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Existence of Hölder Continuous Solutions for a Class of Degenerate Fourth-Order Elliptic Equations

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Abstract. This paper deals with the existence of bounded and locally Hölder continuous weak solutions of a homogeneous Dirichlet problem related to a class of nonlinear fourth-order elliptic equations with strengthened degenerate ellipticity condition.

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1. Introduction

In this paper, we prove the existence of bounded and locally Hölder continuous weak solutions of the homogeneous Dirichlet problem related to a class of nonlinear fourth-order elliptic equations whose model is

$$\sum_{|\alpha|=1,2} (-1)^{|\alpha|} \mathrm{D}^\alpha \left[\frac{|\mathrm{D}^\alpha u|^{p_\alpha -2} \mathrm{D}^\alpha u}{(1+|u|)^{\theta(p_\alpha -1)}} \right] = f \quad \text{ in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is an open-bounded set, $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index with nonnegative integer components and length $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\mathrm{D}^\alpha u(x) = \dfrac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}$. Here, θ and p_α are real numbers, such that $0 \leq \theta < 1$ and

$$p_\alpha = \begin{cases} q & \text{if } |\alpha| = 1 \\ p & \text{if } |\alpha| = 2 \end{cases} \tag{1.2}$$

with $1 < p < \frac{N}{2}$, $2p < q < N$, and $f \in L^t(\Omega)$ with $t > \frac{N}{q}$.

In the case $\theta = 0$, Eq. (1.1) is the fourth-order prototype of a class of nonlinear higher order elliptic equations introduced by I. V. Skrypnik in [42].

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It is well known that for the $2m$-order equation

$$\sum_{|\alpha|\le m}(-1)^{|\alpha|}\mathrm{D}^{\alpha}A_{\alpha}(x,u,\mathrm{D}^{1}u,\ldots,\mathrm{D}^{m}u)=0 \tag{1.3}$$

the ellipticity condition

$$\sum_{|\alpha|=m}A_{\alpha}(x,\xi)\xi_{\alpha}\ge C_1\sum_{|\alpha|=m}|\xi_{\alpha}|^{p}-C_2\sum_{|\beta|<m}|\xi_{\beta}|^{p}$$

does not ensure the boundedness of a solution $u\in W^{m,p}(\Omega)$, unless $mp>N$ (as a consequence of Sobolev's embedding theorem) or $mp=N$ (see [20]) or $N-mp$ is sufficiently small (see [49]), while in the case where $N>mp$, examples of equations with unbounded weak solutions are available.

In [42], I. V. Skrypnik has selected a subclass of (1.3) imposing a strengthened ellipticity condition which, in the model case, takes the following form:

$$\sum_{1\le|\alpha|\le m}A_{\alpha}(x,\eta,\xi)\xi_{\alpha}\ge C_1\Big(\sum_{|\alpha|=m}|\xi_{\alpha}|^{p}+\sum_{|\alpha|=1}|\xi_{\alpha}|^{q}\Big),$$

where $C_1>0$, $p\ge 2$, and $mp<q<N$.

This condition allowed reaching Hölder continuity of any generalized solution $u\in W^{1,q}(\Omega)\cap W^{m,p}(\Omega)$ without any further relation on N, m, p.

Here, we consider a degenerate version of Skrypnik's fourth-order operator in the sense that the differential operator

$$u\to\mathcal{A}(u)=\sum_{|\alpha|=1,2}(-1)^{|\alpha|}\mathrm{D}^{\alpha}\left[\frac{|\mathrm{D}^{\alpha}u|^{p_{\alpha}-2}\mathrm{D}^{\alpha}u}{(1+|u|)^{\theta(p_{\alpha}-1)}}\right],\quad 0<\theta<1,$$

though well defined, is not coercive on $W_0^{1,q}(\Omega)\cap W_0^{2,p}(\Omega)$ when u is large.

Due to this lack of coercivity, standard existence theorems for solutions of nonlinear equations cannot be applied. We overcome this difficulty by approximating our problem with a sequence of homogeneous nondegenerate Dirichlet problems and we will prove an L^{∞}—a priori estimate on the approximating solutions which, in turn, implies an a priori estimate in the energy space. Once this has been accomplished, a compactness result for the approximating solutions allows us to find a bounded weak solution of the problem (1.1) which is, as well, locally Hölder continuous.

It is worthwhile to note that in the case of second-order equations, the existence of solutions of the Dirichlet problem

$$\begin{cases}-\operatorname{div}\left[\dfrac{|\nabla u|^{q-2}\nabla u}{(1+|u|)^{\theta(q-1)}}\right]=f & \text{in }\Omega\\ u=0 & \text{on }\partial\Omega\end{cases}$$

under various assumptions on f, has been studied in the papers [1,2].

We point out that the equation we are dealing with presents two more difficulties: it involves a fourth-order operator which behaves like a system of PDEs, and moreover, it has non smooth coefficients. Many of the well-known techniques which work for one single equation of second order do not hold anymore in the framework of high-order equations and we need to find a suitable method to overcome the issues.

This article is organized as follows. In Sect. 2, we formulate the hypotheses and state the results. In Sect. 3, we prove two a priori estimates will be used in the proof of the main theorem. At last, in Sect. 4, we give the proofs of the existence of bounded solutions as well as their Hölder's continuity.

2. Preliminaries and Statement of the Results

Let Ω be an open-bounded set in $\mathbb{R}^N$ with $N \geq 3$. We denote by $N(2)$ the number of different multi-indices α, such that $|\alpha| = 1, 2$.

Let $A_\alpha(x, \eta, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^{N(2)} \to \mathbb{R}$, with $|\alpha| = 1, 2$, be Carathéodory functions (i.e., $A_\alpha(\cdot, \eta, \xi)$ are measurable on Ω for every $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^{N(2)}$ and $A_\alpha(x, \cdot, \cdot)$ are continuous on $\mathbb{R} \times \mathbb{R}^{N(2)}$ for almost every $x \in \Omega$) satisfying the following structural conditions, for almost every $x \in \Omega$, every $\eta \in \mathbb{R}$ and ξ, $\xi' \in \mathbb{R}^{N(2)}$, $\xi \neq \xi'$:

$$\sum_{|\alpha|=1,2} A_\alpha(x, \eta, \xi)\xi_\alpha \geq \nu_1 \sum_{|\alpha|=1,2} \frac{|\xi_\alpha|^{p_\alpha}}{(1 + |\eta|)^{\theta(p_\alpha - 1)}}, \tag{2.1}$$

$$\sum_{|\alpha|=1,2} |A_\alpha(x, \eta, \xi)|^{\frac{p_\alpha}{p_\alpha - 1}} \leq \nu_2 \Big[\sum_{|\alpha|=1} |\xi_\alpha|^q + \sum_{|\alpha|=2} |\xi_\alpha|^p \Big], \tag{2.2}$$

$$\sum_{|\alpha|=1,2} \big[A_\alpha(x, \eta, \xi) - A_\alpha(x, \eta, \xi')\big](\xi_\alpha - \xi'_\alpha) > 0, \tag{2.3}$$

where ν_1, ν_2 are positive constants, the numbers p_α, $|\alpha| = 1, 2$ are defined by (1.2), and

$$f \in L^t(\Omega) \quad \text{with } t > N/q. \tag{2.4}$$

We set

$$W^{1,q}_{2,p}(\Omega) = W^{1,q}(\Omega) \cap W^{2,p}(\Omega),$$

and

$$\mathring{W}^{1,q}_{2,p}(\Omega) = W^{1,q}_0(\Omega) \cap W^{2,p}_0(\Omega).$$

The assumptions (2.1)–(2.4) allow us to give the following:

Definition 2.1. A weak solution of the problem

$$\begin{cases} \displaystyle\sum_{|\alpha|=1,2} (-1)^{|\alpha|} \mathrm{D}^\alpha A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u) = f & \text{in } \Omega \\ \mathrm{D}^\alpha u = 0, \ |\alpha| = 0, 1 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

is a function $u : \Omega \to \mathbb{R}$, such that

$$\begin{cases} u \in \mathring{W}^{1,q}_{2,p}(\Omega), \\ \displaystyle\sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u) \mathrm{D}^\alpha v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \end{cases} \tag{2.6}$$

for every $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$.

Our first result states the existence of a bounded weak solution of (2.5).

Theorem 2.2. *Let us suppose that conditions* (2.1)–(2.4) *are satisfied and*

$$0 \leq \theta < \frac{q-p}{p(q-1)}. \tag{2.7}$$

Then, there exists a weak solution u of the problem (2.5), in the sense of Definition 2.1*,*

such that

$$||u||_\infty \leq M \tag{2.8}$$

where $M > 0$ is a constant depending on θ, N, q, p, ν_1, ν_2, $|\Omega|$ and $||f||_{L^t(\Omega)}$.

Under the same assumptions, as in the nondegenerate case, it can be readily proved the local Hölder continuity of any weak solution $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$. Namely

Theorem 2.3. *Let us suppose that conditions* (2.1)–(2.4) *and* (2.7) *are satisfied. Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ be a bounded solution of the problem* (2.5)*. Then, there exists $\rho \in (0, 1)$, depending on the data and on $||u||_{L^\infty(\Omega)}$, such that $u \in C^{0,\rho}_{loc}(\Omega)$ and for any domain $\Omega' \subset\subset \Omega$, we have*

$$|u(x) - u(y)| \leq C|x - y|^\rho \quad \textit{for any } x, y \in \Omega',$$

where C is a positive constant depending on the same parameters of ρ and $d' = \mathrm{dist}(\Omega', \partial\Omega)$.

Remark 2.4. In the case $\theta = 0$, operators satisfying condition (2.1) have been studied in connection with many other questions such as homogenization problems, L^1-theory, qualitative properties of the solutions, and removable singularities in the degenerate and nondegenerate case (see [4,5,16–18,26,40,43]). Moreover, a class of nonlinear fourth-order equation with principal part satisfying (2.1) and lower order term having the so-called "natural growth" or a convection term has been studied in [8,12,45–47].

In the framework of second-order elliptic equations with a lower order term having natural growth with respect to Du, the existence of bounded solutions has been studied in [3] assuming f in $L^t(\Omega)$, with $t > \frac{N}{q}$, and in [6,7,10,11,14,15] assuming f in a suitable Morrey space.

For related arguments on elliptic systems with special structural conditions, see also [9,13,19,22–24,28–38].

Remark 2.5. We point out that the assumption (2.4) on f required in Theorems 2.2 and 2.3 is the same which yields to the existence of bounded and Hölder continuous solutions for nondegenerate (i.e., $\theta = 0$) fourth-order equations. In this last case, examples of unbounded solutions of equation (1.1), with $f \in L^{\frac{N}{q}}(\Omega)$ and $f \notin L^{\frac{N}{q}+\varepsilon}(\Omega)$, for any $\varepsilon > 0$, are constructed in [48].

3. A Priori Estimates

We begin this section recalling an algebraic lemma due to Serrin (see Lemma 2 in [41]).

Lemma 3.1. *Let χ be a positive exponent and a_i, β_i, $i=1,\dots,N$, be two sets of N real numbers, such that $0<a_i<+\infty$ and $0<\beta_i<\chi$. Suppose that z is a positive number satisfying the inequality*

$$z^\chi \le \sum_{i=1}^N a_i z^{\beta_i}.$$

Then

$$z \le C\sum_{i=1}^N a_i^{\gamma_i},$$

where C depends only on N, χ, β_i, and $\gamma_i=\frac{1}{\chi-\beta_i}$, $i=1,\dots,N$.

Given $n\in\mathbb{N}$, let $T_n(s)$ be the truncation function defined by

$$T_n(s)=\begin{cases} s & \text{if } |s|\le n\\ n\,\mathrm{sign}(s) & \text{if } |s|>n. \end{cases}$$

Following the technique already used in [1] and [2] in the framework of second-order elliptic equations, let us define the following Dirichlet problems:

$$\begin{cases} u_n\in \mathring{W}^{1,q}_{2,p}(\Omega),\\ \displaystyle\sum_{|\alpha|=1,2}\int_\Omega A_\alpha(x,T_n(u_n),\mathrm{D}^1u_n,\mathrm{D}^2u_n)\mathrm{D}^\alpha v\,\mathrm{d}x=\int_\Omega fv\,\mathrm{d}x\\ \text{for every } v\in \mathring{W}^{1,q}_{2,p}(\Omega). \end{cases} \tag{3.1}$$

Since

$$\sum_{|\alpha|=1,2} A_\alpha(x,T_n(\eta),\xi)\xi_\alpha \ge \nu_1\sum_{|\alpha|=1,2}\frac{|\xi_\alpha|^{p_\alpha}}{(1+|n|)^{\theta(p_\alpha-1)}},$$

for almost every $x\in\Omega$ and for every $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^{N(2)}$, by Leray–Lions existence theorem (see [39]), there exists a solution $u_n\in\mathring{W}^{1,q}_{2,p}(\Omega)$ of problem (3.1). Moreover, every u_n is bounded thanks to the boundedness result of [25] (see also [45]). Now, we are going to prove the following.

Lemma 3.2. *Assume that conditions (2.1)–(2.4) are satisfied. Let u_n be a solution of the problem (3.1) for every $n\in\mathbb{N}$. Then, there exists a positive constant M, depending only on θ, N, q, p, ν_1, ν_2, $|\Omega|$ and $||f||_{L^t(\Omega)}$, such that*

$$||u_n||_\infty \le M \quad \text{for every } n\in\mathbb{N}. \tag{3.2}$$

Next lemma deals with the boundedness of u_n in the energy space.

Lemma 3.3. *Let hypotheses (2.1)–(2.4) be satisfied. Then, there exists a positive constant C, depending only on θ, N, q, p, ν_1, ν_2, $|\Omega|$ and $||f||_{L^t(\Omega)}$, such that*

$$||u_n||_{\mathring{W}^{1,q}_{2,p}(\Omega)} \leq C \ \text{for every } n \in \mathbb{N}. \tag{3.3}$$

To prove the previous two lemmas, we have to state some auxiliary propositions. First of all, we need to ensure that the composition of a suitable function $\zeta(s)$ with a function $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$ belongs to $\mathring{W}^{1,q}_{2,p}(\Omega)$.

Lemma 3.4. *Let $\zeta \in \mathrm{C}^2(\mathbb{R})$ be a function with bounded derivatives ζ' and ζ'', such that $\zeta(0) = 0$. If $u \in \mathring{W}^{1,q}_{2,p}(\Omega)$, then*

$$\zeta(u) \in \mathring{W}^{1,q}_{2,p}(\Omega),$$

and for each multi-index α, such that $|\alpha| = 1, 2$, the following assertion holds:

$$\mathrm{D}^\alpha \zeta(u) = \zeta'(u)\mathrm{D}^\alpha u + R_{|\alpha|}(u), \tag{3.4}$$

where

$$R_{|\alpha|}(u) = \begin{cases} 0 & \text{if} \quad |\alpha| = 1 \\ \zeta''(u) \displaystyle\sum_{|\beta|=|\gamma|=1} \mathrm{D}^\beta u \mathrm{D}^\gamma u & \text{if} \quad |\alpha| = 2. \end{cases}$$

Next, we present a slightly modified version of a well-known Stampacchia's lemma (see [44]), whose proof is contained in [2,25]. See also [21] for new generalizations.

Lemma 3.5. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function, such that*

$$\phi(h) \leq \frac{c_0}{(h-k)^\nu} k^{\theta\nu} [\phi(k)]^{1+\mu}, \quad \text{for all } h > k \geq k_0 > 0,$$

for some positive constants c_0 and k_0, with $\nu > 0$, $0 \leq \theta < 1$ and $\mu > 0$.

Then, there exists $k^* > 0$, depending on c_0, θ, ν, μ and k_0, such that $\phi(k^*) = 0$.

Proof of Lemma 3.2. Given $k \geq 1$ and $\sigma > 1 + \frac{pq}{q-2p}$ (note that $\sigma > 2$), let us consider the function $v = \zeta(u_n)$ with

$$\zeta(s) = \big[|s| - k\big]_+^\sigma \operatorname{sign}(s).$$

Due to the boundedness of u_n, as a consequence of Lemma 3.4, v is an admissible test function in (3.1), and it holds

$$\mathrm{D}^\alpha v = \sigma \big[|u_n| - k\big]_+^{\sigma-1} \mathrm{D}^\alpha u_n + R_{|\alpha|}(u_n) \qquad \text{a.e. in } \Omega, \tag{3.5}$$

with $R_{|\alpha|}(u_n) \equiv 0$ if $|\alpha| = 1$ and

$$|R_{|\alpha|}(u_n)| \leq \sigma^2 \big[|u_n| - k\big]_+^{\sigma-2} \sum_{|\beta|=1} |\mathrm{D}^\beta u_n|^2, \tag{3.6}$$

if $|\alpha| = 2$. □

Choosing $v = \zeta(u_n)$ in (3.1), we obtain

$$\sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n)\Big[\sigma\big[|u_n| - k\big]_+^{\sigma-1} \mathrm{D}^\alpha u_n + R_{|\alpha|}(u_n)\Big]\,\mathrm{d}x$$
$$= \int_\Omega f\big[|u_n| - k\big]_+^{\sigma} \operatorname{sign}(u_n)\,\mathrm{d}x. \tag{3.7}$$

Using the ellipticity condition (2.1), from the above relation, we get

$$\nu_1\sigma \int_\Omega \sum_{|\alpha|=1,2} \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}}\big[|u_n| - k\big]_+^{\sigma-1}\,\mathrm{d}x$$
$$\le c\Big\{\sum_{|\alpha|=2}\int_\Omega |A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n)||R_{|\alpha|}(u_n)|\,\mathrm{d}x$$
$$+ \int_\Omega |f|\big[|u_n| - k\big]_+^{\sigma}\,\mathrm{d}x\Big\}. \tag{3.8}$$

From now on, we will denote by c a positive constant not depending on n (namely, it may depend on N, $|\Omega|$, p, q, ν_1, ν_2, $||f||_t$) and whose value may vary from line to line.

We are going to evaluate the integrals on the right-hand side of (3.8). Due to the growth condition (2.2) and the estimate (3.6), we get

$$I \equiv \sum_{|\alpha|=2}\int_\Omega |A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n)||R_{|\alpha|}(u_n)|\,\mathrm{d}x$$
$$\le c\int_\Omega \Big[\sum_{|\alpha|=2}|\mathrm{D}^\alpha u_n|^p + \sum_{|\alpha|=1}|\mathrm{D}^\alpha u_n|^q\Big]^{\frac{p-1}{p}}\big[|u_n| - k\big]_+^{\sigma-2}\sum_{|\alpha|=1}|\mathrm{D}^\alpha u_n|^2\,\mathrm{d}x. \tag{3.9}$$

Taking into account the inequality

$$1 \le 1 + |T_n(u_n)| \le 1 + |u_n|$$

and using Young's inequality with exponents $\frac{p}{p-1}$, $\frac{q}{2}$ and $\frac{pq}{q-2p}$, for all $\tau > 0$, we obtain

$$I \le c\tau \int_\Omega \sum_{|\alpha|=1,2} \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}}\big[|u_n| - k\big]_+^{\sigma-1}\,\mathrm{d}x$$
$$+C(\tau)\Big[\int_\Omega \big(1+|u_n|\big)^{\theta(q-1)\left(\frac{pq}{q-2p}-1\right)}\big[|u_n| - k\big]_+^{\sigma-1-\frac{pq}{q-2p}}\,\mathrm{d}x\Big]. \tag{3.10}$$

From (3.8)–(3.10), it results

$$\Big[\nu_1\sigma - c\tau\Big]\int_\Omega \sum_{|\alpha|=1,2} \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}}\big[|u_n| - k\big]_+^{\sigma-1}\,\mathrm{d}x$$
$$\le c\Big[\int_\Omega \big(1+|u_n|\big)^{\theta(q-1)\left(\frac{pq}{q-2p}-1\right)}\big[|u_n| - k\big]_+^{\sigma-1-\frac{pq}{q-2p}}\,\mathrm{d}x\Big]$$

$$+\int_\Omega |f|\big[|u_n|-k\big]_+^\sigma\,\mathrm{d}x. \tag{3.11}$$

Now, we set $\delta=\frac{pq}{q-2p}$. Choosing a suitable $\tau>0$ and observing that

$$1+|u_n|\le 2\big(k+\big[|u_n|-k\big]_+\big) \quad \text{if } k\ge,1,$$

we obtain

$$\int_\Omega \sum_{|\alpha|=1,2}\frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}}\big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x$$
$$\le c\Big\{k^{\theta(q-1)(\delta-1)}\int_\Omega \big[|u_n|-k\big]_+^{\sigma-1-\delta}\,\mathrm{d}x$$
$$+\int_\Omega \big[|u_n|-k\big]_+^{\sigma-1-\delta+\theta(q-1)(\delta-1)}\,\mathrm{d}x+\int_\Omega |f|\big[|u_n|-k\big]_+^\sigma\,\mathrm{d}x\Big\}. \tag{3.12}$$

We denote by $A_n(k)$ the level set of $|u_n|$, that is

$$A_n(k)=\big\{x\in\Omega\,:\,|u_n(x)|>k\big\}$$

and by $|A_n(k)|$ the n-dimensional Lebesgue measure of $A_n(k)$.

Let $\sigma>\max\big\{\frac{\theta(q-1)(\delta-1)-\delta-1}{t'-1},1+\delta\big\}$. Using hypothesis (2.4) and Hölder's inequality, we evaluate terms on the right-hand side of the above inequality, as follows:

$$\int_\Omega |f|\big[|u_n|-k\big]_+^\sigma\,\mathrm{d}x\le \|f\|_{L^t(\Omega)}\Big(\int_\Omega \big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{1}{t'}} \tag{3.13}$$

$$\int_\Omega \big[|u_n|-k\big]_+^{\sigma-1-\delta}\,\mathrm{d}x\le c\Big(\int_\Omega \big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1-\delta}{\sigma t'}}|A_n(k)|^{1-\frac{\sigma-1-\delta}{\sigma t'}} \tag{3.14}$$

and

$$\int_\Omega \big[|u_n|-k\big]_+^{\sigma-1-\delta+\theta(q-1)(\delta-1)}\,\mathrm{d}x$$
$$\le c\Big(\int_\Omega \big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1-\delta+\theta(q-1)(\delta-1)}{\sigma t'}}|A_n(k)|^{1-\frac{\sigma-1-\delta+\theta(q-1)(\delta-1)}{\sigma t'}}, \tag{3.15}$$

where t' is the conjugate exponent of t. From (3.12)–(3.15) and dropping the integrals involving second derivatives in the left-hand side of (3.12), we deduce

$$\int_\Omega \sum_{|\alpha|=1}\frac{|\mathrm{D}^\alpha u_n|^{q}}{(1+T_n(u_n))^{\theta(q-1)}}\big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x$$
$$\le c\Big\{k^{\theta(q-1)(\delta-1)}\Big(\int_\Omega \big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1-\delta}{\sigma t'}}|A_n(k)|^{1-\frac{\sigma-1-\delta}{\sigma t'}}$$

$$+\Big(\int_\Omega \big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1-\delta+\theta(q-1)(\delta-1)}{\sigma t'}}|A_n(k)|^{1-\frac{\sigma-1-\delta+\theta(q-1)(\delta-1)}{\sigma t'}}\Big\}$$
$$+\Big(\int_\Omega \big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma}{\sigma t'}}. \tag{3.16}$$

Let $0<\gamma<q$ and choose $\sigma>\max\Big\{\frac{\frac{\gamma}{q-\gamma}\theta(q-1)-1}{t'-1},\frac{\theta(q-1)(\delta-1)-\delta-1}{t'-1},1+\delta\Big\}$.

The use of Hölder's inequality, together with relations

$$1+|T_n(u_n)|\le 1+|u_n|\le 2\big(k+\big[|u_n|-k\big]_+\big),$$

yields to

$$\begin{aligned}&\int_\Omega |\mathrm{D}^1u|^\gamma\big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x\\ &\quad\le c\Big[\int_\Omega\sum_{|\alpha|=1}\frac{|\mathrm{D}^\alpha u_n|^q}{(1+|T_n(u_n)|)^{\theta(q-1)}}\big[|u_n|-k\big]_+^{\sigma-1}\,\mathrm{d}x\Big]^{\frac{\gamma}{q}}\\ &\quad\times\Big[k^{\frac{\gamma}{q-\gamma}\theta(q-1)}\Big(\int_\Omega\big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1}{\sigma t'}}|A_n(k)|^{1-\frac{\sigma-1}{\sigma t'}}\\ &\quad+\Big(\int_\Omega\big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1+\frac{\gamma}{q-\gamma}\theta(q-1)}{\sigma t'}}|A_n(k)|^{1-\frac{\sigma-1+\frac{\gamma}{q-\gamma}\theta(q-1)}{\sigma t'}}\Big]^{\frac{q-\gamma}{q}}.\end{aligned}\tag{3.17}$$

Using (3.17) in (3.16) and Sobolev's embedding theorem, we obtain

$$\begin{aligned}&\Big[\int_\Omega\big[|u_n|-k\big]_+^{\left(\frac{\sigma-1}{\gamma}+1\right)\gamma^*}\,\mathrm{d}x\Big]^{\frac{\gamma}{\gamma^*}}\\ &\le c\Big\{\Big(\int_\Omega\big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma}{\sigma t'}\frac{\gamma}{q}}\\ &\quad+k^{\theta(q-1)(\delta-1)\frac{\gamma}{q}}\Big(\int_\Omega\big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1-\delta}{\sigma t'}\frac{\gamma}{q}}|A_n(k)|^{\left(1-\frac{\sigma-1-\delta}{\sigma t'}\right)\frac{\gamma}{q}}\\ &\quad+\Big(\int_\Omega\big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1-\delta+\theta(q-1)(\delta-1)}{\sigma t'}\frac{\gamma}{q}}|A_n(k)|^{\frac{\gamma}{q}\left[1-\frac{\sigma-1-\delta+\theta(q-1)(\delta-1)}{\sigma t'}\right]}\Big\}\\ &\quad\times\Big\{k^{\frac{\gamma}{q}\theta(q-1)}\Big(\int_\Omega\big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1}{\sigma t'}\frac{q-\gamma}{q}}|A_n(k)|^{\left(1-\frac{\sigma-1}{\sigma t'}\right)\frac{q-\gamma}{q}}\\ &\quad+\Big(\int_\Omega\big[|u_n|-k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{\sigma-1+\frac{\gamma}{q-\gamma}\theta(q-1)}{\sigma t'}\frac{q-\gamma}{q}}|A_n(k)|^{\left[1-\frac{\sigma-1+\frac{\gamma}{q-\gamma}\theta(q-1)}{\sigma t'}\right]\frac{q-\gamma}{q}}\Big\},\end{aligned}\tag{3.18}$$

where $\gamma^*=\frac{N\gamma}{N-\gamma}$.

Now, we choose γ, such that

$$\sigma t' = \big(\sigma - 1 + \gamma\big)\frac{\gamma^*}{\gamma}. \tag{3.19}$$

Note that $0 < \gamma < q$ if $\sigma > \frac{N(t-1)(1-q)}{tq-N}$ and, in turn, this inequality is satisfied, because $\sigma > 0$, $1-q<0$ and $t > N/q$.

Now, we set

$$\varphi_\sigma(u_n) = \Big(\int_\Omega \big[|u_n| - k\big]_+^{\sigma t'}\,\mathrm{d}x\Big)^{\frac{1}{\sigma t'}}$$

and

$$\begin{aligned}
&\chi = \tfrac{\gamma}{\gamma^*}\sigma t',\ \lambda = \tfrac{\gamma}{q}\theta(q-1),\\
&\beta_1 = \sigma - 1 + \tfrac{\gamma}{q},\ \beta_2 = \sigma - 1 + \tfrac{\gamma}{q} + \tfrac{\gamma}{q}\theta(q-1),\\
&\beta_3 = \sigma - 1 - \tfrac{\gamma}{q}\delta,\ \beta_4 = \sigma - 1 + \tfrac{\gamma}{q}[\theta(q-1)-\delta],\\
&\beta_5 = \sigma - 1 + \tfrac{\gamma}{q}[\theta(q-1)(\delta-1)-\delta],\ \beta_6 = \sigma - 1 + \tfrac{\gamma}{q}\delta[\theta(q-1)-1].
\end{aligned}$$

Then, the inequality (3.18) becomes

$$\begin{aligned}
\big[\varphi_\sigma(u_n)\big]^\chi \le c\Big\{ & k^\lambda\big[\varphi_\sigma(u_n)\big]^{\beta_1}|A_n(k)|^{\left(1-\frac{\sigma-1}{\sigma t'}\right)\frac{q-\gamma}{q}}\\
&+\big[\varphi_\sigma(u_n)\big]^{\beta_2}|A_n(k)|^{\left[1-\frac{\sigma-1+\frac{\gamma}{q-\gamma}\theta(q-1)}{\sigma t'}\right]\frac{q-\gamma}{q}}\\
&+k^{\lambda\delta}\big[\varphi_\sigma(u_n)\big]^{\beta_3}|A_n(k)|^{1-\frac{\sigma-1-\frac{\gamma}{q}\delta}{\sigma t'}}\\
&+k^{\lambda(\delta-1)}\big[\varphi_\sigma(u_n)\big]^{\beta_4}|A_n(k)|^{1-\frac{\sigma-1+\frac{\gamma}{q}[\theta(q-1)-\delta]}{\sigma t'}}\\
&+k^{\lambda}\big[\varphi_\sigma(u_n)\big]^{\beta_5}|A_n(k)|^{1-\frac{\sigma-1+\frac{\gamma}{q}[\theta(q-1)(\delta-1)-\delta]}{\sigma t'}}\\
&+\big[\varphi_\sigma(u_n)\big]^{\beta_6}|A_n(k)|^{1-\frac{\sigma-1+\frac{\gamma}{q}\delta[\theta(q-1)-1]}{\sigma t'}}\Big\}.
\end{aligned} \tag{3.20}$$

Note that the exponents $\beta_1, \beta_2, \beta_3, \beta_4$ are less than χ for any $0 \le \theta < 1$, while β_5, β_6 are less than χ thanks to the assumption (2.7). As a consequence, we can apply Lemma 3.1 to the previous inequality with

$$\begin{aligned}
a_1 = k^\lambda|A_n(k)|^{\left(1-\frac{\sigma-1}{\sigma t'}\right)\frac{q-\gamma}{q}},\ a_2 = |A_n(k)|^{\left\{1-\frac{1}{\sigma t'}\left[\sigma-1+\frac{\gamma}{q-\gamma}\theta(q-1)\right]\right\}\frac{q-\gamma}{q}},\\
a_3 = k^{\lambda\delta}|A_n(k)|^{1-\frac{\beta_3}{\sigma t'}},\quad a_4 = k^{\lambda(\delta-1)}|A_n(k)|^{1-\frac{\beta_4}{\sigma t'}},\\
a_5 = k^{\lambda}|A_n(k)|^{1-\frac{\beta_5}{\sigma t'}},\quad a_6 = |A_n(k)|^{1-\frac{\beta_6}{\sigma t'}},
\end{aligned}$$

obtaining

$$\int_\Omega \left[|u_n|-k\right]_+^{\sigma t'}\,\mathrm{d}x \le c\sum_{i=1}^{6} k^{\tau_i\sigma t'}\,|A_n(k)|^{\gamma_i}, \tag{3.21}$$

where

$$\begin{aligned}
\gamma_1 &= \tfrac{q-\gamma}{q}\left(1-\tfrac{\sigma-1}{\sigma t'}\right)\tfrac{\sigma t'}{\chi-\beta_1} = \tfrac{q-\gamma}{\gamma(q-1)}\left(1-\tfrac{\sigma-1}{\sigma t'}\right)\sigma t',\\
\gamma_2 &= \tfrac{q-\gamma}{q}\left[1-\tfrac{\sigma-1+\frac{\gamma}{q-\gamma}\theta(q-1)}{\sigma t'}\right]\tfrac{\sigma t'}{\chi-\beta_2}\\
&= \tfrac{q-\gamma}{\gamma(q-1)(1-\theta)}\left[1-\tfrac{\sigma-1+\frac{\gamma}{q-\gamma}\theta(q-1)}{\sigma t'}\right]\sigma t',\\
\gamma_3 &= \left(1-\tfrac{\beta_3}{\sigma t'}\right)\tfrac{\sigma t'}{\chi-\beta_3} = \tfrac{q}{\gamma(q+\delta)}\left(1-\tfrac{\sigma-1-\frac{\gamma}{q}\delta}{\sigma t'}\right)\sigma t',\\
\gamma_4 &= \left(1-\tfrac{\beta_4}{\sigma t'}\right)\tfrac{\sigma t'}{\chi-\beta_4} = \tfrac{q}{\gamma(q+\delta-\theta(q-1))}\left(1-\tfrac{\sigma-1+\frac{\gamma}{q}[\theta(q-1)-\delta]}{\sigma t'}\right)\sigma t'\\
\gamma_5 &= \left(1-\tfrac{\beta_5}{\sigma t'}\right)\tfrac{\sigma t'}{\chi-\beta_5}\\
&= \tfrac{q}{\gamma(q+\delta-\theta(q-1)(\delta-1))}\left(1-\tfrac{\sigma-1+\frac{\gamma}{q}[\theta(q-1)(\delta-1)-\delta]}{\sigma t'}\right)\sigma t'\\
\gamma_6 &= \left(1-\tfrac{\beta_6}{\sigma t'}\right)\tfrac{\sigma t'}{\chi-\beta_6}\\
&= \tfrac{q}{\gamma[q+\delta-\theta(q-1)\delta]}\left(1-\tfrac{\sigma-1+\frac{\gamma}{q}\delta[\theta(q-1)-1]}{\sigma t'}\right)\sigma t'
\end{aligned}$$

and

$$\begin{gathered}
\tau_1=\frac{\lambda}{\chi-\beta_1},\quad \tau_2=0,\quad \tau_3=\frac{\lambda\delta}{\chi-\beta_3},\\
\tau_4=\frac{\lambda(\delta-1)}{\chi-\beta_4},\quad \tau_5=\frac{\lambda}{\chi-\beta_5},\quad \tau_6=0.
\end{gathered}$$

We observe that under condition $0\le\theta<\frac{q-p}{p(q-1)}$ every number τ_i, $i=1,\dots,6$, is less than 1.

Moreover, the function $\gamma=\gamma(\sigma)$, defined through (3.19), is positive and bounded and

$$\lim_{\sigma\to+\infty}\gamma(\sigma)=\frac{N}{t}<q,$$

whence every γ_i, $i=1,\dots,6$, goes to $+\infty$ as $\sigma\to+\infty$.

Finally, choosing σ sufficiently large, we deduce $\gamma_i>1$ for $i=1,\dots,6$.

Now, for every $h>k\ge1$, being $\left[|u_n|-k\right]_+\ge h-k$ on $A_n(h)$, we have

$$|A_n(h)|\le\sum_{i=1}^{6}\frac{c\,k^{\tau_i\sigma t'}}{(h-k)^{\sigma t'}}|A_n(k)|^{\gamma_i};$$

hence, there exists i, $i=1,\dots,6$, such that

$$\frac{1}{6}|A_n(h)|\le\frac{c\,k^{\tau_i\sigma t'}}{(h-k)^{\sigma t'}}|A_n(k)|^{\gamma_i}.$$

Therefore, using Lemma 3.5, we conclude that there exist two positive constants k^* and d_0 (independent of n), such that

$$|A_n(k)| = 0 \qquad \text{for every } k \geq k^* + d_0.$$

Hence

$$||u_n||_\infty \leq M \qquad \text{for every } n \in \mathbb{N} \tag{3.22}$$

with $M = k^* + d_0$.

Proof of Lemma 3.3. Using $v = u_n$ as test function in the integral identity (3.1), applying the ellipticity condition (2.1), the growth condition (2.2), and Young's inequality, we have

$$\nu_1 \int_\Omega \sum_{|\alpha|=1,2} \frac{|\mathrm{D}^\alpha u_n|^{p_\alpha}}{(1+|T_n(u_n)|)^{\theta(p_\alpha-1)}}\, \mathrm{d}x \leq \int_\Omega |f|\,|u_n|\, \mathrm{d}x.$$

Now, taking into account (3.22), from the above inequality, we obtain

$$\int_\Omega \sum_{|\alpha|=1,2} |\mathrm{D}^\alpha u_n|^{p_\alpha}\, \mathrm{d}x \leq c(M+1)^q \int_\Omega |f|\, \mathrm{d}x, \tag{3.23}$$

and the Lemma follows. □

4. Proofs of the Results

Before proving Theorem 2.2, we have to premise a compactness result for the approximating solutions u_n which, together with the a priori estimates proved in the previous section, will allow us to pass to the limit in the approximate problems (3.1).

As a consequence of Lemma 3.2 and Lemma 3.3, there exist a subsequence, still denoted by $\{u_n\}$, and a function $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$, such that $\{u_n\}$ is bounded in $L^\infty(\Omega)$ and

$$\begin{aligned} &u_n \to u \text{ weakly in } \mathring{W}^{1,q}_{2,p}(\Omega)\\ &u_n \to u \text{ almost everywhere in } \Omega. \end{aligned}$$

We need to prove that the sequences $\{\mathrm{D}^\alpha u_n\}$, $|\alpha| = 1, 2$ are almost everywhere convergent in Ω. To this aim, we exploit the following compactness result whose proof is in [12].

Lemma 4.1. *Assume that hypotheses* (2.1), (2.2), *and* (2.3) *hold, and let* $\{z_n\}$ *be a sequence of functions, such that*

$$z_n \to z \quad \textit{in}\, \mathring{W}^{1,q}_{2,p}(\Omega) \textit{ weakly and a.e. in } \Omega \tag{4.1}$$

and

$$\begin{aligned} \lim_{n\to+\infty} \sum_{|\alpha|=1,2} \int_\Omega \Big[&A_\alpha(x, z_n, \mathrm{D}^1 z_n, \mathrm{D}^2 z_n)\\ &- A_\alpha(x, z, \mathrm{D}^1 z, \mathrm{D}^2 z)\Big] \mathrm{D}^\alpha[z_n - z]\, \mathrm{d}x = 0. \end{aligned} \tag{4.2}$$

Then, $\{z_n\}$ *is relatively compact in the strong topology of* $\mathring{W}^{1,q}_{2,p}(\Omega)$.

We are now in the position to prove the (relatively) compactness of $\{u_n\}$. We take $u_n - u$ as test function in the weak formulation of Problem (3.1), and we obtain

$$\sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n) \mathrm{D}^\alpha [u_n - u] \,\mathrm{d}x = \int_\Omega f(u_n - u) \,\mathrm{d}x.$$

From the above equality, it follows:

$$\sum_{|\alpha|=1,2} \int_\Omega \Big[A_\alpha(x, T_n(u_n), \mathrm{D}^1 u_n, \mathrm{D}^2 u_n) - A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u)\Big] \mathrm{D}^\alpha [u_n - u] \,\mathrm{d}x$$
$$= \int_\Omega f\,(u_n - u) \,\mathrm{d}x - \sum_{|\alpha|=1,2} \int_\Omega A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u)\, \mathrm{D}^\alpha [u_n - u] \,\mathrm{d}x. \tag{4.3}$$

The right-hand side of (4.3) tends to zero as n tends to $+\infty$, since $\{u_n\}$ converges to u weakly* in $L^\infty(\Omega)$, weakly in $\mathring{W}^{1,q}_{2,p}(\Omega)$ and $A_\alpha(x, u, \mathrm{D}^1 u, \mathrm{D}^2 u)$ belongs to $L^{p'_\alpha}(\Omega)$, $|\alpha| = 1, 2$ thanks to (2.2). Due to the boundedness of $\|u_n\|_{L^\infty}$, $T_n(u_n) = u_n$ for sufficiently large n and using Lemma 4.1, we conclude, up to a subsequence, that

$$u_n \to u \quad \text{strongly in } \mathring{W}^{1,q}_{2,p}(\Omega).$$

Proof of Theorem 2.2. For any fixed function $v \in \mathring{W}^{1,q}_{2,p}(\Omega)$, we can pass to the limit as $n \to +\infty$ in the weak formulation (3.1) and we get that u is a weak solution of the problem (2.5), in the sense of the Definition 2.1. $\square$

Proof of Theorem 2.3. Let $u \in \mathring{W}^{1,q}_{2,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of the problem (2.5) and let Ω' be any strictly interior subregion of Ω.

Set $M = ||u||_{L^\infty(\Omega)}$, and Theorem 2.2 gives us

$$|u(x)| \le M \qquad \text{for a.e. } x \in \Omega. \tag{4.4}$$

We fix $x_0 \in \Omega'$, $0 < R < \dfrac{d'}{4}$ and let be r, such that

$$|\alpha| p_\alpha + r(q - p_\alpha) \le q,\ |\alpha| = 1,\ 2, \quad r\Big(q + \frac{pq}{q - 2p}\Big) \le q,$$
$$r < \min\Big\{1 - \frac{N}{qt}, \frac{q - 2p}{q - p}\Big\}. \tag{4.5}$$

We define

$$\omega_1(R) = \text{ess inf}_{B_R(x_0)} u(x), \quad \omega_2(R) = \text{ess sup}_{B_R(x_0)} u(x),$$
$$\omega(R) = \omega_2(R) - \omega_1(R).$$

From now on, thanks to the boundedness of the solution u and following the outlines of the proof of [8] [Theorem 1.4] or [42], we can prove that there exists $0 < \mu < 1$, such that

$$\omega(R) \le \mu\, \omega(2R) + R^r.$$

Once the previous inequality is acquired, Theorem 2.3 follows by virtue of [27] [Lemma 4.8, chap. 2]. □

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