = I_\xi \circ O_{n,\alpha}(z)$.*

For $\alpha = \frac{1}{2}$ and $\alpha = \frac{2n-1}{2n-2}$, the family $O_{n,\alpha}$ degenerates to maps of a lower degree (see [7, Lemma 4.1]). For other values of α, the map $O_{n,\alpha}$ has degree $2n$, hence, it has $4n-2$ critical points. The point $z=0$ maps with degree $n-1$ to the fixed point $z=\infty$. Since the n roots of f_n are superattracting fixed points of local degree 3, there remain precisely n free critical points. The next lemma follows directly from Lemma 2.2, since the orbits of the free critical points are symmetric.

Lemma 2.3. [7, Lemma 3.4] *Let $n \geq 2$ and $\xi \in \mathbb{C}$, such that $\xi^n = 1$. For all $\alpha \in \mathbb{C}$, the basin of attraction $A_{n,\alpha}(\xi)$ contains at most one critical point other than $z = \xi$.*

The following proposition establishes a trichotomy for rational maps with the property described in Lemma 2.3. Based on the existence of the critical point and preimages of the superattracting fixed point in the immediate basin of attraction, we can establish if the immediate basin is simply connected.

Proposition 2.4. [7, Proposition 3.1] *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and let $z = 0$ be a superattracting fixed point of f. Assume that $A_f(0)$ contains at most one critical point other than $z = 0$. Then, exactly one of the following statements holds.*

(1) The set $A^*_f(0)$ contains no critical point other than $z = 0$. Then $A^*_f(0)$ is simply connected.

(2) The set $A^*_f(0)$ contains a critical point $c \neq 0$ and a preimage $z_0 \neq 0$ of $z = 0$. Then $A^*_f(0)$ is simply connected.

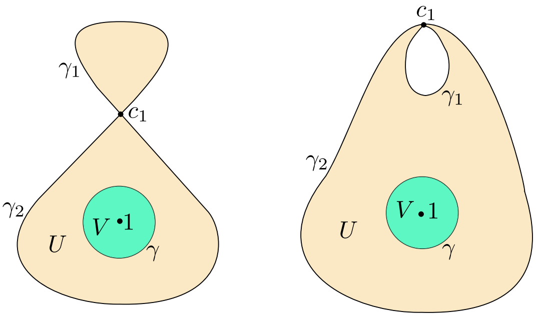
(3) The set $A^*_f(0)$ contains a critical point $c \neq 0$ and no preimage of $z = 0$ other than $z = 0$ itself. Then $A^*_f(0)$ is multiply connected.

Corollary 2.5. [7, Corollary 3.5] *For fixed $n \geq 2$ and $\alpha \in \mathbb{C}$, the immediate basins of attraction of the roots of $z^n - 1$ under $O_{n,\alpha}$ are multiply connected if and only if $A^*_{n,\alpha}(1)$ contains a critical point $c \neq 1$ and no preimage of $z = 1$ other than $z = 1$ itself.*

Remark 1. An immediate attracting basin may only have connectivity 1 or ∞ (see [4]). Hence, if $A^*_{n,\alpha}(1)$ is multiply connected, then all the immediate basins of attraction corresponding to the roots of f_n are infinitely connected.

The following lemma in [18] is the critical criterion used to prove Theorem A (see also [12]).

Lemma 2.6. [18, Lemma 2.2] *Any rational map F of degree d having d distinct superattracting fixed points is conjugate by a Möbius transformation to N_P for a polynomial of degree d. Moreover, if $z = \infty$ is not superattracting for F and F fixes ∞, then $F = N_P$ for some polynomial P of degree d.*



(a) $\partial U = \gamma_2$. (b) $\partial U = \gamma_1 \cup \gamma_2$.

Figure 3. The two possible configurations, of preimages of γ, described in Proposition 3.1

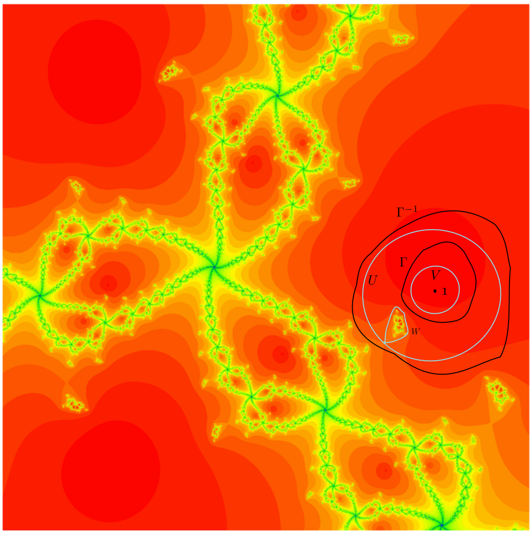


Figure 4. Description of the situation in Proposition 3.1, where $n = 3$ and $\alpha = 0.2 + 1.592i$

3. Proof of Theorem A

We start with a proposition that describes two curves in the dynamical plane of $O_{n,\alpha}$. These curves are used in the proof of Theorem A, as part of a quasiconformal surgery construction. The proof follows closely an argument made in the proof of [7, Proposition 3.1].

Proposition 3.1. *Let $O_{n,\alpha}$ such that $A^*_{n,\alpha}(1)$ is infinitely connected. Then there exist Γ and Γ^{-1}, analytic Jordan curves in $A^*_{n,\alpha}(1)$ which surround $z = 1$, such that $O_{n,\alpha}|_{\Gamma^{-1}} : \Gamma^{-1} \to \Gamma$ is a two-to-one map with $\Gamma \subset \mathrm{Int}(\Gamma^{-1})$.*

Proof. Let U be the maximal domain of definition of the Böttcher coordinates of the superattracting fixed point $z = 1$. By hypothesis and Corollary 2.5, $A^*_{n,\alpha}(1)$ contains the critical point $c_1{:=}c_{n,\,a,\,1}$ (see 3), which lies on ∂U, and no other preimages of $z = 1$. Since $z = 1$ has local degree 3, the map $O_{n,\alpha}|_{A^*_{n,\alpha}(1)} : A^*_{n,\alpha}(1) \to A^*_{n,\alpha}(1)$ has degree 3. Let $V{:=}O_{n,\alpha}(U)$. Then $\gamma{:=}\partial V$ is a Jordan curve. Let γ^{-1} be the preimage of γ contained in $A^*_{n,\alpha}(1)$. Then $\gamma^{-1} = \gamma_1 \cup \gamma_2$ is the union of two simple closed curves which intersect at the critical point c_1. Exactly one of the two curves, say γ_2, contains $z = 1$ in the Jordan domain bounded by it. There exist two possibilities: either $\partial U = \gamma_2$, or $\partial U = \gamma_1 \cup \gamma_2$ (see Fig. 3). Assume that $\partial U = \gamma_2$. By hypothesis, the critical point c_1 lies in $A^*_{n,\alpha}(1)$ and γ_1 is contained in the Fatou set. Therefore, $\gamma_1 \subset A^*_{n,\alpha}(1)$ and there exists a preimage of V which lies in $\mathrm{Int}(\gamma_1)$. Hence, $A^*_{n,\alpha}(1)$ contains a preimage of $z = 1$ other than itself, which is impossible according to Corollary 2.5. Consequently, we have that $\partial U = \gamma_1 \cup \gamma_2$. Let W be the connected component of $\hat{\mathbb{C}} \setminus \gamma_1$ which does not contain $z = 1$. Then W is mapped to an open set which contains $\hat{\mathbb{C}} \setminus U$, so W contains a pole. Since $z = 0$ is mapped to $z = \infty$ with degree $n-1$, $z = \infty$ is a fixed point, and the map has degree $2n$, there remain exactly n preimages for $z = \infty$. By symmetry, the pole in W is simple; therefore, ∂W is mapped onto ∂V with degree 1. Hence, γ_1 is mapped onto γ with degree 1 and γ_2 is mapped onto γ with degree 2.

Let Γ be an analytic Jordan curve which surrounds $z = 1$ such that $\Gamma \subset U \setminus V$, and let $\mathcal{A}$ be the open annulus bounded by Γ and γ. Then $\mathcal{A}$ has precisely 3 preimages in $A^*_{n,\alpha}(1)$. Since $\mathcal{A}$ does not contain any critical value, its preimages do not contain critical points. It follows from the Riemann–Hurwitz formula (see, for instance, [17]) that any preimage of $\mathcal{A}$ is also an annulus. One preimage of $\mathcal{A}$ lies in W and is mapped onto $\mathcal{A}$ with degree 1. There exists precisely one other preimage of $\mathcal{A}$ in $A^*_{n,\alpha}(1)$, which we denote by $\mathcal{A}^{-1}$. It lies in $A^*_{n,\alpha}(1) \setminus \mathrm{Fill}(U)$, surrounds $z = 1$, and is mapped onto $\mathcal{A}$ with degree 2. Let Γ^{-1} be the boundary component of $\mathcal{A}^{-1}$ which is mapped onto Γ. Observe that Γ^{-1} is an analytic Jordan curve. Since Γ^{-1} surrounds $z = 1$ and lies outside U, we have that $\Gamma \subset \mathrm{Int}(\Gamma^{-1})$ (see Fig. 4). □

The main tool used in the proof of Theorem A is quasiconformal surgery. For an introduction of the topic, we refer to [5]. The strategy of the proof is as follows. We start by defining a quasiregular map $f : A^*_{n,\alpha}(1) \to A^*_{n,\alpha}(1)$ on the immediate basin of attraction of $z = 1$, which we later extend to a quasiregular map $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Secondly, we construct a *symmetric* F-invariant Beltrami coefficient and prove, using the Integrability Theorem (see, for instance, [5, Theorem 1.28]), the existence of a map N_P quasiconformally conjugate to F. Then, we use Lemma 2.6 to show that N_P is a map obtained by applying Newton's method to a polynomial of degree n, and it is quasiconformally conjugated to N_{f_n}. Finally, we compare the filled Julia sets of N_{f_n} and $O_{n,\alpha}$.

Proof of Theorem A. Let $0 < \rho < 1$. Let $R : \mathrm{Int}(\Gamma) \to \mathbb{D}_{\rho^2}$ be a Riemann map such that $R(1) = 0$. Since Γ is an analytic curve, the Riemann map R extends analytically to the boundary (see, for instance, [5], Theorem 2.9c). Let ψ_2 :

$\Gamma \to \mathbb{S}_{\rho^2}$ be the restriction of R to its boundary. Let $\psi_1 : \Gamma^{-1} \to \mathbb{S}_\rho$ be an analytic lift map such that $\psi_2\big(O_{n,\alpha}(z)\big) = \big(\psi_1(z)\big)^2$. Let $\mathcal{A} = \mathrm{Int}(\Gamma^{-1})\backslash\overline{\mathrm{Int}(\Gamma)}$ and $\mathcal{A}_{\rho^2,\rho} = \mathbb{D}_\rho\backslash\overline{\mathbb{D}_{\rho^2}}$. Let $\psi : \partial\mathcal{A} \to \partial\mathcal{A}_{\rho^2,\,\rho}$, such that $\psi|_{\Gamma^{-1}} = \psi_1$ and $\psi|_\Gamma = \psi_2$. Since ψ_1 and ψ_2 are analytic maps, ψ extends quasiconformally to $\psi : \overline{\mathcal{A}} \to \overline{\mathcal{A}}_{\rho^2,\,\rho}$ (see, for instance, [5, Proposition 2.30b]).

We now define $f : A^*_{n,\alpha}(1) \to A^*_{n,\alpha}(1)$ quasiregular, as follows:

$$f(z) := \begin{cases} R^{-1}\big((R(z))^2\big) & \text{if } z \in \mathrm{Int}(\Gamma), \\ R^{-1}\left(\big(\psi(z)\big)^2\right) & \text{if } z \in \overline{\mathcal{A}}, \\ O_{n,\alpha}(z) & \text{if } z \in A^*_{n,\alpha}(1) \setminus \overline{\mathrm{Int}(\Gamma^{-1})}. \end{cases}$$

Now let $\xi := e^{\frac{2\pi i}{n}}$. We have that $I_{\xi^j}(z) = I^j_\xi(z)$, for $j \in \{0,\, 1,\, \dots\; n-1\}$, where I_ξ is defined as in Lemma 2.2. We extend to a quasiregular map $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined on the Riemann sphere, as follows:

$$F(z) := \begin{cases} I_{\xi^j} \circ f \circ I^{-1}_{\xi^j}(z) & z \in A^*_{n,\alpha}(\xi^j),\ j \in \{0,\, 1,\, \dots, n-1\}, \\ O_{n,\alpha}(z) & \text{otherwise.} \end{cases}$$

Observe that F is a quasiregular map which coincides with $O_{n,\alpha}$ outside the immediate basins of the roots of unity. We intend to construct an $F-$invariant and $I_\xi-$ invariant Beltrami coefficient μ. We first define an F-invariant Beltrami coefficient, say μ_1, in $A_{n,\alpha}(1)$, as follows:

$$\mu_1(z) := \begin{cases} \psi^*\mu_0(z) & \text{if } z \in \mathcal{A}, \\ (F^m)^*\mu_0(z) & \text{if } F^{m-1}(z) \in \mathcal{A},\ \text{for } m \geq 2, \\ \mu_0(z) & \text{otherwise.} \end{cases}$$

Observe that for $z \in \mathcal{A}$, we have that $\psi^*\mu_0(z) = F^*\mu_0(z)$. Now, we extend the previous construction to the rest of the Fatou set, that is, the basins of attraction of the nth root of unity $\xi^j \neq 1$, for $1 \leq j \leq n-1$. In the following, instead of using I_{ξ^j}, we will only refer to invariance with respect to I_ξ. Since $I_{\xi^j} = \underbrace{I_\xi \circ I_\xi \circ \cdots \circ I_\xi}_{\text{j times}}$, it suffices to prove the symmetry for I_ξ. We define an $I_\xi-$invariant Beltrami coefficient in $A_{n,\alpha}(\xi^j)$:

$$\mu(z) := \begin{cases} \mu_1(z) & \text{if } z \in A_{n,\alpha}(1), \\ (I^{-1}_{\xi^j})^*\mu_1(z) & \text{if } z \in A_{n,\alpha}(\xi^j), \\ \mu_0(z) & \text{otherwise.} \end{cases}$$

For $z \in A_{n,\alpha}(\xi)$ we have that

$$(F)^*\mu = (F)^*(I^{-1}_{\xi^j})^*\mu_1 = (I^{-1}_{\xi^j}F)^*\mu_1 = (FI^{-1}_{\xi^j})^*\mu_1 = (I^{-1}_{\xi^j})^*F^*\mu_1 = (I^{-1}_{\xi^j})^*\mu_1 = \mu.$$

It follows that μ is also $F-$invariant. By hypothesis, the map $O_{n,\alpha}$ is hyperbolic, hence, the Julia set has measure 0. Since $I^n_\xi(z) = z$, by construction, μ is both $F-$invariant and $I^{-1}_\xi-$invariant, with bounded dilatation. By the Integrability Theorem (see, for instance, [5, Theorem 1.28]), there exists $\phi_0 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ quasiconformal map such that $\phi_0^*\mu_0 = \mu$. We normalise ϕ_0 such that $\phi_0(0) = 0$, $\phi_0(\infty) = \infty$, and ϕ_0 is tangent to the identity at ∞. Let $\phi_\xi := I_\xi\phi_0 I^{-1}_\xi$. We prove that ϕ_ξ and ϕ_0 coincide by using the uniqueness part of the Integrability Theorem. First, we have that ϕ_ξ satisfies the same equation as ϕ_0:

$$\phi_\xi^*\mu_0 = (I_\xi^{-1})^*\phi_0^* I_\xi^*\mu_0 = (I_\xi^{-1})^*\phi_0^*\mu_0 = (I_\xi^{-1})^*\mu = \mu.$$

We have that ϕ_ξ satisfies $\phi_\xi^*\mu_0 = \mu$, $\phi_\xi(\infty) = \infty$, $\phi_\xi(0) = 0$, and ϕ_ξ is tangent to the identity at ∞. It follows from the uniqueness up to post-composition with Möbius transformations of the Integrability Theorem that $\phi_\xi = \phi_0$; therefore, $I_\xi \circ \phi_0 = \phi_0 \circ I_\xi$.

Now let $N_P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $N_P := \phi_0 \circ F \circ \phi_0^{-1}$. Observe that, by construction, $N_P \circ I_\xi = I_\xi \circ N_P$. The map N_P is quasiregular and satisfies $(N_P)^*\mu_0 = \mu_0$, therefore, by Weyl's lemma (see, for instance, [5, Theorem 1.14]) it is holomorphic and quasiconformally conjugated to F by ϕ_0. Since $z = \infty$ is a fixed point of F which is topologically repelling, $z = \infty$ is a repelling (therefore, not superattracting) fixed point of N_P. It also follows from the conjugacy that N_P has precisely n distinct superattracting fixed points, given by the set $\{\xi^j\phi_0(1)\}$, where $j \in \{0, 1, \ldots n-1\}$.

By Lemma 2.6, the map N_P is the map obtained by applying Newton's method to

$$P(z) = \prod_{j=0}^{n-1}\left(z - \xi^j\phi_0(1)\right) = z^n - \phi_0(1)^n.$$

We prove that N_P and N_{f_n} are linearly conjugated by $\eta(z){:=}\phi_0(1)z$. Analogously to the proof of Lemma 2.1, we first compute

$$N_P = \frac{n-1}{n}z + \frac{\phi_0(1)^n}{nz^{n-1}}.$$

Then,

$$\begin{aligned} N_P\left(\eta(z)\right) &= \frac{n-1}{n}\phi_0(1)z + \frac{\phi_0(1)^n}{n\phi_0(1)^{n-1}z^{n-1}} \\ &= \phi_0(1)\left(\frac{n-1}{n}z + \frac{1}{nz^{n-1}}\right) = \eta\left(N_{f_n}(z)\right) \end{aligned}$$

completes the proof of the linear conjugation.

The Julia set of N_{f_n}, $J(N_{f_n})$, is connected (see [16, Theorem 3.1]). Moreover, by construction, N_{f_n} and $O_{n,\alpha}$ are quasiconformally conjugate in a neighborhood of $J(N_{f_n})$, by a conjugacy, say ϕ. Since the conjugacy sends ∞ to ∞, we can conclude that there is an unbounded connected component Π of $J(O_{n,\alpha})$, which is a quasiconformal copy of $J(N_{f_n})$. The fact that $\phi(J(N_{f_n}))$ is a connected component of $J(O_{n,\alpha})$ follows from the surgery construction, since the surgery is done on the Fatou set of $O_{n,\alpha}$. □

4. Proof of Theorem B

We begin by studying the case of $n = 2$. Let $\alpha > 2$ and let $M_2(z){:=}\frac{z+1}{z-1}$ be the Möbius transformation which maps the superattracting fixed points

$z = 1$ and $z = -1$, to $z = \infty$ and $z = 0$. Finally, set $a = 2(\alpha - 1) > 2$, and consider the map

$$B_a(z) = z^3 \frac{z-a}{1-az} \tag{4}$$

which is conjugated to $O_{2,\alpha}$ by M_2. Indeed, for $n = 2$, the map $O_{n,\alpha}$ is

$$O_{2,\alpha}(z) = \frac{(2\alpha-3)z^4 - 6z^2 + (1-2\alpha)}{4(\alpha-2)z^3 - 4\alpha z}.$$

We remark that if $z = \frac{a}{b}$, then $M_2(z) = \frac{a+b}{a-b}$. This gives us

$$M_2\left(O_{n,\alpha}(z)\right) = \frac{(2\alpha-3)z^4 + 4(\alpha-2)z^3 - 6z^2 - 4\alpha z + (1-2\alpha)}{(2\alpha-3)z^4 - 4(\alpha-2)z^3 - 6z^2 + 4\alpha z + (1-2\alpha)},$$

and

$$B_a\left(M_2(z)\right) = \frac{(z+1)^3\left[z(3-2\alpha) + (2\alpha-1)\right]}{(z-1)^3\left[z(3-2\alpha) - (2\alpha-1)\right]} = M_2\left(O_{n,\alpha}(z)\right).$$

The map $B_a(z) = z^3 \frac{z-a}{1-az}$ is a rational map of degree 4 studied in [8,9], and [6]. In [6, Section 4] it is proven that for $a \in \mathbb{C}$, $|a| > 15.133$, $c_+ \in A_{B_a}(\infty)$. More precisely, it is shown that there exists a critical point c_+, such that $B_a(c_+) \in A^*_{B_a}(\infty)$. We will prove that this is a sufficient condition for $A^*_{B_a}(\infty)$ to be infinitely connected. Therefore, to prove Theorem B when $n = 2$, it suffices to prove the statements for the family B_a, and by conjugacy, they hold for $O_{2,\alpha}$.

The map B_a is a rational map of degree 4, and it is symmetric with respect to $\mathbb{S}^1$. The points $z = 0$ and $z = \infty$ are superattracting fixed points of local degree 3. Moreover, $z_\infty = \frac{1}{a} \in (0, 1)$ is a pole, and $z_0 = a$ is a preimage of $z = 0$. Consequently, there are two free critical points given by

$$c_\pm = \frac{1}{3a}\left(2 + a^2 \pm \sqrt{(a^2-4)(a^2-1)}\right). \tag{5}$$

The following lemma is a particular case of [6, Proposition 4.5].

Lemma 4.1. *Let $a > 1$. If $|z| > 2a$, then $z \in A^*_{B_a}(\infty)$. Equivalently, for $a > 1$, we have that $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}(0, 2a) \subset A^*_{B_a}(\infty)$.*

Proof. If $|z| > 2a$ then

$$|B_a(z)| = |z^3| \frac{|z-a|}{|1-az|} > |z-a||z| \frac{2a|z|}{|1-az|} > a|z| \frac{2a|z|}{1+a|z|} > a|z|.$$

Since $|B_a(z)| > |z|$, it follows that $z \in A^*_{B_a}(\infty)$. $\square$

In the proof of Proposition 4.6 in [6], the authors show that for $a \in \mathbb{C}$ with $|a|$ large enough (indeed $|a| > 16$), we have $B_a(c_+) \in A^*_{B_a}(\infty)$. A similar proof was previously done in [9, Lemma 2.6] for a family that includes B_a (but without providing an explicit bound). Here we present an easier proof, only for real values of the parameter a.

Lemma 4.2. *If $a \in \mathbb{R}_+$ is large enough, then $B_a(c_+) \in A^*_{B_a}(\infty)$.*

Proof. It follows from (5) that if $a > 2$, then $\frac{a}{2} < c_+ < a$. We write $B_a(z)$ as $B_a(z) = z^3\, h(z)$, where $h(z) = \frac{z-a}{1-az}$, and $h'(z) = -\frac{(a+1)(a-1)}{(az-1)^2}$. Then

$$B_a'(z) = 3z^2\, h(z) + z^3\, h'(z), \quad \text{so} \quad B_a(z) = \frac{zB'(z)}{3} - \frac{z^4}{3} h'(z).$$

We have that

$$B_a(c_+) = c_+^4 \frac{(a+1)(a-1)}{3(ac_+ - 1)^2} > c_+^4 \frac{a(a-1)}{3a^4}.$$

Since $c_+ > \frac{a}{2} > 1$, it follows that

$$B_a(c_+) > \frac{a-1}{48} a.$$

So, for $a > 97$, we have that $B_a(c_+) > 2a$. According to Lemma 4.1, we conclude that $B_a(c_+) \in A^*_{B_a}(\infty)$. □

Proposition 4.3. *Assume $a \in \mathbb{R}_+$ is large enough such that Lemma 4.2 applies. Then $c_+ \in A^*_{B_a}(\infty)$ and $A^*_{B_a}(\infty)$ is infinitely connected.*

Proof. Observe that, for $a > 1$, we have that $0 < z_\infty < z_0 < 2a$. By Lemma 4.2, $B_a(c_+) \in A^*_{B_a}(\infty)$. Therefore, the critical point c_+ lies either in $A^*_{B_a}(\infty)$ or in a preimage of $A^*_{B_a}(\infty)$.

Assume that c_+ lies in a preimage of $A^*_{B_a}(\infty)$, distinct from $A^*_{B_a}(\infty)$, say U. Since U contains a critical point, it is mapped onto $A^*_{B_a}(\infty)$ with degree at least 2. Hence, U contains at least 2 preimages of $z = \infty$ (different from itself), a contradiction with $\deg(B_a) = 4$, and $z = \infty$ being a superattracting fixed point with local degree 3.

Since the map is real, by the Schwarz Reflexion Principle, the map is conjugated to itself by complex conjugation, i.e. $I(z) = \bar{z}$. Then, Fatou components intersecting the real line are symmetric with respect to the real line. Since $0 < c_+ < z_0$, it follows that 0 and z_0 belong to different connected components of the complement of $A^*_{B_a}(\infty)$. Thus, $A^*_{B_a}(\infty)$ is multiply connected, therefore, by Remark 1 it is infinitely connected. □

Remark 2. It follows from [6, Proposition 4.6] that all $a \in \mathbb{C}$, with $|a| > 15.133$, belong to the same hyperbolic component. Since the connectivity of $A^*_{B_a}(\infty)$ is an invariant topological property within hyperbolic components, we conclude from Proposition 4.3 that if $|a| > 15.133$, then $A^*_{B_a}(\infty)$ is infinitely connected. This completes the proof of Theorem B for $n = 2$.

To finish the proof of Theorem B we now consider $n \geq 3$. As we did before, we consider a new map $R_{n,\alpha}$ which is conjugated to $O_{n,\alpha}$ via the Möbius map $M(z) = \frac{1}{z-1}$. More specifically, we consider $R_{n,\alpha} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, given by $R_{n,\alpha} = M \circ O_{n,\alpha} \circ M^{-1}$. Since M sends $z = 1$ to $z = \infty$ and $z = \infty$ to $z = 0$, it is clear from (2) that $z = \infty$ is a superattracting fixed point of $R_{n,\alpha}$ with local degree 3 and $z = 0$ is a fixed point of $R_{n,\alpha}$.

We have $M(z) = \frac{1}{z-1}$, $M^{-1}(z) = \frac{z+1}{z}$, and

$$O_{n,\alpha}(z) = \frac{(1-2\alpha)(n-1) + (2 - 4\alpha - 4n + 6\alpha n - 2\alpha n^2)z^n + (n-1)(1 - 2\alpha - 2n + 2\alpha n)z^{2n}}{2nz^{n-1}(\alpha(1-n) + (-\alpha - n + \alpha n)z^n)}.$$

We write

$$O_{n,\alpha}(z) = \frac{E_3(n,\alpha) + E_4(n,\alpha)z^n + E_5(n,\alpha)z^{2n}}{2nz^{n-1}[E_1(n,\alpha) + E_2(n,\alpha)z^n]},$$

where $E_i(n,\alpha)$ are polynomials of degree 1 in α. It follows that $R_{n,\alpha}(z) = M \circ O_{n,\alpha} \circ M^{-1}(z) =$

$$= M\left(\frac{E_3(n,\alpha)z^{2n} + E_4(n,\alpha)z^n(z+1)^n + E_5(n,\alpha)(z+1)^{2n}}{2nz(z+1)^{n-1}[E_1(n,\alpha)z^n + E_2(n,\alpha)(z+1)^n]}\right)$$

$$= \frac{2nz(z+1)^{n-1}[E_1(n,\alpha)z^n + E_2(n,\alpha)(z+1)^n]}{2nz(z+1)^{n-1}[E_1(n,\alpha)z^n + E_2(n,\alpha)(z+1)^n] - E_3(n,\alpha)z^{2n} + E_4(n,\alpha)z^n(z+1)^n + E_5(n,\alpha)(z+1)^{2n}}$$

which we finally write as

$$R_{n,\alpha}(z) = \frac{2nz(z+1)^{n-1}[E_1(n,\alpha)z^n + E_2(n,\alpha)(z+1)^n]}{Q^1_{2n-3}(z) + \alpha Q^2_{2n-3}(z)} =: \frac{zP(z)}{Q(z)}, \tag{6}$$

where $Q^j_{2n-3}(z)$, $j = 1, 2$ are degree $2n-3$ polynomials, with coefficients independent of α. We can use this expression of $R_{n,\alpha}$, without further computations, since $R_{n,\alpha}$ is a rational map of degree $2n$, being conjugated to $O_{n,\alpha}$. Furthermore, $z = \infty$ corresponds to the superattracting fixed point of local degree 3 of $O_{n,\alpha}$, $z = 1$. So the denominator of $R_{n,\alpha}$ has degree $2n-3$ in z. Since $E_i(n,\alpha)$ are polynomials of degree 1 in α, this concludes the argument of writing $R_{n,\alpha}$ in this form.

We split the proof of the case $n \geq 3$ in several lemmas. We start by giving an estimate for a zero of $R_{n,\alpha}$, which lies on the positive real line.

Lemma 4.4. *Let $\alpha > 2$ and let $S(z) = E_1(\alpha)z^n + E_2(\alpha)(z+1)^n$. Then $S(\alpha(n-1)) < 0 < S(\alpha n - \alpha - n)$. In particular, $R_{n,\alpha}$ has a zero on the interval $(\alpha(n-1) - n, \alpha(n-1))$, for all $\alpha > 2$.*

Proof. Direct computations show that S writes as

$$S(z) = -nz^n + [\alpha(n-1) - n)] \sum_{k=0}^{n-1} \binom{n}{k} z^k. \tag{7}$$

On one hand,

$$S(\alpha(n-1) - n) = \sum_{k=0}^{n-2} \binom{n}{k} (\alpha n - \alpha - n)^k > 0.$$

On the other hand,

$$\begin{aligned} S(\alpha(n-1)) &= -n[\alpha(n-1)]^n + [\alpha(n-1) - n] \sum_{k=0}^{n-1} \binom{n}{k} [\alpha(n-1)]^k \\ &= -n[\alpha(n-1)]^n + [\alpha(n-1)] \sum_{k=0}^{n-2} \binom{n}{k} [\alpha(n-1)]^k \\ &\quad + [\alpha(n-1)]n[\alpha(n-1)]^{n-1} - n \sum_{k=0}^{n-1} \binom{n}{k} [\alpha(n-1)]^k \\ &= \sum_{k=0}^{n-2} \left[\binom{n}{k} - n\binom{n}{k+1}\right] [\alpha(n-1)]^{k+1} - n\binom{n}{0} < 0. \end{aligned}$$

$\square$

The following technical lemma will be useful later.

Lemma 4.5. *Let $m, k \in \mathbb{N}^*$, $m > k$. Let $u, v_j \in \mathbb{C}$, $j = 1, \ldots, m$. If $|u| - \sum_{j=1}^{m} |v_j| > 0$, then*

$$\left| u - \sum_{j=1}^{k} v_j \right| > \left| \sum_{j=k+1}^{m} v_j \right|.$$

Proof. Since $|u| - \sum_{j=1}^{m} |v_j| > 0$, we have that

$$\left| u - \sum_{j=1}^{k} v_j \right| \geq |u| - \sum_{j=1}^{k} |v_j| > \sum_{j=k+1}^{m} |v_j| \geq \left| \sum_{j=k+1}^{m} v_j \right|.$$

□

We give a sufficient condition for points to lie in $A^*_{R_{n,\alpha}}(\infty)$.

Lemma 4.6. *Let $\alpha > 0$ large enough. If $|z| > n\alpha$, then $z \in A^*_{R_{n,\alpha}}(\infty)$.*

Proof. We show that if $|z| > n\alpha$, then $|R_{n,\alpha}(z)| > |z|$, which is a sufficient condition for $z \in A^*_{R_{n,\alpha}}(\infty)$. According to (6), we have to prove that, for α large enough, $\left|\frac{P(z)}{Q(z)}\right| > 1$. We write P as

$$P(z) = 2n\left[-nz^{2n-1} + n\alpha(n-1)z^{2n-2} + P_{2n-2}(z) + \alpha P_{2n-3}(z)\right].$$

Observe that $P_{2n-2}(z)$ and $P_{2n-3}(z)$ are polynomials of degree $2n-2$ and $2n-3$, respectively, with coefficients independent of α. For α large enough (recall that we are assuming $|z| > n\alpha$), the following statements hold:

(1) $(n-1)|z|^{2n-1} > n\alpha(n-1)|z|^{2n-2}$.

(2) $\frac{1}{3}|z|^{2n-1} > |P_{2n-2}(z)|$, since

$$\lim_{\alpha\to\infty} \frac{n(n-1)|z|^{2n-2}}{|z|^{2n-1}} = 0.$$

(3) $\frac{1}{3}|z|^{2n-1} > |\alpha P_{2n-3}(z)|$, since

$$\lim_{\alpha\to\infty} \frac{P_{2n-3}(z)}{|z|^{2n-2}} = 0.$$

(4) $\frac{1}{3}|z|^{2n-1} > |Q(z)|$, since Q^1_{2n-3} and Q^2_{2n-3} are polynomials of degree $2n-3$ with coefficients independent of α.

All together imply that

$$n|z|^{2n-1} > n\alpha(n-1)|z|^{2n-2} + |P_{2n-2}(z)| + \alpha|P_{2n-3}(z)| + |Q(z)|.$$

By using Lemma 4.5 (recall that for α large enough, there is no root of Q for $|z| > n\alpha$), we get that

$$\left|\frac{P(z)}{Q(z)}\right| = \left| 2n\frac{-nz^{2n-1} + \alpha(n-1)z^{2n-2} + P_{2n-2}(z) + \alpha P_{2n-3}(z)}{Q(z)} \right| > 2n > 1.$$

Thus, for $|z| > n\alpha$, we have that $|R_{n,\alpha}(z)| > |z|$ and $z \in A^*_{R_{n,\alpha}}(\infty)$. □

The following proposition concludes the proof of Theorem B.

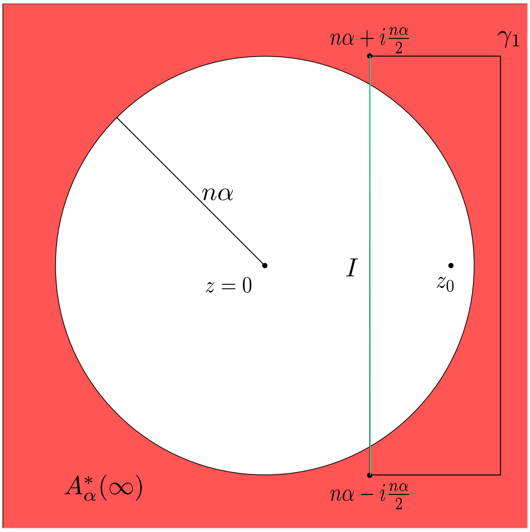


Figure 5. Description of the situation in proof of Proposition 4.7. The zero z_0 is separated by $(I \cup \gamma_1) \subset A^*_{R_{n,\alpha}}(\infty)$ from $z = 0$. Therefore, $A^*_{R_{n,\alpha}}(\infty)$ is multiply connected

Proposition 4.7. *Let $\alpha > 0$ large enough. Then $A^*_{R_{n,\alpha}}(\infty)$ is infinitely connected.*

Proof. If $z \in (0,\, \alpha n - \alpha - n)$, then $nz^n < (\alpha n - \alpha - n)nz^{n-1}$. It follows that $S(z)$ (see 7) has no zeros in $(0,\, \alpha n - \alpha - n)$. In particular, $R_{n,\alpha}$ has no zeros in $(0,\, \alpha n - \alpha - n)$. Let

$$I = \left\{ z \in \mathbb{C} \middle| z = n\alpha \left(\frac{1}{2} + it \right),\, t \in [-1,\, 1] \right\}.$$

We claim that $R_{n,\alpha}(I) \subset A^*_{R_{n,\alpha}}(\infty)$.

Let $T_{n,\alpha}(z) := \frac{1}{(1+z)^2} R_{n,\alpha}(z)$. Firstly, we prove that there exists a constant $\kappa > 0$ such that for $z \in I$, we have that $|T_{n,\alpha}(z)| > \kappa$. A direct computation shows that

$$\left| T_{n,\alpha} \left(n\alpha \left(\frac{1}{2} + it \right) \right) \right| := \frac{N(\alpha)}{M(\alpha)},$$

where N and M are polynomials of degree $2n - 2$ in α with coefficients depending on t. Moreover, if we denote by $c(t)$ the degree $2n - 2$ coefficient of N, we have:

$$c(t) = 2n^{2n-1} \left(\frac{1}{2} + it \right)^{2n-2} \left[-n \left(\frac{1}{2} + it \right) + n - 1 \right].$$

Observe that

$$\min_{t \in [-1,\, 1]} |c(t)| = |c(0)| := C > 0.$$

We denote by $d(t)$ the degree $2n-2$ coefficient of M. Let $D := \max_{t\in[-1,\,1]} |d(t)|$, and let $\kappa := \frac{C}{2D}$. For large enough α, we have that $\left|T_{n,\alpha}\left(n\alpha\left(\frac{1}{2}+it\right)\right)\right| > \kappa$ and that

$$\left|R_{n,\alpha}\left(n\alpha\left(\frac{1}{2}+it\right)\right)\right| > \kappa\left|\frac{n\alpha}{2}+1+n\alpha ti\right|^2 > \frac{n^2}{4}\kappa\alpha^2 > n\alpha.$$

It follows from Lemma 4.6 that $R_{n,\alpha}(I) \subset A^*_{R_{n,\alpha}}(\infty)$. Hence, I is a subset of $A^*_{R_{n,\alpha}}(\infty)$ or a preimage of this Fatou component. Moreover, for $z_\pm = \frac{n}{2}\alpha \pm in\alpha$ we have $|z_\pm| > n\alpha$. We conclude from Lemma 4.6 that $z_\pm \in A^*_{R_{n,\alpha}}(\infty)$. Therefore, $I \subset A^*_{R_{n,\alpha}}(\infty)$. By Lemma 4.4, there exists a zero z_0 of $R_{n,\alpha}$ such that $\frac{n\alpha}{2} < z_0 < n\alpha$. Therefore, there exists a piece-wise smooth Jordan curve $\Gamma = I \cup \gamma_1 \subset A^*_{R_{n,\alpha}}(\infty)$ such that $z_0 \in \mathrm{Int}(\Gamma)$ and $0 \in \mathrm{Ext}(\Gamma)$ (see Fig. 5). It follows that $A^*_{R_{n,\alpha}}(\infty)$ is multiply connected. By Remark 1, it is infinitely connected.

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