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FC-groups with Few Subnormal Non-normal Subgroups

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Abstract. A group G is said to be an FC-group if every conjugacy class of G has finite order and is said to be a T-group if every subnormal subgroup is normal in G. In this paper we give a characterization of groups that are both FC-groups and T-groups and go further in the study of FC-groups with few subnormal non-normal subgroups in the sense of the chain conditions. The structure of FC-groups satisfying the maximal, the minimal and the double chain condition on subnormal non-normal subgroups will be described.

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1. Introduction

A group G is called an FC-group if every conjugacy class of G has finite order or, equivalently, if $|G : C_G(g)|$ is finite for each element g of G. The theory of FC-groups has been extensively developed; for instance, it is well-known that if G is an FC-group, then both $G/Z(G)$ and G' are locally finite and these two results are due to Baer and Neumann, respectively. For a reference on the subject, one may refer to [8]. On the other hand, a group G is called a T-group if normality is a transitive relation in G, i.e. if all the subnormal subgroups of G are normal in G. The structure of soluble T-groups was described by Gaschütz [11] in the finite case and by Robinson [14] for arbitrary groups. It turns out that all soluble groups with the T-property are metabelian and hypercyclic, while any finitely generated soluble T-group either is finite or abelian. Relevant classes of generalized T-groups can be introduced by imposing that the set of all subnormal non-normal subgroups of the group is small in some sense (see, among the others, [3–5]). In this connection goes the imposition of the so-called *double chain conditions on subnormal non-normal*

subgroups. Generalizing the classical results of Černikov and O. Schmidt on groups satisfying the minimal and the maximal condition on subgroups, respectively, Shores [19] and Zaicev [20] independently proved that if G is a (generalized) soluble group admitting no chains of subgroups with the same order type of the set of the integers, then G is soluble-by-finite and is either a Černikov or a polycyclic-by-finite group. Conditions of this type can of course be considered for special systems of subgroups: if θ is a subgroup-theoretical property, we shall say that a group G satisfies the double chain condition on θ-subgroups if for each double chain

$$\cdots \leq X_{-n} \leq \cdots \leq X_{-1} \leq X_0 \leq X_1 \leq \cdots \leq X_n \leq \cdots$$

of θ-subgroups of G there exists an integer k such that either $X_n = X_k$ for all $n \leq k$ or $X_n = X_k$ for all $n \geq k$. For some values of θ already studied, see for instance [1,2,6,7]. In the present work, θ will be the subgroup-theoretical property of being subnormal and not normal. Clearly, every T-group, every Černikov and every polycyclic group satisfies the double chain condition on subnormal non-normal subgroups. A group satisfying this condition will be sometimes called for short a DC_{snn}-group.

Joining together these classes of problems, this work deals with a thorough study of FC-groups having few subnormal non-normal subgroups. In particular, we present in Sect. 2 a characterization of FC-groups all of whose subnormal subgroups are normal, which is a natural extension of some results in [18]; in the same section a description of soluble FC-groups will also be made explicit: although it can be retrieved without too much effort from what is already known (for instance from [14]), so far it had no place in literature. In Sect. 3, we will give a characterization of FC-groups satisfying the double chain condition on subnormal non-normal subgroups and from this we will derive the structure of FC-groups satisfying the maximal or the minimal condition on subnormal non-normal subgroups. Finally, in Sect. 4, we will present some examples aiming to show how our results cannot be improved and giving concrete instances to the groups studied here.

Along the paper, it will be evident that some parts of this work rely on the Classification of Finite Simple Groups, in that, for instance, we need precise information on the Schur multipliers of every finite simple group. Recall here that the Schur multiplier of a group G can be defined as the second homology group $H_2(G, \mathbb{Z})$ for trivial group action and that it has strong connections with the theory of projective representations and with central extensions. For an account on this, the reader may for instance refer to [13].

Most of our notation is standard and can be found in [15].

2. T-groups with Finite Conjugacy Classes

We begin studying FC-groups in which normality is a transitive relation. The proof of the first result is indeed akin to the theorem of Gaschütz classifying finite T-groups. For the well-known properties of soluble T-groups, we refer to [14].

There are basic examples showing that T-groups do not constitute a formation. On the other hand, something can be easily said for quotients over central subgroups.

Lemma 2.1. *Let G be a group and let A and B be central subgroups of G. If G/A and G/B are T-groups, the same holds for $G/A \cap B$.*

Proof. First, we may clearly assume without loss of generality that $A \cap B = \{1\}$. Let X be a subnormal subgroup of G. To show that X is normal in G, we can further assume that X has trivial intersection with AB, as by hypothesis $AB \leq Z(G)$. Let now x_1 and x_2 be elements of X and let a and b be elements of A and B, respectively, such that $x_1 a = x_2 b$. Since $X \cap AB = \{1\}$, $ab^{-1} = 1$ and hence $a = b = 1$ because $A \cap B = \{1\}$. We have just shown that $X = XA \cap XB$ and hence that X is normal in G. □

The following proposition deals with the soluble case.

Proposition 2.2. *Let G be a soluble group and let L be the last term of the lower central series of G. Then G is a T-group with finite conjugacy classes if and only if either*

(i) *G is abelian or*
(ii) *G is periodic, every subgroup of L is normal in G, L is the direct product of cyclic subgroups, L_p is finite for each $p \in \pi(L)$ and $G = K \ltimes L$, where K is a Dedekind group, $\pi(K) \cap \pi(L) = \emptyset$, $2 \notin \pi(L)$ and $[k, L]$ is finite for each element k of $K \setminus Z(G)$.*

Proof. Assume that G is a T-group with finite conjugacy classes, let $Z = Z(G)$ and let F be the Fitting subgroup of G. Since G is a T-group, every subgroup of F is normal in G and hence $G/Z(F)$ acts on F as a group of power automorphisms; moreover, the FC-property implies that G/Z is locally finite and residually finite and G' is periodic, so that in particular G, if not periodic, is generated by its elements of infinite order. If G is not periodic, then such is Z and hence F, so that, since every subgroup of F is normal in G and there are central elements of infinite order, it follows that F is central in G and hence $Z = F = C_G(F) = G$. Then we may assume that G is periodic and recall that L_2 is divisible. Since G/Z is residually finite, L_2 must be trivial and hence $L \cap Z = \{1\}$, so that L is reduced. Moreover, if we let P be a non-trivial primary component of L, G must contain an x element acting as a non-trivial (universal) power automorphism on P, so that P must be finite, otherwise x would have infinitely many conjugates. In particular, L is the direct product of cyclic subgroups. Let $C = C_G(L)$ and let g be an element in $G \setminus C$. As g has finitely many conjugates in G, it follows that $[g, L]$ is finite. From this and from the fact that every subgroup of L is normal in G, it follows that G/C is isomorphic with a subgroup of

$$\operatorname*{Dr}_{p \in \pi(L)} PAut(L_p),$$

where L_p is the p-component of L. Then G/C is countable and by Theorem 5.1.2 in [14] we may find a subgroup K of G such that $G = K \ltimes L$ and $\pi(K) \cap \pi(L) = \emptyset$. Clearly, $K \simeq G/L$ is a nilpotent T-group and hence is a

Dedekind group. Finally, just as above, the FC-property implies that $[k, L]$ is finite for each element k of $K \setminus Z(G)$.

Conversely, one sees immediately that G is an FC-group, while it is a T-group by a straightforward application of Lemma 5.2.2 in [14].   □

We notice that one thing we can deduce from the proof is that that last condition in point (ii) is just another way of saying that $K/Z(G)$ acts on L as a subgroup of the direct product $\mathrm{Dr}_{p \in \pi(L)} PAut(L_p)$, where L_p is the p-component of L for a prime p in $\pi(L)$ and $PAut(L_p)$ is the group of power automorphisms of L_p, i.e. the group of all automorphisms of L_p which fix every cyclic subgroup of L_p.

Corollary 2.3. *Every non-periodic T-group with finite conjugacy classes is abelian.*

We continue with a descriptive lemma which relies on the Classification of Finite Simple Groups and allows us to be more specific about our exposition. Its proof is a straightforward application of the theory of Schur covers together with a knowledge of Schur multipliers of finite simple groups (see, for instance, Table 4.1 in [12]).

Lemma 2.4. *Let G be a group which is a central extension of a subgroup Z by a finite simple non-abelian group. Then $G' \cap Z$ either is isomorphic with a subgroup of $\mathbb{Z}_l \times \mathbb{Z}_m \times \mathbb{Z}_n$ with*

$$(l, m, n) \in \{(3, 3, 4), (3, 4, 4)\}$$

or is a periodic cyclic group if G/Z is of projective or unitary type.

We pass now to a characterization of T-groups with finite conjugacy classes. This is a natural generalization of Theorem 4.1 in [18].

Theorem 2.5. *Let G be a group and let S be the subsoluble radical of G. Then G is a T-group with finite conjugacy classes if and only if there exists a perfect subgroup P of G such that*

  (i) *$P \cap Z(S) = Z(P)$ and $P/Z(P)$ is the direct product of finite simple normal subgroups of G;*

  (ii) *if there is a bound over the order of the fields or over the dimensions of every finite simple subgroup of $P/Z(P)$ which is of projective or unitary type, then $Z(P)$ has finite exponent;*

 (iii) *G/P is a soluble T-group with finite conjugacy classes;*

 (iv) *$[P, S] = \{1\}$ and each subnormal subgroup of S is normal in G;*

  (v) *$[g, P]$ is finite and non-trivial for each element g of $G \setminus PS$.*

Proof. Suppose that G is a T-group with finite conjugacy classes. Let S be the subsoluble radical of G, which is also the soluble radical of G, since subsoluble T-groups are easily seen to be soluble. As $G/Z(G)$ is locally finite and any finite subset of a periodic FC-group can be embedded in a finite normal subgroup, it is clear that G contains no infinite simple sections. The soluble case has already been treated in Proposition 2.2, so we may suppose that G is not soluble. Let H be a subgroup of G such that HS/S is non-abelian and

simple, so that it is also finite. Since $H/H \cap S$ is perfect and G' is periodic, H contains a finite perfect subgroup K such that $H = K(H \cap S)$. As every subgroup of both $S/\gamma_3(S)$ and $\gamma_3(S)$ is G-invariant and K is perfect, the latter must centralize the whole S and hence, in particular, it is a central extension of $K \cap Z(S)$ by a finite simple non-abelian group. Let now $\{S_i/Z(S)\}_{i \in I}$ be the collection of all simple non-abelian normal subgroups of $G/Z(S)$ and let $Z = \langle S'_i \cap Z(S) \,|\, i \in I \rangle$. Since G' is periodic, Z is a periodic subgroup of $Z(S)$ and if there is a bound over the order of the fields or over the dimensions of every $S_i/Z(S)$ which is of projective or unitary type, then Z has finite exponent by Lemma 2.4. Let now P/Z be the subgroup of G/Z generated by all simple non-abelian normal subgroups of G/Z and let C/Z be the centralizer in G/Z of P/Z. Clearly, $Z = P \cap Z(S) = Z(P)$. Then P/Z is the direct product of (possibly infinitely many) finite simple non-abelian groups (see, for instance, [15, Lemma 5.44]). If C contains a normal subgroup D such that DS/S is simple and non-abelian, then by a previous argument C/Z will contain a simple non-abelian subgroup, against the fact that $P \cap C = Z$. So we have that C/Z is soluble and hence $C = S$. Now, by the Classification of Finite Simple Groups the outer automorphism group of any finite simple group is soluble and hence G/P is a soluble group, which is of the type described by Proposition 2.2, because G/P is still a T-group.

Since now $[P, S]$ lies in the centre of P, it follows that each mapping $g \in P \to [g, s] \in Z$ for any element s of S is a homomorphism, which shows that $[P, S]$ must be trivial. Moreover, it is obvious that every subnormal subgroup of S is normal in G. Finally, $[g, P]$ is finite for any element g of $G \setminus PS$, because G is an FC-group and so g must act non-trivially on finitely many simple direct factors of P/Z.

Conversely, let G be a group satisfying our description. As G/P and G/S are both FC-groups by (iii) and (v), respectively, and since $[P, S] = \{1\}$ by (iv), it follows that G is isomorphic with a subgroup of the direct product of FC-groups and hence it is an FC-group itself. Assume now for a contradiction that there exists a subnormal non-normal subgroup H of G. By (iv), $H \cap S$ is normal in G and hence we may assume that H and S have trivial intersection. Moreover, the same holds for P by (i) and hence we take $H \cap PS = \{1\}$, so that H has to contain an element in $G \setminus PS$. However, any such element must act non-trivially on some simple direct factor of P/Z, otherwise it would be contained in $C_G(P) = S$, and this implies that P and H have non-trivial intersection by means of the subnormality of H and the fact that P/Z is perfect, a contradiction. $\qquad\square$

With the notation of Theorem 2.5, the subgroup $Z(P)$ seems to have quite an unrestricted structure, excepted for the case in which there is some sort of bound over the finite simple subgroups of $P/Z(P)$. Example 4.1 shows that $Z(P)$ cannot be taken in general to be a subgroup of $Z(G)$. Other reasonable questions are about the possible rank and exponent of $Z(P)$. To answer this, we will show in Example 4.5 that $Z(P)$ can have infinite exponent and also infinite rank.

3. *FC*-groups with Few Subnormal Non-normal Subgroups

We now want to inspect the structure of FC-groups with few subnormal non-normal subgroups in the sense of satisfying either the maximal, the minimal or the double chain condition on subnormal non-normal subgroups. As a general classification of FC-groups satisfying the latter property will easily enable us to specialize to the other two properties, we begin studying FC-groups satisfying the double chain condition on subnormal non-normal subgroups. In dealing with this case, just as in the previous section, finite simple groups will naturally show up and a use of the Classification of Finite Simple Groups seems to be unavoidable.

Just as in the case of a subsoluble T-group, if a subsoluble group satisfies the double chain condition on subnormal non-normal subgroups, then it is soluble. This has been proved in [4].

Proposition 3.1. *Every subsoluble DC_{snn}-group is soluble.*

Before stating our results, we need some tools to construct double chains of subnormal non-normal subgroups out of some particular infinite direct products. To this aim, let G be a group such that

$$G = \operatorname*{Dr}_{n\in\mathbb{N}} G_n \tag{1}$$

is a decomposition of G into the direct product of countably many non-trivial subgroups. Then G admits the double chain

$$\cdots < U_{-k} < \cdots < U_{-1} < U_0 < U_1 < \cdots < U_k < \cdots \tag{2}$$

where

$$U_k = \left(\operatorname*{Dr}_{n\in\mathbb{N}} G_{2n-1}\right) \times \left(\operatorname*{Dr}_{1\leq n\leq k} G_{2n}\right) \quad \text{and} \quad U_{-k} = \operatorname*{Dr}_{k<n+1} G_{2n-1}$$

for each non-negative integer k. We shall say that (2) is the *double chain associated* to the direct decomposition (1).

Remember here that a *subnormal section* of an arbitrary group G is a factor group of the form X/Y, where X is a subnormal subgroup of G and Y is a normal subgroup of X.

Lemma 3.2. *Let G be a DC_{snn}-group and let H/K be a subnormal section of G which is a direct product of an infinite family of non-trivial subgroups $\mathcal{F} = \{S_i/K\}_{i\in I}$. Then K is normal in G and every subnormal subgroup of G/K having trivial intersection with infinitely many elements of $\mathcal{F}$ is normal. In particular, each S_i is normal in G. Moreover, if G is an FC-group, then G/K is a T-group.*

Proof. Assume first without loss of generality that $I = \mathbb{N}$ and split the collection $\mathcal{F}$ into the two infinite subcollections $\{S_{2n}\}_{n\in\mathbb{N}}$ and $\{S_{2n-1}\}_{n\in\mathbb{N}}$, so that

$$H/K = U/K \times V/K$$

where

$$U/K = \operatorname*{Dr}_{n\in\mathbb{N}} (S_{2n}/K) \quad \text{and} \quad V/K = \operatorname*{Dr}_{n\in\mathbb{N}} (S_{2n-1}/K).$$

Since the group satisfies the double chain condition on subnormal non-normal subgroups, taking into account the double chains associated with U/K and V/K, respectively, one can find two G-invariant subgroups U^* and V^* of G such that $K \leq U^* \leq U$ and $K \leq V^* \leq V$. Obviously, we have $U^* \cap V^* = K$ and hence K is normal in G. On the other hand, by a similar reasoning we see that each direct term of H/K is normal in G/K and in particular H is normal in G.

Let now X/K be any subnormal subgroup of G/K with trivial intersection with infinitely many S_i/K. Then we may find a positive integer m such that $X \cap H \leq \langle S_1, \ldots, S_m \rangle$. Consider now the double chains

$$\cdots < W_{-n} < \cdots < W_{-1} < W_0 < W_1 < \cdots < W_n < \cdots$$

and

$$\cdots < Y_{-n} < \cdots < Y_{-1} < Y_0 < Y_1 < \cdots < Y_n < \cdots$$

respectively associated to the collections of normal subgroups $\{S_{2m}, S_{2m+2}, S_{2m+4}, \ldots\}$ and $\{S_{2m+1}, S_{2m+3}, S_{2m+5}, \ldots\}$. Notice that any two distinct members of the two collections have intersection equal to K. Then

$$\cdots < XW_{-n} < \cdots < XW_{-1} < XW_0 < XW_1 < \cdots < XW_n < \cdots$$

and

$$\cdots < XY_{-n} < \cdots < XY_{-1} < XY_0 < XY_1 < \cdots < XY_n < \cdots$$

are double chains consisting of subnormal subgroups of G, and so there exist two integers r and s such that XW_r and XY_s are normal in G. On the other hand, $X = XW_r \cap XY_s$ and hence X is normal in G.

Finally, suppose that G is an FC-group and let Y be a subgroup of G such that YK is a subnormal non-normal subgroup of G. If we now let y be an element of Y such that $\langle y \rangle^G$ is not contained in YK, by replacing the latter with $(Y \cap \langle y \rangle^G)K$ we may assume that YK/K is finitely generated by the FC-property. However, one immediately finds in H/K an infinite family of G-invariant subgroups with trivial intersection with YK/K and this leads to a contradiction by what proved in the previous paragraph. $\square$

Lemma 3.3. *Let G be an FC-group satisfying the double chain condition on subnormal non-normal subgroups, let S be the subsoluble radical of G and let T be the torsion subgroup of $Z(G)$. If G is not a T-group, G/S is infinite and $S/Z(G)$ is finite, then*

(i) *T has finite total rank and G contains subgroups A, B and P such that $A \leq B \leq T$, G/A is a maximal T-quotient of G, P/B is an infinite direct product of finite simple non-abelian normal subgroups of G/B and G''/P is finite; moreover, if A is infinite, then $A = B$, while if A is finite $A \leq N'$ for each normal factor N/B of P/B;*

(ii) *A is isomorphic either with a subgroup of $\mathbb{Z}_l \times \mathbb{Z}_m \times \mathbb{Z}_n$, where*

$$(l, m, n) \in \{(3, 3, 4), (3, 4, 4)\},$$

or with a locally cyclic periodic group; if moreover there is a bound over the order of the fields or over the dimensions of every simple direct

factor of P/B which is of projective or unitary type, then B has finite exponent.

Proof. By Proposition 3.1, S is also the soluble radical of G, while $G/Z(G)$ is periodic and residually finite since G is an FC-group. Let $Z = Z(G)$, let X be a subnormal non-normal subgroup of G and let x be an element of X such that $\langle x \rangle^G$ is not contained in X. By changing X with $X \cap \langle x \rangle^G$, which is finitely generated because G is an FC-group, we may also assume that X is finitely generated. By this and Lemma 3.2, it follows that the torsion subgroup T of Z has finite total rank.

Let H/Z be a normal subgroup of finite index of G/Z such that $H \cap S = Z$. Then H/Z has no non-trivial soluble subnormal subgroup. Let F/Z be a finite non-abelian simple subnormal subgroup of H/Z. Then $E_1/Z = F_1/Z = F^G/Z$ is the direct product of finitely many non-abelian simple subgroups of H/Z (see, for instance, [15, Lemma 5.44]). Now consider H_1/Z to be the normal core of $C_{H/Z}(F_1/Z)$ in G/Z. Clearly, H_1 is a normal subgroup of finite index of G and $F_1 \cap H_1 = Z$, because F_1/Z has trivial centre. As H_1/Z does not contain soluble subnormal subgroups, arguing as above, there exists a G-invariant subgroup E_2/Z of H_1/Z, which is the direct product of finitely many non-abelian finite simple groups. Clearly, $F_2/Z = E_1E_2/Z$ is still a normal subgroup of G/Z which is the direct product of finitely many non-abelian finite simple groups and has trivial intersection with the normal core H_2/Z of $C_{H/Z}(F_2/Z)$ in G/Z. Hence, by recursion we construct the normal subgroup $\bigcup F_i/Z$, which is the direct product of infinitely many subnormal non-abelian simple subgroups, any of them being normal in G/Z by Lemma 3.2. If we now let $\{S_i/Z\}_{i \in I}$ be the collection of all simple non-abelian normal subgroups of G/Z, from Lemma 2.4 we get that each $S_i' \cap Z$ is a periodic central subgroup of G which can have only a very restricted structure. Put $Z_i = S_i' \cap Z$ and let U be the subgroup of Z generated by every Z_i. As G/U contains a direct product of infinitely many subgroups, from Lemma 3.2 it follows that G/U is a T-group. Moreover, since U satisfies the minimal condition on subgroups, we may find a subgroup A of U such that G/A is a maximal T-quotient of G. If A is finite, set $J = \{i \in I \mid A \leq Z_i\}$. If $I \setminus J$ is infinite, we may let $H = \langle Z_i \mid i \in I \setminus J \rangle$, so that the quotient G/H contains an infinite direct product of finite simple non-abelian groups and it is a T-group by Lemma 3.2. However, Lemma 2.1 shows that $G/H \cap A$ is a T-group and this goes against the maximality of G/A. Then in this case J is infinite and $I \setminus J$ is finite. On the other hand, if A is infinite, this means that there are infinitely many projective or unitary members of $\{S_i/Z_i\}$ and, correspondingly, infinitely many cyclic members of $\{Z_i\}$. Since U has finite total rank, we may find an infinite subfamily of $\{S_{i_j}\}_{j \in J_1}$ of $\{S_i\}$ whose corresponding $\{Z_{i_j}\}_{j \in J_1}$ generate an infinite locally cyclic group of finite total rank. This, together with the minimality of A, shows that A is exactly the subgroup $\langle S_{i_j} \mid j \in J_1 \rangle$ and in particular it is divisible. In this case, set $J = \{i \in I \mid Z_i < A\}$. As before, $I \setminus J$ is finite.

Put now $P = \langle S_j \mid j \in J \rangle$, $R = \langle S_i \mid i \in I \rangle$ and $B = \langle Z_j \mid j \in J \rangle$. Notice that $P_1 \leq G''$, since P is perfect, and that P/B is an infinite direct product of

finite simple non-abelian normal subgroups of G/B, since J is infinite. Now the finiteness of $I \setminus J$ yields that R/P is finite, while G/Z is a T-group by Lemma 3.2 since P/B is infinite. Then by Theorem 2.5 we may find a perfect subgroup P_1/Z containing $Z(S)/Z$ such that G/P_1 is a soluble T-group, so that $G'' = P_1$. Remember that $|G : H|$ is finite, so that $|G'' : G'' \cap H|$ is finite as well. However, $H \cap S = Z$ and hence $G'' \cap H \leq R$, which shows that $|G'' : R|$ is finite. Then G''/P is finite.

If now A is infinite, we already saw that $A \leq \langle Z_j \,|\, j \in J \rangle$ and hence $A = B$. On the other hand, if A is finite, by definition it is contained in each S_j' for $j \in J$, and hence we are done with point (i).

Finally, if A is infinite, we have already showed that it is locally cyclic, while if it is finite it is defined in a way to be contained in each Z_j with $j \in J$. In particular, Lemma 2.4 applied to the subgroups S_j (with $j \in J$) yields the structural information on A needed in point (ii). The same application of Lemma 2.4 gives the condition for B to have bounded exponent and this completes (ii) and the proof of our lemma. □

Notice here that Example 4.5 shows that in this lemma the hypothesis on G being not a T-group cannot be omitted. In that case, G is a T-group, so $A = \{1\}$ and B can even have infinite rank. Moreover, some further information about the relation between A and B in an FC-group satisfying the DC_{snn}-property but not the T-property is given in Example 4.4, as they can be as far away as possible with regard to cardinality.

We prove a splitting criterion which makes use of cohomology. It can be put in a more general form, but we present it here in a way, that is more than enough for our scopes. From now on, we denote with $\mathbb{Q}$ the group of rational numbers and with $\mathbb{Q}_p$ the group of rational numbers whose denominators are integer powers of a prime number p.

Lemma 3.4. *Let G be a group whose torsion elements form a subgroup T and let p be a prime number. If G/T is isomorphic with a subgroup of $\mathbb{Q}$, $Z(T)$ is finite, G acts trivially on the p'-component of $Z(T)$ and either fixed-point-freely or trivially on the p-component of $Z(T)$, then G splits over T.*

Proof. Let $Q = G/T$, let $Z = Z(T)$, make Z a Q-module with the induced G-action and let $B = [Q, Z]$ and $A = C_Z(Q)$. We first claim that $H^2(Q, Z) = 0$. By hypothesis, B is either trivial or equal to the p-component Z_p of Z, so we can take into account the extension

$$A \rightarrowtail Z \twoheadrightarrow B$$
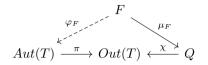
and take the long exact cohomology sequence

$$\cdots \longrightarrow H^2(Q, A) \longrightarrow H^2(Q, Z) \longrightarrow H^2(Q, B) \longrightarrow \cdots$$

If we let

$$Ext(Q, A) \rightarrowtail H^2(Q, A) \twoheadrightarrow Hom(M(Q), A)$$

be the short exact sequence derived by the Universal Coefficients Theorem, we see that $Hom(M(Q), A) = 0$, because Q is locally cyclic and hence $M(Q) = 0$, and that $Ext(Q, A)$ is zero, too, since Q is torsion-free and A is finite (see

[10, p. 223]). If $B = \{1\}$ our claim is proved, so we may assume that Q acts fixed-point-freely on B. On the other hand, now the results in [16] yield that also $H^2(Q, B)$ is zero, so that $H^2(Q, Z) = 0$ in any case. Let now χ be a coupling of Q to T, let F a free subgroup of Q and consider the following diagram

$$\xymatrix{ & F & \\ Aut(T) \ar[r]^{\pi} & Out(T) & Q \ar[l]_{\chi} }$$

where μ_F is the inclusion, π is the canonical epimorphism over $Inn(G)$ and φ_F is the homomorphism from F to $Aut(T)$ which exists by the projective property of free groups. Since Q is locally free, its free subgroups form a directed system, so that, by the universal property of direct limits, there exists a homomorphism from Q to $Aut(T)$ which induces χ over $Out(T)$. This shows that there is a semidirect product of T by Q with χ as its coupling. Since we have already seen that $H^2(Q, Z)$ vanishes, it follows that there is only one equivalence class of extensions of T by Q with coupling χ (see, for instance, [17, 11.4.10]) and hence G splits over T. $\qquad\square$

Recall here that an abelian group A is said to be *minimax* if it contains a finitely generated subgroup X such that A/X is the product of finitely many Prüfer subgroups. It is customary to define the *spectrum* of A, $Sp(A)$ to be the set of primes dividing orders of elements in A/X. We are now ready to prove the main theorem of this section.

Theorem 3.5. *Let G be an FC-group and let T be the torsion subgroup of G. Then G satisfies the double chain condition on subnormal non-normal subgroups if and only if it satisfies one of the following conditions:*

(a) *G is a Černikov or a polycyclic group;*
(b) *G is T-group;*
(c) *G is not central-by-finite and*
　(c.1) *G contains subgroups A, B and P such that $A \leq B \leq T \cap Z(G)$, G/A is a maximal T-quotient of G, P/B is an infinite direct product of finite simple non-abelian normal subgroups of G/B and G''/P is finite; moreover, if A is infinite, then $A = B$ and G/A is maximal with respect to containing infinite direct products of finite simple non-abelian G-invariant subgroups, while if A is finite $A \leq N'$ for each normal factor N/B of P/B;*
　(c.2) *if S is the subsoluble radical G, then either*
　　(i) *S is a Černikov group or*
　　(ii) *S is a polycyclic group or*
　　(iii) *S is a polycyclic extension of a Prüfer group properly containing A;*
　(c.3) *A is isomorphic either with a subgroup of $\mathbb{Z}_l \times \mathbb{Z}_m \times \mathbb{Z}_n$, where*

$$(l, m, n) \in \{(3, 3, 4), (3, 4, 4)\},$$

> or with a locally cyclic periodic group; if moreover there is a bound
> over the order of the fields or over the dimensions of every simple
> direct factor of P/B which is of projective or unitary type, then B
> has finite exponent;

(d) G is central-by-finite and $Z(G) = H \times P$, where H is finitely generat-
ed and infinite and P is a Prüfer group containing a finite non-trivial
subgroup A such that G/A is a T-group;

(e) $G = H \ltimes T$, where H is torsion-free minimax group of rank 1 such that
$Sp(H) = \{p\}$ for a prime p, T is finite, every subnormal subgroup of T
is normal in G and G acts trivially on each p'-subgroup of $Fit(T)$ and
$Fit(T/T'')$.

Proof. Suppose first that G satisfies the double chain condition on subnormal
non-normal subgroups. Assume that G is neither Černikov nor polycyclic nor
a T-group, let X be a subnormal non-normal subgroup of G, put $Z = Z(G)$
and let S be the subsoluble radical of G, which is also the soluble radical
of G by Proposition 3.1. If S is either Černikov or polycyclic, we have that
G/S is infinite and that S/Z is finite, since G/Z is locally and residually
finite, so that G satisfies (c.1) and (c.3) by Lemma 3.3 and clearly also (c.2).
Suppose hence that S is neither Černikov nor polycyclic. If we let x be an
element of X such that $\langle x \rangle^G$ is not contained in X, then we may replace
X with $X \cap \langle x \rangle^G$ and assume from now on that X is finite if and only if x
is periodic and that X^G is finitely generated, because of the FC-property.
By this and Lemma 3.2, we have that any abelian subgroup of $F = Fit(S)$
has finite total rank, so that, in particular, the torsion subgroup T of F is a
Černikov group. Then, if Z is torsion, the whole G is torsion, so that $T = F$
and S is a Černikov group, as $S/C_S(F)$ is periodic group of automorphisms
of a Černikov group. This contradiction to our assumption on S gives that Z
is not periodic.

Suppose now for a contradiction that G does not satisfy the maxi-
mal condition on subnormal non-normal subgroups and let M be a mini-
mal subnormal non-normal subgroup of G. Notice here that any minimal
subnormal non-normal subgroup of an FC-group is finite. For if m is an el-
ement of M such that $\langle m \rangle^G$ is not contained in M and m has finite order,
$M = M \cap \langle m \rangle^G < \langle m \rangle^G$ is finite, while if m has infinite order, then M is eas-
ily seen to contain some proper subgroup which is subnormal and not normal
in G, as G/Z is periodic. Let now

$$M = G_0 < G_1 < \cdots < G_n < \cdots$$

be an ascending chain of subnormal non-normal subgroups of G with M
minimal subnormal non-normal and G_1 minimal subnormal non-normal with
respect to the condition of containing M. We remark that the existence of
G_1 is guaranteed by the double chain condition. Let $N = M^{G_1}$, which is
finite because M itself is finite. Since M is subnormal in G_1, N is a proper
subgroup of G_1, so that we may write $G_1 = \langle \langle g \rangle^{G_1} N \mid g \in G_1 \setminus N \rangle$; then by
the minimality of G_1 and since it is not normal in G, we have as before that
G_1/N must coincide with the normal closure in itself of one of its elements,
say $G_1 = \langle g_1 \rangle^{G_1} N$. If now $g_1 N$ has infinite order, as before it contains a

proper subgroup containing M which is subnormal non-normal in G and this contradicts the minimality of G_1. So G_1/N is finite and thus G_1 is finite. Hence we may assume that every G_i is finite. If a is a central element of infinite order of G, then

$$\cdots < \langle a^2 \rangle G_0 < \langle a \rangle G_0 < \langle a \rangle G_1 < \cdots < \langle a \rangle G_n < \cdots$$

is a double chain of subnormal non-normal subgroups of G, so that G is periodic, against what we already proved. Then G satisfies the maximal condition on subnormal non-normal subgroups. Moreover, notice that F is nilpotent (see [4, Corollary 2.8]) and that in particular it satisfies the maximal condition on non-normal subgroups. Let now P be a Prüfer p-subgroup of Z for a prime p, if any. It follows from the maximal condition on subnormal non-normal subgroups that for each subnormal non-normal subgroup Y of G we may find a non-trivial cyclic subgroup C_Y contained in P such that YC_Y is normal in G. Assume by a contradiction that the orders of these C_Y have no upper bound. In this case, let $\langle c_i \rangle$ be the subgroup of order p^i of P for any positive integer i and let J be an infinite subset of $\mathbb{N}$ such that $\{Y_j\}_{j \in J}$ is a family of subnormal non-normal subgroups of G for which $C_{Y_j} = \langle c_j \rangle$. Since the normal closure of Y_j is precisely $Y_j\langle c_j \rangle$, we have that each subgroup of the following ascending chain

$$Y_{j_1} < \langle Y_{j_1}, Y_{j_2} \rangle < \cdots < \langle Y_{j_1}, Y_{j_2}, \ldots, Y_{j_i} \rangle < \cdots$$

is subnormal and not normal and this goes against our last assumption on G. Then there is a maximal cyclic subgroup C of P such that G/C is a T-group. We now want to show that if such P exists, it is the only Prüfer subgroup of G. To this aim, let P_1 be a Prüfer subgroup of Z with trivial intersection with P. By what just proved, there exists a subgroup C_1 of P_1 such that G/C_1 is a T-group. However, Lemma 2.1 yields that G itself is a T-group and this case has been already ruled out.

We split the proof in two cases: whether F/T is finitely generated or not.

Suppose first that F/T is finitely generated. This, together with the fact that S is not polycyclic, implies that T contains the Prüfer p-subgroup P. Now, $S/Z(F)$ is isomorphic with a periodic group of automorphisms of F and hence is finite. Since P is the only Prüfer subgroup of S, S/P is polycyclic. Thence, if G/S is infinite, S satisfies (c.2) and G satisfies (c.1) and (c.3) again by Lemma 3.3, while if G/S is finite, G satisfies (d).

Suppose finally that F/T is not finitely generated. As F satisfies the maximal condition on non-normal subgroups and T is a Černikov group, by the results in [9] we may write $F = Q \times T$, where Q is abelian, torsion-free and not finitely generated. Notice that by Lemma 2.2 in [4], every subgroup of Q is normal in G and hence Q is central in G by the FC-property. By this, we get that $S/Z(S)$ is isomorphic with a periodic group of automorphisms of T, which is a Černikov group and hence is itself finite. Assume by a contradiction that Q has rank at least 2, let K be a maximal locally cyclic subgroup of Q which is not finitely generated and let a be an element of Q such that $\langle a \rangle$ has trivial intersection with K. Notice that $X \cap K < \langle x \rangle$. If $\langle x \rangle$ has non-trivial

intersection with K, then $\langle xa \rangle \cap K = \{1\}$, so that we may assume that X and K have trivial intersection. However, if

$$\langle k_1 \rangle < \langle k_2 \rangle < \cdots < \langle k_i \rangle < \cdots$$

is an ascending chain in K, then the family of subgroups of G defined by $\{X\langle k_i \rangle\}_{i \geq 1}$ forms an ascending chain of subnormal non-normal subgroups. Hence Q has rank 1. Moreover, if Q contains a subgroup R such that Q/R has infinite total rank, than G/R contains an infinite direct product of locally cyclic subgroups and hence G/R is a T-group by Lemma 3.2 and even G is a T-group by Lemma 2.1. This shows in particular that Q is minimax and, since Q is not finitely generated, it is divisible for at least one prime, call it q. Let

$$Q_0 < Q_1 < \cdots < Q_n < \cdots$$

be an ascending chain of subgroups of Q. Every finitely generated periodic subnormal subgroup W of G has trivial intersection with Q, and hence there exists a non-negative integer m such that WB_m is normal in G, so that $W = W \cap WQ_m$ is also normal in G. Then every periodic subnormal subgroup of G is normal in G and in particular x has infinite order. Let $X = \langle x \rangle \ltimes X_1$ for a periodic subgroup X_1 of G. Notice that we can do this since $X = \langle x \rangle^G \cap X$ and since X' is periodic. Let now T_p and $T_{p'}$ be the p-component and the p'-component of T, respectively, and assume the existence of a q'-element t of T such that $\langle ta \rangle$ is different from $\langle ta \rangle^g$ for some g in G and a in Q. As Q is q-divisible, we may find a subset $\{a = a_0, a_1, \ldots\}$ of Q such that $a_{i+1}^q = a_i$ for any non-negative integer i. Let e be the order of $\langle t \rangle$ and consider now the following family of subgroups: $\{\langle t^{k_i} a_i \rangle\}_{i \geq 0}$, where $k_0 = 1$ and $qk_i = k_{i-1} \pmod{e}$ for $i > 0$. Then this family is easily seen to form a proper ascending chain of subgroups which are not normal in G and this is a contradiction. So every subgroup of $Q \times T_{q'}$ is normal in G. In particular, as the same reasoning can be made in G/X_1 and not every subgroup of XF/X_1 is normal in G, Q is divisible only by q.

As we want to show that T is finite, which is to say that it contains no Prüfer subgroups, and that G/S is finite, we will assume for simplicity in this paragraph that $X_1 = \{1\}$; in fact, the Fitting subgroup of G/X_1 is still a direct product of a subgroup which is torsion-free, minimax, divisible only by q and has rank 1, with a torsion subgroup all of whose cyclic subgroups are normal in G/X_1. Now, F contains $\langle x \rangle$, which is subnormal and non-normal, so that we have that T_p is not central in G, otherwise F itself would be central, and may write $x = ab$ for elements a of Q and b of $T_p \setminus Z(G)$. In particular, T_p is reduced, otherwise P would not be central in G. However, if $T_{p'}$ is infinitely generated, one steadily finds an ascending chain of subnormal non-normal subgroups containing x and this cannot be. Thence T is finite. Assume now for a contradiction that G/S is infinite. Then Lemma 3.3 yields the existence of a periodic subgroup $A \leq Z(G)$ such that G/A is a T-group. Notice that we can keep on assuming $X_1 = \{1\}$, since it clearly does not contain A. However, $\langle a, b \rangle$ has trivial intersection with A, so that $\langle x \rangle A$ is still subnormal and not normal in G/A. Then G/S is finite. Let E be the

torsion subgroup of G. Since Q has rank 1 and $Sp(Q) = \{q\}$, the same holds for G/E, which is hence a torsion-free minimax group of rank 1 with $Sp(G/E) = \{q\}$. As G/E is q-divisible, it either acts trivially or fixed-point-freely on T_p and hence we can use Lemma 3.4 to find a subgroup H of G such that $G = H \ltimes T$ and H is torsion-free minimax group of rank 1 such that $Sp(H) = \{q\}$. Finally, assume for a contradiction that U is a periodic normal non-central subgroup of G containing a normal subgroup V such that $U/V = \langle u \rangle V/V$ is cyclic of order q^n for a prime $q \neq p$ and a positive integer n. Since $A \cap V = \{1\}$, we may assume without loss of generality that $V = \{1\}$. Let h be a non-trivial element of Q and let $y = hu$. Now, $F/\langle y^{q^n} \rangle$ contains a Prüfer p-group and $\langle y \rangle \langle y^{q^n} \rangle$ is a subnormal non-normal subgroup of $G/\langle y^{q^n} \rangle$, which makes it possible to construct an ascending chain of subnormal non-normal subgroups. Hence G is of type (e) and this concludes the necessary condition.

Conversely, an FC-group of type (a) or (b) clearly satisfies the double chain condition on subnormal non-normal subgroups, while a group of type (d) certainly satisfies the maximal one. Hence we may assume that G is an FC-group of type (c) or (e). Begin with type (c) and notice that the subsoluble radical S of G is also the soluble radical of G by (c.2). Denote here with a "bar" a quotient group modulo S. Let $\overline{G}'' = \overline{P} \times \overline{Q}$, where $\overline{P} = \mathrm{Dr}\overline{P_i}$ and $\overline{Q} = \mathrm{Dr}\overline{Q_i}$. Since G is an FC-group, G/A is a T-group and $\overline{G}''$ has trivial centralizer in $\overline{G}$, G/G'' can be embedded into the direct product of $\mathrm{Dr}Aut(\overline{P_i}) \times \mathrm{Dr}Aut(\overline{Q_i})$, so that $C_{\overline{G}}(\overline{P})$ is a finite group by (c.1). Let $\overline{C} = C_{\overline{G}}(\overline{P})$ and let X be any subnormal non-normal subgroup of G. Assume first that A is finite and let N/B be any finite simple direct factor of P/B. Since $A \leq N'$ by (c.1), X has to centralise N and hence the whole P by the generality of N. In particular, X is contained in C. If S is either polycyclic or Černikov, every subnormal non-normal subgroup of G is contained in a polycyclic or Černikov group (i.e. the subgroup C), respectively, so G satisfies the double chain condition on subnormal non-normal subgroups. Hence we may assume that S is a polycyclic extension of a Prüfer p-group D for a prime p such that D properly contains A. In this case, since C/D satisfies the maximal condition on subgroups, for any ascending chain

$$G_1 < G_2 < \cdots < G_n < \cdots$$

of subnormal non-normal subgroup of G, the subgroups of which we have shown are contained in C, there exists a positive integer m such that G_m has to contain A itself. Hence G_m is normal in G and this contradiction shows that G satisfies again the maximal condition on subnormal non-normal subgroups. Assume now that A is infinite, so that, by (c.2) and (c.3), S is a Černikov group. Since G/A is a T-group, $X \cap A$ is a proper subgroup of A. Since $A = B$ and G/A is maximal with respect to containing infinite direct products of finite simple non-abelian G-invariant subgroups, we have that only finitely many P_i' are contained inside $X \cap A$, so we may collect all of these in a finite subgroup P_1 of P and write $\overline{P} = \overline{P_1} \times \overline{P_2}$. Since $\overline{C}$ is finite, $C_{\overline{G}}(\overline{P_2})$ is finite as well. Now, if we let $\overline{C_1} = C_{\overline{G}}(\overline{P_2})$, just as in the case where A was finite,

X is contained in C_1, which is Černikov since S is such. Hence in this case G satisfies the minimal condition on subnormal non-normal subgroups. Finally, let G be a group described in (e), assume without loss of generality that X is finitely generated and let $X = \langle x \rangle \ltimes X_1$ be a subnormal non-normal subgroups of G for an element x of infinite order and a periodic subgroup X_1. Since we want to show that only finitely many subnormal non-normal subgroups contain X, using the facts that T is a T-group with structure as in Theorem 2.5 and that moreover every subnormal subgroup of T is normal in G, it is not difficult to show that we can assume that $X_1 = \{1\}$ and that $\langle xt \rangle$ is central in G for an element t of $F = Fit(T)$ of order a power of a prime q. Now we know that every p'-subgroup of F is central in G, so that $q = p$. Then, in particular, the element $x(Z(G) \cap \langle x \rangle)$ has order a (positive) power of p. However, $x(Z(G) \cap \langle x \rangle)$ is not contained in the only Prüfer p-subgroup of $G/(Z(G) \cap \langle x \rangle)$ and hence cannot have infinite height. From this it follows that G has only finitely many subgroups containing $\langle x \rangle$ and our proof is concluded. $\square$

From the proof of this theorem it is not difficult to derive the following two corollaries pertaining the maximal and the minimal condition on subnormal non-normal subgroups.

Corollary 3.6. *Let G be an FC-group. Then G satisfies the maximal chain condition on subnormal non-normal subgroups if and only if it satisfies one of the following*

(a) *G is a polycyclic group;*

(b) *G is T-group;*

(c) *if G is not central-by-finite, then*

    (c.1) *G contains subgroups A, B and P such that $A \leq B \leq T \cap Z(G)$, G/A is a maximal T-quotient of G, P/B is an infinite direct product of finite simple non-abelian normal subgroups of G/B, G''/P is finite and $A \leq N'$ for each normal factor N/B of P/B;*

    (c.2) *the subsoluble radical S of G is either polycyclic or a polycyclic extension of a Prüfer group properly containing A;*

    (c.3) *A is isomorphic with a subgroup of $\mathbb{Z}_l \times \mathbb{Z}_m \times \mathbb{Z}_n$, with*

$$(l, m, n) \in \{(3, 3, 4), (3, 4, 4)\};$$

(d) *if G is central-by-finite, then $Z(G) = H \times P$, where H is finitely generated and infinite and P is a Prüfer group containing a finite non-trivial subgroup A such that G/A is a T-group.*

(e) *$G = H \ltimes T$, where H is torsion-free minimax group of rank 1 such that $Sp(H) = \{p\}$ for a prime p, T is finite, every subnormal subgroup of T is normal in G and G acts trivially on each p'-subgroup of $Fit(T)$ and $Fit(T/T'')$.*

Corollary 3.7. *Let G be an FC-group. Then G satisfies the minimal condition on subnormal non-normal subgroups if and only if it satisfies one of the following*

(a) *G is a Černikov group;*

(b) *G is T-group;*

(c) *G is not central-by-finite and*

(c.1) *G contains subgroups A, B and P such that $A \leq B \leq T \cap Z(G)$, G/A is a maximal T-quotient of G, P/B is an infinite direct product of finite simple non-abelian normal subgroups of G/B and G''/P is finite; moreover, if A is infinite, then $A = B$ and G/A is maximal with respect to containing infinite direct products of finite simple non-abelian G-invariant subgroups, while if A is finite $A \leq N'$ for each normal factor N/B of P/B;*

(c.2) *the subsoluble radical S of G is a Černikov group;*

(c.3) *A is isomorphic either with a subgroup of $\mathbb{Z}_l \times \mathbb{Z}_m \times \mathbb{Z}_n$, where*

$$(l, m, n) \in \{(3, 3, 4), (3, 4, 4)\},$$

or with a locally cyclic periodic group; if moreover there is a bound over the order of the fields or over the dimensions of every simple direct factor of P/B which is of projective or unitary type, then B has finite exponent;

Notice finally that from the latter corollary it follows that any non-periodic FC-group satisfying the minimal condition on subnormal non-normal subgroups has to be a T-group.

4. Some Examples

In this section, we present some examples on how our results cannot be improved. We begin showing that in Theorem 2.5 the subgroup Z cannot be required to be central in G. To this aim, let f be a primitive root of $GF(25)$, let $H = SL(3, 25)$ and let σ be the element of $Aut(H)$ induced by the Frobenius map $f \rightarrow f^5$. Now, the centre of H has order 3 and is generated by the diagonal element $diag(f^8)$, so that, if we consider the group $G = \langle \sigma \rangle \ltimes H$ under the usual action of σ, we immediately see that G is a T-group with trivial centre. Clearly, we immediately get an infinite T-group of the same kind if we take the direct product of G by itself infinitely many times.

Example 4.1. There exist finite and infinite T-groups with finite conjugacy classes, trivial centre and non-trivial soluble radical.

Notice that this behaviour is not typical of the case in which G is a T-group. Indeed, if G is the direct product of a group constructed above by any DC_{snn}-group H such that $3 \notin \pi(H)$, then we have that G satisfies the double chain condition on subnormal non-normal subgroups and that for any subgroup K of G such that G/K contains a maximal direct product of finite simple non-abelian G-invariant subgroups, K cannot be contained in the centre.

In this connection, we deal now with questions regarding the structure of the centre of an FC-group satisfying the double chain condition on subnormal non-normal subgroups.

Example 4.2. It is possible to construct, for any prime p, an FC-group satisfying the double chain condition on subnormal non-normal subgroups which is a non-split central extension of a Prüfer p-group P and such that each factor of G over a finite subgroup of P is not a T-group.

Proof. Fix a prime p and use Dirichlet's theorem on arithmetic progressions to find, for every integer $m > 1$, a prime q_m such that $l_m p^m = q_m - 1$ for a positive integer l_m. Let now $n_m = p^{m-1}$, define (for $r \in \{1,2\}$) $^r S_m = SL(n_m, q_m)$, $^r G_m = \langle ^r g_m \rangle \ltimes {}^r S_m \simeq GL(n_m, q_m)$, $Z(^r G_m) = \langle ^r z_m \rangle$ and $P = \bigcup_{i \in \mathbb{N}} \langle c_i \rangle$ a group of type p^∞ with $|\langle c_i \rangle| = p^i$. Notice that the order of $^r z_m$ is p^{m-1} and that the order of g_m, which is $q_m - 1$, is divided by p^m for each m. Let

$$H = (\operatorname*{Dr}_{m>1} {}^2 G_m) \ltimes (\operatorname*{Dr}_{m>1} {}^1 G_m \times P),$$

where $[P, {}^2 G_m] = 1$, $[{}^1 G_i, {}^2 G_j] = 1$ if $i \neq j$, $[{}^1 S_i \langle g_i^{p^i} \rangle, {}^2 g_i] = 1$ and $[{}^1 g_i^{l_i}, {}^2 g_i] = c_i$. Let K be the (central) subgroup of H generated by the elements $^r z_m^{-1} c_{m-1}$ (for each $r \in \{1, 2\}$ and $m > 1$) and let $G = H/K$. Then G is the FC-group which we were looking for. Notice that G clearly satisfies the minimal condition on subnormal non-normal subgroups, while it does not satisfy the maximal one, as the consideration of the family $\{{}^1 G_2 H/H, \ldots, {}^1 G_m H/H, \ldots\}$ shows. $\square$

A little modification in what just constructed (namely taking $n_m = p^m$ and, consequently, H being generated by the elements of the form $^r z_m^{-1} c_m$) leads to the following

Example 4.3. It is possible to construct, for any prime p, a T-group with finite conjugacy classes in which every proper central factor is a non-split central extension of a Prufer p-group P.

With the same notations as in Example 4.2, let

$$H = (^2 G_2 \times \operatorname*{Dr}_{m>2} {}^2 S_m) \ltimes (^1 G_2 \times \operatorname*{Dr}_{m>2} {}^1 S_m \times P),$$

where $[P, {}^2 S_m] = [P, {}^2 G_2] = 1$, $[{}^1 S_m, {}^2 G_2] = [{}^2 S_m, {}^1 G_2] = 1$ for $m > 2$, $[{}^1 S_i, {}^2 S_j] = 1$ for all i and j and $[{}^1 g_2^{l_2}, {}^2 g_2] = c_2$. Let K be the (central) subgroup of H generated by the elements $^r z_m^{-1} c_{m-1}$ (for each $r \in \{1, 2\}$ and $m > 2$) and let $G = H/K$. Then G is an FC-group satisfying the minimal condition on subnormal non-normal subgroups with A and B respectively finite and infinite in the notation of Theorem 3.5. Then, we have proved the following

Example 4.4. It is possible to construct, for any prime p, an FC-group G satisfying the double chain condition on subnormal non-normal subgroups which is not a T-group and which contains a finite subgroup A and an infinite subgroup B such that $A \leq B \leq Z(G)$, G/A is a maximal T-quotient of G and G/B is maximal with respect to containing an infinite direct product of finite simple non-abelian G-invariant subgroups.

Another question arises naturally from Theorem 2.5 and it concerns the rank and the exponent of the subgroup $Z(P)$. Again with the same notations as in Example 4.2, let $G_p = (\operatorname*{Dr}_{m>1} {}^1 S_m \times P)K/K$ and let $G = \operatorname*{Dr}_{p \in \mathbb{P}} G_p$. Then we get more information on $Z(P)$.

Example 4.5. There exists a T-group G with finite conjugacy classes which contains a perfect subgroup P such that $Z(P)$ has infinite exponent and infinite total rank.

We remark here that one can construct a similar example where $Z(P)$ is not even central in G.

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