Mediterr. J. Math. (2022) 19:175
https://doi.org/10.1007/s00009-022-02087-2
1660-5446/22/040001-24
published online June 24, 2022

Mediterranean Journal of Mathematics

Dual-Band General Toeplitz Operators

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Abstract. We relate dual-band general Toeplitz operators to block truncated Toeplitz operators and, via equivalence after extension, with Toeplitz operators with 4×4 matrix symbols. We discuss their norm, their kernel, Fredholmness, invertibility and spectral properties in various situations, focusing on the spectral properties of the dual-band shift, which turns out to be considerably complex, leading to new and nontrivial connections with the boundary behaviour of the associated inner function.

Mathematics Subject Classification. 47B35, 30H10, 94A12.

Keywords. General Wiener–Hopf operator, Toeplitz operator, Truncated Toeplitz operator, dual-band signal, Riemann–Hilbert problem, Wiener–Hopf factorization .

1. Introduction

Multiband spaces occur naturally in applications. First, multiband signals are seen in speech processing (see [2,4,6,24] for example), as an alternative to the Paley–Wiener space $PW(b)$ of inverse Fourier transforms of functions in $L^2(-b,b)$, when both high and low frequencies are to be ignored. Second, multiplex signal transmission, as a way of sending several signals down the same channel, has many practical applications, and we refer the reader to [7] for a detailed history of the subject with 137 references, tracing the analysis back to work of Raabe and Shannon in the 1930s and 1940s. Furthermore, in recent years, dual-band filters have become key components in ubiquitous wireless communication devices such as cellular phones [26,29].

To see a basic example of a multiband space, which in this case is a dual-band space, choose $0 < a < b$ and consider the inverse Fourier transform of the space $L^2((-b,-a) \cup (a,b))$, which is a space $M \subset L^2(\mathbb{R})$. Indeed $M = PW(b) \ominus PW(a)$.

If we define the inverse Fourier transform formally by

$$\hat{f}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{ist} \, \mathrm{d}t,$$

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then this extends to an isomorphism between $L^2(0,\infty)$ and the Hardy space $H^2(\mathbb{C}^+)$ on the upper half-plane, and $L^2(0,b-a)$ corresponds to the model space $K_\theta := H^2(\mathbb{C}^+)\ominus\theta H^2(\mathbb{C}^+)$ where θ is the inner function $\theta(s) = e^{i(b-a)s}$.

It is now clear that the space M above has the orthogonal decomposition

$$M = \phi K_\theta \oplus \psi K_\theta, \tag{1.1}$$

where

$$\phi(s) = e^{-ibs} \quad \text{and} \quad \psi(s) = e^{ias}. \tag{1.2}$$

More generally, let θ be an inner function in $H^\infty(\mathbb{C}^+)$, and ϕ, ψ unimodular functions in $L^\infty(\mathbb{R})$ such that $\phi K_\theta \perp \psi K_\theta$. Then we shall consider dual-band spaces $M := \phi K_\theta \oplus \psi K_\theta$ and general Wiener–Hopf operators [18,36] on those spaces.

Toeplitz operators are a basic example of a general Wiener–Hopf operator where the operator on L^2, of the circle or the real line, which is compressed (in this case, to the Hardy space H^2) is a multiplication operator; in the latter case we say that we have a general Toeplitz operator.

Truncated Toeplitz operators of the form $A_g^\theta u = P_\theta(gu)$, where P_θ denotes the orthogonal projection onto K_θ, are another example of general Toeplitz operators. They have been much studied, since being formally defined by Sarason [35], although they occur much earlier, for example in [1,34]. Some recent surveys on the subject are in [17,20].

The dual-band general Toeplitz operator (abbreviated to dual-band Toeplitz operator in what follows) with symbol $g \in L^\infty$, T_g^M, is defined on the space $M := \phi K_\theta \oplus \psi K_\theta$ by

$$T_g^M u = P_M(gu),$$

where P_M is the orthogonal projection onto M.

In the particular case of the two-interval example given earlier, these are unitarily equivalent to convolution operators restricted to the union of two intervals. Clearly, the same definition can also be made in the more usual situation of $H^2(\mathbb{D})$, except that now ϕ, ψ are unimodular in $L^\infty(\mathbb{T})$ and $g \in L^2(\mathbb{T})$.

Two degenerate cases may appear: the decomposition $M = \overline{\theta} K_\theta \oplus K_\theta$ gives the Paley–Wiener space as a special case, and the decomposition $K_{\theta^2} = K_\theta \oplus \theta K_\theta$ is also a special case. In this paper we will assume that $\bar{\phi}\psi$ and $\phi\bar{\psi}$ are not constant multiples of the inner function θ to avoid those limit cases.

Note that, in contrast with Toeplitz operators and truncated Toeplitz operators, dual-band Toeplitz operators do not act on spaces of holomorphic functions and M is not a direct sum of model spaces, unless ϕ, ψ are constant.

In this paper we relate dual-band Toeplitz operators to block truncated Toeplitz operators of a particular form, allowing for the information on the unimodular functions ϕ, ψ and the symbol g to be encoded in the various components of the matrix symbol of a block truncated Toeplitz operator. By using the concept of equivalence after extension, these are in their turn related to Toeplitz operators with 4×4 matrix symbols. We are thus able to discuss their norm, their kernel, Fredholmness, invertibility and spectral properties in various important situations.

We consider in particular the spectral properties of the dual-band shift. In fact this is a very natural particular case to consider since it has the simplest possible symbol apart from constants and it is an important model to understand the different roles played by θ, on the one hand, and by ϕ, ψ, on the other. There are also links with Volterra operators, as first noted by Sarason [33].

This study also highlights the importance of using the equivalence after extension between dual-band Toeplitz operators and block Toeplitz operators. Indeed, it allows one to use the powerful Riemann–Hilbert method, used in a variety of mathematical and physical problems [8,15,25], as well as the theory of Wiener–Hopf factorization [9,30], in order to study kernels, invertibility and Fredholmness of dual-band Toeplitz operators, to obtain explicit expressions for the inverse operators and the resolvent operators and, for the first time, to investigate certain boundary properties of inner functions, such as the existence of an angular derivative in the sense of Carathéodory for θ in terms of an L^2 factorization [28].

Our results are presented in the context of L^2 and L^∞ spaces on the circle $\mathbb{T}$, or H^2 and H^∞ spaces on the disc $\mathbb{D}$, but they apply also to the case of L^2 and L^∞ spaces on the real line $\mathbb{R}$ and H^2 and H^∞ spaces on the upper half-plane $\mathbb{C}^+$.

2. General Toeplitz Operators on Dual-Band Spaces

2.1. The Dual-Band Space M

Proposition 2.1. *Let $\theta \in H^\infty$ be inner, and let $K_\theta = H^2 \ominus \theta H^2$ be the corresponding model space. Then for $\phi, \psi \in L^\infty$ unimodular, the spaces ϕK_θ and ψK_θ are orthogonal if and only if the truncated Toeplitz operator $A^\theta_{\overline{\phi}\psi}$ is the zero operator.*

Proof. If $f = \psi k_1$ and $g = \phi k_2$ with $k_1, k_2 \in K_\theta$, then

$$\langle f, g\rangle = \langle \psi k_1, \phi k_2\rangle = \langle \overline{\phi}\psi k_1, k_2\rangle = \langle P_\theta \overline{\phi}\psi k_1, k_2\rangle$$

and this is zero for all f, g of that form if and only if $A^\theta_{\overline{\phi}\psi} = 0$. $\square$

Let now $M := \phi K_\theta \oplus^\perp \psi K_\theta$, where θ is inner, $\phi, \psi \in L^\infty$ are unimodular and we assume that $A^\theta_{\overline{\phi}\psi} = 0$, i.e., $\overline{\phi}\psi \in \theta H^2 + \overline{\theta H^2}$ [35, Thm. 3.1]. We also assume throughout the paper that $\phi\bar{\psi}$ and $\bar{\phi}\psi$ are not constant multiples of θ.

M is a closed subspace of L^2 and the operator P_M defined by

$$P_M f = \phi P_\theta \overline{\phi} f + \psi P_\theta \overline{\psi} f \quad (f \in L^2) \tag{2.1}$$

is the orthogonal projection from L^2 onto M.

One can define a conjugation C_M on L^2 which keeps M invariant (and is therefore a conjugation on M when restricted to that space). Recall that a conjugation C on a complex Hilbert space $\mathcal{H}$ is an antilinear isometric involution, i.e.,

$$C^2 = I_{\mathcal{H}} \quad \text{and} \quad \langle Cf, Cg\rangle = \langle g, f\rangle \quad \text{for all} \quad f, g \in \mathcal{H}.$$

The study of conjugations, which generalize complex conjugation, is motivated by applications in physics, in connection with the study of *complex symmetric operators* [21–23]. These are the operators $A \in \mathcal{L}(\mathcal{H})$ such that $CAC = A^*$ for a conjugation C on $\mathcal{H}$. We can define a natural conjugation C_θ on any model space K_θ by

$$C_\theta f = \theta \overline{z} \overline{f}, \tag{2.2}$$

and it is known that any bounded truncated Toeplitz operator is C_θ-symmetric [19, Chap. 8]. For $\mathcal{H} = M$, we have the following result.

Proposition 2.2. *The antilinear operator C_M defined by*

$$C_M f = \theta\phi\psi\overline{z}\overline{f} = \phi\psi C_\theta f \tag{2.3}$$

is a conjugation on L^2 preserving M as an invariant subspace.

Proof. If $f \in M$ has the form $f = \phi k_1 + \psi k_2$ with $k_1, k_2 \in K_\theta$, then

$$C_M f = \psi C_\theta k_1 + \phi C_\theta k_2 \in M, \tag{2.4}$$

from which the conjugation properties are easily verified. $\square$

2.2. A Matrix Representation

Let now T_g^M for $g \in L^2$ be the operator densely defined in M by

$$T_g^M f = P_M g f \quad (f \in L^\infty \cap M),$$

the density of $L^\infty \cap M$ in M following easily from the density of $L^\infty \cap K_\theta$ in K_θ, which was given in [35].

If this operator is bounded, we also denote by T_g^M its unique bounded extension to M. The operator T_g^M is bounded, in particular, whenever $g \in L^\infty$. It is easy to see that $(T_g^M)^* = T_{\overline{g}}^M$.

Theorem 2.3. *Let T_g^M be a bounded general Toeplitz operator on the dual-band space $M := \phi K_\theta \oplus^\perp \psi K_\theta$, where θ is inner and $\phi, \psi \in L^\infty$ are unimodular. Then T_g^M is unitarily equivalent to the block truncated Toeplitz operator*

$$W = \begin{pmatrix} A_g^\theta & A_{\overline{\phi}\psi g}^\theta \\ A_{\overline{\psi}\phi g}^\theta & A_g^\theta \end{pmatrix}, \tag{2.5}$$

on $K_\theta \oplus K_\theta$. Hence $T_g^M = 0$ if and only if each of the four truncated Toeplitz operators composing W is 0.

Proof. Let M_ϕ denote the operator of multiplication by ϕ, and similarly for other multiplication operators. We have the factorization

$$\begin{aligned} T_g^M &= \begin{pmatrix} M_\phi A_g^\theta M_{\overline{\phi}} & M_\phi A_{\overline{\phi}\psi g}^\theta M_{\overline{\psi}} \\ M_\psi A_{\overline{\psi}\phi g}^\theta M_{\overline{\phi}} & M_\psi A_g^\theta M_{\overline{\psi}} \end{pmatrix} \\ &= \begin{pmatrix} M_\phi & 0 \\ 0 & M_\psi \end{pmatrix} \begin{pmatrix} A_g^\theta & A_{\overline{\phi}\psi g}^\theta \\ A_{\overline{\psi}\phi g}^\theta & A_g^\theta \end{pmatrix} \begin{pmatrix} M_{\overline{\phi}} & 0 \\ 0 & M_{\overline{\psi}} \end{pmatrix}. \end{aligned} \tag{2.6}$$

This has the form U^*AU, where U is a unitary operator from M onto $K_\theta \oplus K_\theta$. Using the fact that $M_{\overline{\phi}}$ maps ϕK_θ bijectively to K_θ and $P_{\phi K_\theta} = M_\phi P_{K_\theta} M_{\overline{\phi}}$, it is easy to verify that the identity (2.6) holds. □

There are some simplifications possible here, since some of these four blocks may be 0. The basic properties of matrix-valued truncated Toeplitz operators were studied in [27].

Theorem 2.4. *If T_g^M is bounded, then it is C_M-symmetric.*

Proof. We wish to check the identity $T_g^M C_M = C_M (T_g^M)^*$. Note that by Proposition 2.2, we have

$$UC_M U^* = \begin{pmatrix} 0 & C_\theta \\ C_\theta & 0 \end{pmatrix},$$

where U is the unitary mapping given in (2.6). Hence

$$UT_g^M C_M U^* = \begin{pmatrix} A_g^\theta & A^\theta_{\overline{\phi}\psi g} \\ A^\theta_{\overline{\psi}\phi g} & A_g^\theta \end{pmatrix} \begin{pmatrix} 0 & C_\theta \\ C_\theta & 0 \end{pmatrix} = \begin{pmatrix} A^\theta_{\overline{\phi}\psi g} C_\theta & A_g^\theta C_\theta \\ A_g^\theta C_\theta & A^\theta_{\overline{\psi}\phi g} C_\theta \end{pmatrix}$$

while

$$UC_M (T_g^M)^* U^* = \begin{pmatrix} 0 & C_\theta \\ C_\theta & 0 \end{pmatrix} \begin{pmatrix} A^\theta_{\overline{g}} & A^\theta_{\psi\overline{\phi}\overline{g}} \\ A^\theta_{\phi\overline{\psi}\overline{g}} & A^\theta_{\overline{g}} \end{pmatrix} = \begin{pmatrix} C_\theta A^\theta_{\phi\overline{\psi}\overline{g}} & C_\theta A^\theta_{\overline{g}} \\ C_\theta A^\theta_{\overline{g}} & C_\theta A^\theta_{\psi\overline{\phi}\overline{g}} \end{pmatrix},$$

and these are equal since $AC_\theta = C_\theta A^*$ for any truncated Toeplitz operator A. □

3. Equivalence After Extension

Definition 3.1. [3,37–39] The operators $T : X \to \widetilde{X}$ and $S : Y \to \widetilde{Y}$ are said to be *(algebraically and topologically) equivalent* if and only if $T = ESF$ where E, F are invertible operators. More generally, T and S are *equivalent after extension* if and only if there exist (possibly trivial) Banach spaces X_0, Y_0, called *extension spaces*, and invertible bounded linear operators $E : \widetilde{Y} \oplus Y_0 \to \widetilde{X} \oplus X_0$ and $F : X \oplus X_0 \to Y \oplus Y_0$, such that

$$\begin{pmatrix} T & 0 \\ 0 & I_{X_0} \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_{Y_0} \end{pmatrix} F. \tag{3.1}$$

In this case we say that $T \overset{*}{\sim} S$.

It was shown in [14] that for $g \in L^\infty$ the scalar Toeplitz operator A_g^θ is equivalent by extension to the block Toeplitz operator with symbol

$$\begin{pmatrix} \overline{\theta} & 0 \\ g & \theta \end{pmatrix}.$$

This result was used in [13] to study spectral properties of A_g^θ and, more generally, to study asymmetric truncated Toeplitz operators.

Motivated by the result of Theorem 2.3, we now consider the truncated Toeplitz operator A_G^θ acting on $K_\theta \oplus K_\theta$, where $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in (L^\infty)^{2\times 2}$, and link it with the Toeplitz operator $T_{\mathcal{G}}$ acting on $(H^2)^4$, where

$$\mathcal{G} = \begin{pmatrix} \overline{\theta} & 0 & 0 & 0 \\ 0 & \overline{\theta} & 0 & 0 \\ g_{11} & g_{12} & \theta & 0 \\ g_{21} & g_{22} & 0 & \theta \end{pmatrix}. \tag{3.2}$$

Clearly, for $p, q, r, s \in H^2$, we have $(p, q, r, s) \in \ker T_{\mathcal{G}}$ if and only if $p, q \in K_\theta$ and $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \theta \begin{pmatrix} r \\ s \end{pmatrix} \in \overline{H_0^2} \oplus \overline{H_0^2}$. So $(p, q) \in \ker A_G^\theta$, and likewise given $(p, q) \in \ker A_G^\theta$ there exist $r, s \in H^2$ with $(p, q, r, s) \in \ker T_{\mathcal{G}}$.

The following theorem shows that the result in [13, Thm. 2.3] can be extended to block truncated Toeplitz operators and in fact we can give the result more generally for $n \times n$ blocks. We shall write P_θ for the orthogonal projection from $(H^2)^n$ onto $(K_\theta)^n$, and Q_θ for the complementary projection from $(H^2)^n$ onto $\theta(H^2)^n$.

Theorem 3.2. *Let $G \in (L^\infty)_{n\times n}$ and let I_n be the $n \times n$ identity matrix. The operator $A_G^\theta = P_\theta G P_\theta : K_\theta^n \to K_\theta^n$ is equivalent after extension to $T_{\mathcal{G}} : (H^2)^{2n} \to (H^2)^{2n}$ with*

$$\mathcal{G} = \begin{pmatrix} \overline{\theta} I_n & 0 \\ G & \theta I_n \end{pmatrix}. \tag{3.3}$$

Proof. We have, following the proof of [13, Thm. 2.3],

$$A_G^\theta \overset{*}{\sim} P_\theta G P_\theta + Q_\theta$$

because

$$\begin{pmatrix} A_G^\theta & 0 \\ 0 & I_{\theta(H^2)^n} \end{pmatrix} = E_1 \begin{pmatrix} P_\theta G P_\theta + Q_\theta & 0 \\ 0 & I_{\{0\}^n} \end{pmatrix} F_1,$$

where

$F_1 : K_\theta^n \oplus \theta(H^2)^n \to (H^2)^n \oplus \{0\}^n$ and $E_1 : (H^2)^n \oplus \{0\}^n \to K_\theta^n \oplus \theta(H^2)^n$ are invertible operators, defined in the obvious way. On the other hand, it is clear that, denoting by P^+ the orthogonal projection from L^2 onto H^2,

$$P_\theta G P_\theta + Q_\theta \overset{*}{\sim} \begin{pmatrix} P_\theta G P_\theta + Q_\theta & 0 \\ 0 & P^+ \end{pmatrix} : (H^2)^{2n} \to (H^2)^{2n}.$$

Now,

$$\begin{pmatrix} P_\theta G P_\theta + Q_\theta & 0 \\ 0 & P^+ \end{pmatrix} = \underbrace{\begin{pmatrix} T_{\theta I_n} - P_\theta G T_{\theta I_n} & P_\theta \\ -P^+ & T_{\overline{\theta} I_n} \end{pmatrix}}_{E} T_{\mathcal{G}} \underbrace{\begin{pmatrix} P^+ & 0 \\ T_{\overline{\theta} I_n}(P^+ - T_{\mathcal{G}}) & P^+ \end{pmatrix}}_{F},$$

where $E, F : (H^2)^n \to (H^2)^n$ are invertible operators with

$$E^{-1} = \begin{pmatrix} T_{\overline{\theta} I_n} & 0 \\ P^+ + P_\theta G Q_\theta & T_{\theta I_n} \end{pmatrix}$$

and

$$F^{-1} = \begin{pmatrix} P^+ & 0 \\ -T\overline{\theta} I_n (P^+ - T_G) & P^+ \end{pmatrix}.$$

□

Corollary 3.3. *For $g \in L^\infty$, one has $T_g^M \overset{*}{\sim} T_{\mathcal{G}}$ with*

$$\mathcal{G} = \begin{pmatrix} \overline{\theta} & 0 & 0 & 0 \\ 0 & \overline{\theta} & 0 & 0 \\ g & g\overline{\phi}\psi & \theta & 0 \\ g\phi\overline{\psi} & g & 0 & \theta \end{pmatrix}. \tag{3.4}$$

Proof. This is an immediate consequence of Theorems 2.3 and 3.2 . □

We clearly have the following corollary of the above.

Corollary 3.4. *The operators T_g^M and W are invertible (resp., Fredholm) if and only if $T_{\mathcal{G}}$ is invertible (resp., Fredholm), with $\mathcal{G}$ given by* (3.4).

More general results will be proved later.

4. Kernels, Ranges and Solvability Relations

The equivalence after extension proved in Theorem 3.2 implies certain relations between the kernels, the ranges, and the invertibility and Fredholm properties of the two operators T_g^M and $T_{\mathcal{G}}$ with $\mathcal{G}$ given by (3.4) [3,37], and therefore it implies certain relations between the solutions of

$$T_g^M f_M = h_M \quad \text{for a given} \quad h_M \in M \tag{4.1}$$

and those of

$$T_{\mathcal{G}} F_+ = H_+ \quad \text{for a given} \quad H_+ \in (H^2)^4. \tag{4.2}$$

In this section we study these relations, which also allow for a better understanding of the equivalence after extension obtained in the previous section.

Theorem 4.1. *$T_g^M f_M = h_M$ with $f_M, h_M \in M$ if and only if $T_{\mathcal{G}} F_+ = H_+$ with $F_+ = (f_{j+}) \in (H_2^+)^4$, $H_+ \in (H_2^+)^4$ given by*

$$\begin{aligned} f_{1+} &= f_{1\theta} = P_\theta \overline{\phi} f_M, f_{3+} = -P^+ \overline{\theta} g(P_\theta \overline{\phi} f_M + \overline{\phi}\psi P_\theta \overline{\psi} f_M), \\ f_{2+} &= f_{2\theta} = P_\theta \overline{\psi} f_M, f_{4+} = -P^+ \overline{\theta} g(\phi\overline{\psi} P_\theta \overline{\phi} f_M + P_\theta \overline{\psi} f_M). \end{aligned} \tag{4.3}$$

and

$$H_+ = (0, 0, h_{1\theta}, h_{2\theta}). \tag{4.4}$$

Consequently, if $f_M \in \ker T_g^M$, then $F_+ \in \ker T_{\mathcal{G}}$.

Proof. First note that the equation (4.2) is equivalent to the Riemann–Hilbert problem

$$\mathcal{G}F_+ = F_- + H_+, \quad F_\pm \in (H_2^\pm)^4, \tag{4.5}$$

where $H_2^+ = H^2$ and $H_2^- = \overline{H_0^2}$. Let

$$\begin{aligned} f_{1\theta} &= P_\theta \overline{\phi} f_M, h_{1\theta} = P_\theta \overline{\phi} h_M, \\ f_{2\theta} &= P_\theta \overline{\psi} f_M, h_{2\theta} = P_\theta \overline{\psi} h_M, \end{aligned} \tag{4.6}$$

so that $f_M = \phi f_{1\theta} + \psi f_{2\theta}$ and $h_M = \phi h_{1\theta} + \psi h_{2\theta}$. Given $h_M \in M$ we can write, by (2.1), $T_g^M f_M = h_M$ if and only if $P_M\left(g\left(\phi f_{1\theta} + \psi f_{2\theta}\right)\right) = \phi h_{1\theta} + \psi h_{2\theta}$, or equivalently,

$$\begin{aligned} &\phi P_\theta \overline{\phi}(g\phi f_{1+} + g\psi f_{2+}) + \psi P_\theta \overline{\psi}(g\phi f_{1+} + g\psi f_{2+}) \\ &\quad = \phi h_{1\theta} + \psi h_{2\theta}\,, \ \ f_{j+} = f_{j\theta} \in K_\theta. \end{aligned}$$

Since $\phi K_\theta \perp \psi K_\theta$, this is equivalent to

$$\begin{aligned} \overline{\theta} f_{1+} &= f_{1-} \in H_2^-\,, \quad P_\theta(g f_{1+} + g\overline{\phi}\psi f_{2+}) = h_{1\theta}, \\ \overline{\theta} f_{2+} &= f_{2-} \in H_2^-\,, \quad P_\theta(g\phi\overline{\psi} f_{1+} + g f_{2+}) = h_{2\theta}, \end{aligned} \tag{4.7}$$

which, in its turn, is equivalent to

$$\bar{\theta} F_1^+ = F_1^-, \quad P_\theta\, g\, G_1 F_1^+ = H_\theta$$

where

$$F_{1\pm} = (f_{1\pm}, f_{2\pm})\ ,\ G_1 = \begin{pmatrix} 1 & \bar{\phi}\psi \\ \phi\bar{\psi} & 1 \end{pmatrix}\ ,\ H_\theta = (h_{1\theta}, h_{2\theta}). \tag{4.8}$$

Equivalently, there exist $F_{2\pm} \in H^2_{2\pm}$ such that

$$\bar{\theta}\, F_{1+} = F_{1-}\ \ ,\ g\, G_1 F_{1+} = H_\theta + F_{2-} - \theta F_{2+}. \tag{4.9}$$

This system determines F_{2+} in terms of F_{1+} as $F_{2+} = P^+(\bar{\theta} g\, G_1 F_{1+})$. So, identifying $F_\pm \in (H^2_\pm)^4$ with (F_{1+}, F_{2+}), (4.9) is equivalent to

$$\mathcal{G}F_+ = F_- + H_+$$

with $H_+ = (0, 0, h_{1\theta}, h_{2\theta})$ and $F_+ = (f_{j+})$ where

$$\begin{aligned} f_{1+} &= P_\theta \overline{\phi} f_M, \quad f_{3+} = -P^+\overline{\theta} g(P_\theta \overline{\phi} f_M + \overline{\phi}\psi P_\theta \overline{\psi} f_M), \\ f_{2+} &= P_\theta \overline{\psi} f_M, \quad f_{4+} = -P^+\overline{\theta} g(\phi\overline{\psi} P_\theta \overline{\phi} f_M + P_\theta \overline{\psi} f_M). \end{aligned} \tag{4.10}$$

□

Theorem 4.2. *$T_{\mathcal{G}} F_+ = H_+$ with $F_+ = (\tilde{f}_{j+}), H_+ = (h_{j+}) \in (H^2)^4$, if and only if $T_g^M f_M = h_M$, where*

$$f_M = \phi(\tilde{f}_{1+} - \theta h_{1+}) + \psi(\tilde{f}_{2+} - \theta h_{2+}) \tag{4.11}$$

and

$$h_M = \phi P_\theta(h_{3_+} - g\theta(h_{1+} + \overline{\phi} h_{2_+})) + \psi P_\theta(h_{4+} - g\theta(\phi\overline{\psi} h_{1+} + h_{2+})) \tag{4.12}$$

Proof. We have $T_{\mathcal{G}}F_+ = H_+$ if and only if $\mathcal{G}F_+ = F_- + H_+$, with $F_- \in (H_2^-)^4$. Let $\tilde{F}_{1\pm} = (\tilde{f}_{1\pm}, \tilde{f}_{2\pm})$ and $\tilde{F}_{2\pm} = (\tilde{f}_{3\pm}, \tilde{f}_{4\pm})$ where $(\tilde{f}_{j\pm}) = F_\pm$ and let $H_{1+} = (h_{1+}, h_{2+})$, $H_{2+} = (h_{3+}\, , \, h_{4+}$. Then $\mathcal{G}F_+ = F_- + H_+$ if and only if

$$\begin{cases} \overline{\theta}\tilde{F}_{1+} = \tilde{F}_{1-} + H_{1+} \\ g\, G_1\tilde{F}_{1+} + \theta\tilde{F}_{2+} = \tilde{F}_{2-} + H_{2+} \end{cases} \tag{4.13}$$

where G_1 is given in (4.8).

This in turn is equivalent to the system of equations

$$\begin{cases} \overline{\theta}(\tilde{F}_{1+} - \theta H_{1+}) = \tilde{F}_{1-} \\ g\, G_1(\tilde{F}_{1+} - \theta H_{1+}) + \theta(\tilde{F}_{2+} + P^+ g\, G_1 H_{1+} - P^+\bar{\theta}H_{2+}) \\ -P_\theta H_{2+} - P_\theta\theta g\, G_1 H_{1+} = \tilde{F}_{2-} + P^-\theta g\, G_1 H_{1+} \end{cases} \tag{4.14}$$

Taking $F_{1+} = \tilde{F}_{1+} - \theta H_{1+}$, $F_{2+} = \tilde{F}_{2+} + P^+ g\, G_1 H_{1+} - P^+\bar{\theta}H_{2+}$ we get

$$\begin{cases} \overline{\theta}F_{1+} = \tilde{F}_{1-} \\ g\, G_1 F_{1+} + \theta F_{2+} - P_\theta H_{2+} - P_\theta\theta g\, G_1 H_{1+} \\ = \tilde{F}_{2-} + P^-\theta g\, G_1 H_{1+} + P_\theta(H_{2+} + \theta g\, G_1 H_{1+}) \end{cases} \tag{4.15}$$

By Theorem 4.1 this is equivalent to $T_g^M f_M = h_M$ with f_M and h_M given by (4.11) and (4.12). $\square$

Corollary 4.3. *If $F_+ \in \ker T_{\mathcal{G}}$, with $F_+ = (\tilde{f}_{j+}) \in (H^2)^4$, then $f_M \in \ker T_g^M$ with $f_M = \phi\tilde{f}_{1+} + \psi\tilde{f}_{2+}$.*

Note that (4.13) shows that any element of the kernel of $T_{\mathcal{G}}$ is determined by its first two components $\tilde{f}_{1+}$ and $\tilde{f}_{2+}$, since

$$\tilde{f}_{3+} = P^+\overline{\theta}g(\tilde{f}_{1+} + \overline{\phi}\psi\tilde{f}_{2+})$$

and

$$\tilde{f}_{4+} = P^+\overline{\theta}g(\phi\overline{\psi}\tilde{f}_{1+} + \tilde{f}_{2+}).$$

Let $P_{1,2}$ be the projection defined by $P_{1,2}(x, y, u, v) = (x, y)$.

Corollary 4.4. *The map*

$$\mathcal{K} : \ker T_g^M \to \ker T_{\mathcal{G}}\ ,\ \ \mathcal{K}f_M = (f_{1+}, f_{2+}, f_{3+}, f_{4+})$$

with f_{j+} given by (4.3) is an isomorphism. We have

$$\ker T_g^M = \mathcal{K}^{-1}\ker T_{\mathcal{G}} = \{\phi f_{1+} + \psi f_{2+} : (f_{1+}, f_{2+}) \in P_{1,2}\ker T_{\mathcal{G}}\}.$$

From Theorems 4.1 and 4.2 we also obtain the following regarding ranges.

Corollary 4.5. *With the same notation as above,*

(i) $(h_{1+}, h_{2+}, h_{3+}, h_{4+}) \in \operatorname{ran} T_{\mathcal{G}}$ *if and only if*

$$\phi P_\theta\left(h_{3+} - g\theta\left(h_{1+} + \overline{\phi}\psi h_{2+}\right)\right) + \psi P_\theta\left(h_{4+} - g\theta\left(\phi\overline{\psi}h_{1+} + h_{2+}\right)\right) \in \operatorname{ran} T_g^M;$$

(ii) $\phi h_{1\theta} + \psi h_{2\theta} \in \operatorname{ran} T_g^M$ *if and only if* $(0, 0, h_{1\theta}, h_{2\theta}) \in \operatorname{ran} T_{\mathcal{G}}$.

Moreover, we obtain a relation between the inverses of T_g^M and $T_{\mathcal{G}}$ when these operators are invertible.

Corollary 4.6. *T_g^M is invertible if and only if $T_{\mathcal{G}}$ is invertible and, in that case, $(T_g^M)^{-1} = [\phi P_1, \psi P_2, 0, 0] T_{\mathcal{G}}^{-1} U_0$, where $P_j(x_1, x_2, x_3, x_4) = x_j$ and $U_0 : M \to (H_2^+)^4$, is given by $U_0 h_M = (0, 0, P_\theta \overline{\phi} h_M, P_\theta \overline{\psi} h_M)$.*

We can also relate the kernels of T_g^M and its adjoint as follows.

Theorem 4.7. *If $g \in L^\infty$, then $\ker T_g^M \simeq \ker (T_g^M)^* = \ker T_{\overline{g}}^M$.*

Proof. Since $T_g^M \overset{*}{\sim} T_{\mathcal{G}}$, we have that $\ker T_g^M \simeq \ker T_{\mathcal{G}}$ and $\ker (T_g^M)^* \simeq \ker T_{\mathcal{G}}^*$. So it is enough to prove that $\ker T_{\mathcal{G}}$, with $\mathcal{G}$ given by (3.4), is isomorphic to $\ker T_{\mathcal{G}}^* = \ker T_{\overline{\mathcal{G}}^T}$. Since

$$\ker T_{\mathcal{G}} = \left\{ \phi_+ \in (H^2)^4 : \mathcal{G}\phi_+ = \phi_- \in \left(\overline{H_0^2}\right)^4 \right\},$$

we have

$$\mathcal{G}\phi_+ = \phi_- \iff \overline{z}\overline{\phi_+} = \overline{\mathcal{G}^{-1}}(\overline{z}\overline{\phi_-}) \iff \overline{\mathcal{G}}^{-1}\psi_+ = \psi_-,$$

where $\psi_+ = \overline{z}\overline{\phi_-} \in (H^2)^4$ and $\psi_- = \overline{z}\overline{\phi_+} \in (\overline{H_0^2})^4$. Since

$$\begin{aligned}
\overline{\mathcal{G}}^{-1} &= \begin{pmatrix} \overline{\theta} & 0 & 0 & 0 \\ 0 & \overline{\theta} & 0 & 0 \\ -\overline{g} & -\overline{g}\phi\overline{\psi} & \theta & 0 \\ -\overline{g}\overline{\phi}\psi & -\overline{g} & 0 & \theta \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \theta & 0 & \overline{g} & \overline{g}\overline{\phi}\psi \\ 0 & \theta & \overline{g}\phi\overline{\psi} & \overline{g} \\ 0 & 0 & \overline{\theta} & 0 \\ 0 & 0 & 0 & \overline{\theta} \end{pmatrix}}_{\overline{\mathcal{G}}^T} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}$$

it is clear that $\ker T_{\overline{\mathcal{G}}^{-1}} \simeq \ker T_{\overline{\mathcal{G}}^T}$. □

Corollary 4.8. *If T_g^M is Fredholm, then it has index 0. Consequently, T_g^M is invertible if and only if it is Fredholm and injective.*

5. The Norm and the Spectrum for Analytic Symbols

Clearly the norm of the dual-band Toeplitz operator T_g^M is the same as the norm of the block truncated Toeplitz operator W. One important case that can be analysed is when g is in H^∞ (in the language of dual-band signals, this corresponds to a causal convolution on $L^2((-b, -a) \cup (a, b))$).

The following is an easy generalization of scalar results which apparently go back to [34].

Proposition 5.1. *Suppose that the symbol*

$$\Phi := \begin{pmatrix} g & \overline{\phi}\psi g \\ \overline{\psi}\phi g & g \end{pmatrix}$$

is in $(H^\infty)_{2\times 2}$. *Then*

$$\|T_g^M\| = \|W\| = \operatorname{dist}(\overline{\theta}\Phi, (H^\infty)_{2\times 2} = \|\Gamma_{\overline{\theta}\Phi}\|,$$

where the vectorial Hankel operator $\Gamma_{\overline{\theta}\Phi} : (H^2)^2 \to (L^2 \ominus H^2)^2$ *is defined by* $\Gamma_{\overline{\theta}\Phi} v = P_{(L^2\ominus H^2)^2}\overline{\theta}\Phi v$.

Proof. Since the symbol Φ is analytic, if we write $(u_1, u_2) \in H^2 \oplus H^2$ as $(k_1 + \theta\ell_1, k_2 + \theta\ell_2)$ with the k_j in K_θ and $\ell_j \in H^2$, then we have $W(u_1, u_2) = W(k_1, k_2)$ implying that the norm of the truncated Toeplitz operator W is the same when the domain is $H^2 \oplus H^2$ or $K_\theta \oplus K_\theta$.

But $Wu = \theta(P_- \oplus P_-)\overline{\theta}\Phi u = \theta\Gamma_{\overline{\theta}\Phi}u$, and so

$$\|W\| = \|\Gamma_{\overline{\theta}\Phi}\| = \operatorname{dist}(\overline{\theta}\Phi, H^\infty(M_2(\mathbb{C})),$$

by the vectorial form of Nehari's theorem [32, Sec. 2.2]. $\square$

Some results on the spectrum of W can be derived using known results on the scalar case, particularly in the context of Proposition 2.1. Note that the hypotheses of this theorem are satisfied in the original example given by (1.1) and (1.2).

Theorem 5.2. *Suppose that* $g \in H^\infty$ *and that* $\overline{\phi}\psi \in \theta H^\infty$ *or* $\overline{\phi}\psi \in \overline{\theta H^\infty}$. *Then*

$$\sigma(T_g^M) = \sigma(A_g^\theta) = \{\lambda \in \mathbb{C} : \inf_z(|\theta(z)| + |g(z) - \lambda|) = 0\},$$

where the infimum is taken over $\mathbb{D}$.

Proof. Assume that $g \in H^\infty$ and that $\overline{\phi}\psi \in \theta H^\infty$. Note that W has the form

$$W = \begin{pmatrix} A_g^\theta & 0 \\ A^\theta_{\overline{\psi}\phi g} & A_g^\theta \end{pmatrix}$$

and we claim that W is invertible if and only if A_g^θ is. For the necessity note that for arbitrary block operator matrices, if we have

$$\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} A & 0 \\ B & A \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

then $AP = I$ and $SA = I$, so $S = SAP = P$, and A is invertible with inverse P.

The sufficiency follows from the formula

$$W^{-1} = \begin{pmatrix} (A_g^\theta)^{-1} & 0 \\ -(A_g^\theta)^{-1}A^\theta_{\overline{\psi}\phi g}(A_g^\theta)^{-1} & (A_g^\theta)^{-1} \end{pmatrix}.$$

The spectrum of A_g^θ for $g \in H^\infty$ is described in [31, p. 66], and the $H^\infty(\mathbb{C}^+)$ case may be found in [13]. $\square$

For the essential spectrum of A_g^θ we may similarly prove the following result.

Theorem 5.3. *Suppose that* $g \in H^\infty$ *and that* $\overline{\phi}\psi \in \theta H^\infty$. *Then*

$$\sigma_e(T_g^M) = \sigma_e(A_g^\theta) = \left\{ \lambda \in \mathbb{C} : \liminf_{z \to \xi} (|\theta(z)| + |g(z) - \lambda|) = 0 \text{ for some } \xi \in \mathbb{T} \right\},$$

where z *is taken in* $\mathbb{D}$.

Proof. The method of proof of Theorem 5.2 adapted to inversion modulo the compact operators (i.e., in the Calkin algebra) shows directly that $\sigma_e(T_g^M) = \sigma_e(A_g^\theta)$, and an expression for this is known from results in [5,13]. □

These results are of particular interest in the case of the *restricted shift* or *truncated shift* S_M on M, with $g(z) = z$. We thus have

Corollary 5.4. *If* $\overline{\phi}\psi \in \theta H^\infty$, *then for the restricted shift* S_M *on* M *we have*

$$\sigma(S_M) = \{\lambda \in \mathbb{C} : \inf_{z \in \mathbb{D}} (|\theta(z)| + |z - \lambda|) = 0\},$$

and

$$\begin{aligned}\sigma_e(S_M) &= \{\lambda \in \mathbb{C} : \liminf_{z \to \xi} (|\theta(z)| + |z - \lambda|) = 0 \text{ for some } \xi \in \mathbb{T}\} \\ &= \{\lambda \in \mathbb{T} : \liminf_{z \to \lambda} (|\theta(z)|) = 0\}.\end{aligned}$$

6. The Double-Band Shift: Spectral Properties

In order to have matrix symbols which are essentially bounded, and since $\bar{\psi}\phi \in \overline{\theta H^2} + \theta H^2$, we assume here that $\overline{\psi}\phi = A_-\overline{\theta} + A_+\theta$ with $A_+ \in H^\infty$ and $A_- \in \overline{H^\infty}$. In this case, for $g = z - \lambda$, we have in W (see (2.5)) $A^\theta_{\bar{\phi}\psi g} = A^\theta_{\bar{A}_+\bar{\theta}(z-\lambda)}$ and $A^\theta_{\bar{\psi}\phi g} = A^\theta_{A_-\bar{\theta}(z-\lambda)}$. Using the result of Corollary 3.3 we thus associate to the operator $T^M_{z-\lambda}$, with $\lambda \in \mathbb{C}$, the matrix symbol

$$\mathcal{G}_\lambda = \begin{pmatrix} \overline{\theta} & 0 & 0 & 0 \\ 0 & \overline{\theta} & 0 & 0 \\ z - \lambda & (z-\lambda)\overline{A_+\theta} & \theta & 0 \\ (z-\lambda)A_-\overline{\theta} & z - \lambda & 0 & \theta \end{pmatrix}. \tag{6.1}$$

6.1. Eigenvalues and Eigenspaces

By Corollaries 4.3 and 4.4 we have that $\ker T^M_{z-\lambda} \simeq \ker T_{\mathcal{G}_\lambda}$ and $f^M \in \ker T^M_{z-\lambda}$ if and only if

$$f^M = \phi f_{1+} + \psi f_{2+} \tag{6.2}$$

where f_{1+} and f_{2+} are the two first components of $\ker T_{\mathcal{G}_\lambda} \subset (H^2)^4$. Note that $\ker T_{\mathcal{G}_\lambda}$ consists of the solutions $f_+ \in (H^2)^4$ of

$$\mathcal{G}_\lambda f_+ = f_-, \quad \text{with } f_+ \in (H^2)^4, f_- \in (\overline{H^2_0})^4. \tag{6.3}$$

It is easy to see from (6.3) that all components of f_+ and f_- are determined by f_{1+} and f_{2+}.

We will consider the cases $\lambda \in \mathbb{D}$, $\lambda \in \mathbb{D}^- = \{z \in \mathbb{C} : |z| > 1\}$ and $\lambda \in \mathbb{T}$ separately.

Proposition 6.1. *If* $\lambda \in \mathbb{D}$ *then:*

(i) $\ker T^M_{z-\lambda} = \{0\} \iff \Delta_\lambda := \theta(\lambda)^2 - \overline{A_+(0)}\,\overline{\overline{A_-}(0)}(1-\overline{\theta(0)}\theta(\lambda))^2 \neq 0.$
(ii) $\dim\ker T^M_{z-\lambda} = 1 \iff \Delta_\lambda = 0$ *and* $|\theta(\lambda)| + |A_+(0)| + |\overline{A_-}(0)| \neq 0.$
(iii) $\dim\ker T^M_{z-\lambda} = 2 \iff \Delta_\lambda = 0$ *and* $\theta(\lambda) = A_+(0) = \overline{A_-}(0) = 0.$

Proof. From (6.1) and (6.3) we get, for $f_\pm = (f_{1\pm}, f_{2\pm}, f_{3\pm}, f_{4\pm}) \neq 0$,

$$(z-\lambda)\begin{pmatrix} f_{1+} \\ f_{2+}\end{pmatrix} + \theta\begin{pmatrix} f_{3+} \\ f_{4+}\end{pmatrix} = -(z-\lambda)\begin{pmatrix} \overline{A_+}f_{2-} \\ A_-f_{1-}\end{pmatrix} + \begin{pmatrix} f_{3-} \\ f_{4-}\end{pmatrix} = \begin{pmatrix} k_1 \\ k_2\end{pmatrix} \tag{6.4}$$

with $k_1, k_2 \in \mathbb{C}$, since the left-hand side is in $(H^2)^2$ and the right-hand side of the first equality is in $\overline{(H^2)}^2$. Therefore

$$(z-\lambda)f_{1+} + \theta f_{3+} = k_1$$

and, since $\overline{\theta}f_{1+} = f_{1-}$, we have

$$(z-\lambda)f_{1-} - \overline{\theta}k_1 = -f_{3+} = C_1 \in \mathbb{C}, \tag{6.5}$$

so $(z-\lambda)f_{1+} = k_1 + C_1\theta$ and we have $k_1 = -C_1\theta(\lambda)$. It follows that

$$f_{1+} = C_1\frac{\theta - \theta(\lambda)}{z-\lambda}. \tag{6.6}$$

Analogously,

$$f_{2+} = C_2\frac{\theta - \theta(\lambda)}{z-\lambda} \tag{6.7}$$

and it follows that

$$\begin{aligned} f_{1-} &= C_1\frac{1-\overline{\theta}\theta(\lambda)}{z-\lambda}\ , & f_{2-} &= C_2\frac{1-\overline{\theta}\theta(\lambda)}{z-\lambda} \\ f_{3+} &= -C_1\ , & f_{4+} &= -C_2 \\ f_{3-} &= -\theta(\lambda)C_1 + \overline{A_+}(1-\overline{\theta}\theta(\lambda))C_2\ , \\ f_{4-} &= -\theta(\lambda)C_2 + A_-(1-\overline{\theta}\theta(\lambda))C_1\ . \end{aligned}$$

Since $f_{3-}, f_{4-} \in \overline{H^2_0}$, we must have

$$\begin{aligned} -\theta(\lambda)C_1 + \overline{A_+(0)}(1-\overline{\theta(0)}\theta(\lambda))C_2 &= 0 \\ -\theta(\lambda)C_2 + \overline{\overline{A_-}(0)}(1-\overline{\theta}(0)\theta(\lambda))C_1 &= 0. \end{aligned} \tag{6.8}$$

If $C_1 = C_2 = 0$, we get from (6.4) that $f_+ = f_- = 0$. A necessary and sufficient condition for (6.8) to have non zero solutions, C_1, C_2, is that the determinant of the system is zero, i.e., $\Delta_\lambda = 0$. So (i) holds.

If $\Delta_\lambda = 0$, then (6.8) is equivalent to

$$\theta(\lambda)C_1 = \overline{A_+(0)}(1-\overline{\theta(0)}\theta(\lambda))C_2. \tag{6.9}$$

If $\theta(\lambda) \neq 0$, we must have also $A_+(0), \overline{A_-}(0) \neq 0$ and the system (6.8) is equivalent to $C_1 = \overline{A_+(0)}\,\frac{1-\overline{\theta(0)}\theta(\lambda)}{\theta(\lambda)}\,C_2$. If $\theta(\lambda) = 0$ then $A_+(0)\overline{A_-}(0) = 0$.

If $A_+(0) = 0, \overline{A_-}(0) \neq 0$, then

$$C_1 = 0,\ f_{1+} = 0,\quad f_{2+} = \beta_2\frac{\theta}{z-\lambda},\ \text{with } \beta_2 \in \mathbb{C};$$

if $A_+(0) \neq 0, \overline{A_-}(0) = 0$, then

$$C_2 = 0,\ f_{1+} = \beta_1 \frac{\theta}{z-\lambda},\ \text{with } \beta_1 \in \mathbb{C},\ f_{2+} = 0;$$

if $A_+(0) = \overline{A_-}(0)$, we have

$$f_{1+} = \beta_1 \frac{\theta}{z-\lambda},\ f_{2+} = \beta_2 \frac{\theta}{z-\lambda} \text{ with } \beta_1, \beta_2 \in \mathbb{C}.$$

So (ii) and (iii) hold. □

Proposition 6.2. *If $\lambda \in \mathbb{D}^-$ then :*

(i) $\ker T^M_{z-\lambda} = \{0\} \iff \widetilde{\Delta_\lambda} := 1 - \overline{A_+(0)}\overline{\overline{A_-}(0)}(\overline{\theta(0)} - \overline{\theta}(\lambda))^2 \neq 0,$

(ii) $\dim \ker T^M_{z-\lambda} = 1 \iff \widetilde{\Delta_\lambda} = 0.$

Proof. From (6.4) we get

$$(z-\lambda)f_{1-} - \overline{\theta}k_1 = C_1 \iff (z-\lambda)f_{1-} = C_1 + \overline{\theta}k_1.$$

Replacing z ($\in \mathbb{D}^-$) by λ we see that

$$C_1 = -\overline{\theta}(\lambda)k_1,\ \text{with } \overline{\theta}(\lambda) = \overline{\theta(1/\overline{\lambda})}.$$

Analogously, we get $C_2 = -\overline{\theta}(\lambda)k_2$, so

$$f_{1-} = k_1 \frac{\overline{\theta} - \overline{\theta}(\lambda)}{z-\lambda},\ f_{2-} = k_2 \frac{\overline{\theta} - \overline{\theta}(\lambda)}{z-\lambda}.$$

It follows that

$$\begin{aligned}
f_{1+} &= k_1 \frac{1 - \theta\overline{\theta}(\lambda)}{z-\lambda}, & f_{2+} &= k_2 \frac{1 - \theta\overline{\theta}(\lambda)}{z-\lambda} \\
f_{3+} &= \overline{\theta}(\lambda)k_1, & f_{4+} &= \overline{\theta}(\lambda)k_2, \\
f_{3-} &= k_1 + \overline{A_+}k_2(\overline{\theta} - \overline{\theta}(\lambda)), & f_{4-} &= k_2 + A_-k_1(\overline{\theta} - \overline{\theta}(\lambda))\ .
\end{aligned} \tag{6.10}$$

Since $f_{3-}, f_{4-} \in \overline{H^2_0}$, we must have, from (6.10),

$$\begin{aligned}
&k_1 + \overline{A_+(0)}k_2(\overline{\theta(0)} - \overline{\theta}(\lambda)) = 0 \\
&k_2 + \overline{\overline{A_-}(0)}k_1(\overline{\theta(0)} - \overline{\theta}(\lambda)) = 0.
\end{aligned} \tag{6.11}$$

and discussing this system as in the proof of Proposition 6.1 we conclude that (i) and (ii) hold. Moreover, if $\widetilde{\Delta_\lambda} = 0$ we see that

$$f_{1+} = -\beta\overline{A_+(0)}(\overline{\theta(0)} - \overline{\theta}(\lambda))\frac{1 - \theta\overline{\theta}(\lambda)}{z-\lambda},$$

$$f_{2+} = \beta\frac{1 - \theta\overline{\theta}(\lambda)}{z-\lambda},$$

with $\beta \in \mathbb{C}$, which determines the kernel of $T^M_{z-\lambda}$ by (6.2). □

To study $\ker T^M_{z-\lambda}$ where $\lambda \in \mathbb{T}$ we will need the following two lemmas.

Lemma 6.3. *If $\phi_+ \in H^2$ then $\phi_+(z)(z-\lambda) \to 0$ when $z \to \lambda \in \mathbb{T}$ nontangentially in $\mathbb{D}$.*

Proof. For $\phi_+ \in H^2$ we have $|\phi_+(w)| \le \|\phi_+\|\,\|k_w\|$, where k_w is the reproducing kernel function $k_w(z) = 1/(1-\overline{w}z)$, with $\|k_w\| = 1/\sqrt{1-|w|^2}$. Hence

$$|\phi(z)(z-\lambda)| \le \frac{|z-\lambda|\,\|\phi_+\|}{\sqrt{1-|z|^2}},$$

and this tends to 0 if z tends nontangentially to $\lambda \in \mathbb{T}$, as this means that $|z-\lambda| \le C(1-|z|)$ for some constant C. □

We say that an inner function θ has an angular derivative in the sense of Carathéodory (ADC) if and only if θ has a nontangential limit

$$\theta(\lambda) = \lim_{z\to\lambda \text{ n.t.}} \theta(z)$$

with $|\theta(\lambda)| = 1$ and the difference quotient $\dfrac{\theta(z)-\theta(\lambda)}{z-\lambda}$ has a nontangential limit $\theta^{'}(\lambda)$ at λ [19,35]. By Theorem 7.4.1 in [19], θ has an ADC at $\lambda \in \mathbb{T}$ if and only if there exists $a \in \mathbb{T}$ such that $\dfrac{\theta(z)-a}{z-\lambda} \in H^2$, which implies, by Lemma 6.3, that there exists the limit $\theta(\lambda)$ and we have $\theta(\lambda) = a$. Thus we have:

Lemma 6.4. *θ has an ADC at $\lambda \in \mathbb{T}$ if and only if:*
- (i) $\lim_{z\to\lambda\ n.t.} \theta(z)$ *exists in $\mathbb{C}$ (denoted by $\theta(\lambda)$) and*
- (ii) $\dfrac{\theta(z)-\theta(\lambda)}{z-\lambda} \in H^2$.

We denote by $\mathbb{T}_{\text{ADC}}$ the set of all $\lambda \in \mathbb{T}$ where θ has an ADC.

Proposition 6.5. *Let $\lambda \in \mathbb{T}$. If* $\ker T^M_{z-\lambda} \neq \{0\}$ *then $\lambda \in \mathbb{T}_{ADC}$.*

Proof. Consider again (6.3), now with $\lambda \in \mathbb{T}$, and assume that $f_+, f_- \neq 0$. then, as in the proof of Proposition 6.1, we get (6.4) and, if $k_1 = k_2 = 0$, we have that $(z-\lambda)f_{1+} + \theta f_{3+} = 0$, which implies that

$$(z-\lambda)f_{1-} = -f_{3+} = C \in \mathbb{C}.$$

It follows that $f_{1-} = \frac{C}{z-\lambda}$, so $C = 0$ and therefore $f_{1-} = f_{1+} = 0$. Analogously, from (6.4) we get $f_{2+} = 0$ if $k_2 = 0$. Thus, to have a non-zero solution to (6.3) either k_1 or k_2 must be different from 0. Assume that $k_1 \neq 0$. Then, from (6.4),

$$\begin{aligned}(z-\lambda)f_{1+} + \theta f_{3+} = k_1 \implies (z-\lambda)f_{1-} - \overline{\theta}k_1 = -f_{3+} = C \in \mathbb{C} \implies \\ (z-\lambda)f_{1-} = C + \overline{\theta}k_1 \implies (z-\lambda)f_{1+} = C\theta + k_1.\end{aligned}$$

From Lemma 6.3 it follows now that there exists $\theta(\lambda)$ and $f_{1+} = C\dfrac{\theta-\theta(\lambda)}{z-\lambda} \in H^2$, so that by Lemma 6.4 θ has an ADC at λ. □

Corollary 6.6. *If $\lambda \in \mathbb{T}\setminus\mathbb{T}_{ADC}$ then* $\ker T^M_{z-\lambda} = \{0\}$.

Proposition 6.7. *Let $\lambda \in \mathbb{T}_{ADC}$. Then, for Δ_λ defined as in Proposition 6.1,*
- (i) *if $\Delta_\lambda \neq 0$ then* $\ker T^M_{z-\lambda} = \{0\}$,

(ii) *if* $\Delta_\lambda = 0$ *then* $\dim\ker T^M_{z-\lambda} = 1$.

Proof. Analogous to the proof of Proposition 6.1, taking into account Proposition 6.5 and noting that now we cannot have $\theta(\lambda) = 0$. □

We summarise the previous results as follows.

Theorem 6.8. (i) $\ker T^M_{z-\lambda}$ *and* $\ker(T^M_{z-\lambda})^*$ *have the same finite dimension for all* $\lambda \in \mathbb{C}$.

(ii) $\lambda \in \sigma_p(T^M_z)$ *if and only if* $\lambda \in \mathbb{D}$, $\Delta_\lambda = 0$, *or* $\lambda \in \mathbb{D}^-$, $\widetilde{\Delta_\lambda} = 0$, *or* $\lambda \in \mathbb{T}_{ADC}$, $\Delta_\lambda = 0$, *where* Δ_λ *and* $\widetilde{\Delta_\lambda}$ *are defined in Proposition 6.1 and 6.2 , respectively.*

Remark 6.9. For $\lambda \in \sigma_p(T^M_z)$, the previous results provide a description of the eigenspace in each case.

The following corollary applies in particular to the case considered in Theorem 5.2.

Corollary 6.10. *If* $A_+(0)\overline{A_-}(0) = 0$ *then* $\lambda \in \sigma_p(T^M_z) \iff \lambda \in \mathbb{D}$, $\theta(\lambda) = 0$.

Corollary 6.11. *If* $A_+(0)\overline{A_-}(0) \neq 0$ *then*

$$\lambda \in \sigma_p(T^M_z) \cap \mathbb{D} \implies \theta(\lambda) = 0$$
$$\lambda \in \sigma_p(T^M_z) \cap \mathbb{D}^- \implies \theta(1/\overline{\lambda}) \neq 0$$

6.2. Fredholmness and Essential Spectrum

It is easy to see, using equivalence after extension and the theory of Wiener–Hopf factorization (WH factorization) that $T^M_{z-\lambda}$ is Fredholm for all $\lambda \in \mathbb{C}\setminus\mathbb{T}$. Indeed, this is a particular case of the following more general result. Here we denote by $\mathcal{R}$ the space of all rational functions without poles on $\mathbb{T}$.

Theorem 6.12. *Let* R *be a rational function without zeros or poles on* $\mathbb{T}$, *i.e.,* $R \in \mathcal{GR}$. *Then* A^M_R *is Fredholm.*

Proof. It is enough to prove that T_{G_R}, with

$$G_R = \begin{pmatrix} \overline{\theta} & 0 & 0 & 0 \\ 0 & \overline{\theta} & 0 & 0 \\ R & R\overline{A_+\theta} & \theta & 0 \\ RA_-\overline{\theta} & R & 0 & \theta \end{pmatrix}$$

is Fredholm. This follows from the fact that G_R admits a meromorphic factorization ([10,12], see also [11], Theorem 3.3) of the form $G_R = M_-M_+^{-1}$ with $M_-^{\pm 1} \in (\overline{H^\infty} + \mathcal{R})^{4\times 4}$ and $M_+^{\pm 1} \in (H^\infty + \mathcal{R})^{4\times 4}$ given by:

$$M_+ = \begin{pmatrix} 1 & 0 & \theta & 0 \\ 0 & 1 & 0 & \theta \\ 0 & 0 & -R & 0 \\ 0 & 0 & 0 & -R \end{pmatrix},$$

$$M_- = \begin{pmatrix} \overline{\theta} & 0 & 1 & 0 \\ 0 & \overline{\theta} & 0 & 1 \\ R & R\overline{A_+\theta} & 0 & \overline{A_+}R \\ RA_-\overline{\theta} & R & A_-R & 0 \end{pmatrix}.$$

as can be easily verified. □

Corollary 6.13. $\sigma_e(T_z^M) \subset \mathbb{T}$.

Now, to study the Fredholmness of $A^M_{z-\lambda}$ for $\lambda \in \mathbb{T}$ we use another factorization of $\mathcal{G}_\lambda$. Indeed, more generally, for any $R \in \mathcal{R}$ we can factorise

$$G_R = \underbrace{\begin{pmatrix} I_{2\times 2} & 0_{2\times 2} \\ R\begin{pmatrix} 1 & \overline{A_+} \\ A_- & 1 \end{pmatrix} & I_{2\times 2} \end{pmatrix}}_{H_R^-} \underbrace{\begin{pmatrix} \overline{\theta} I_{2\times 2} & 0_{2\times 2} \\ R(1-\overline{\theta}) I_{2\times 2} & \theta I_{2\times 2} \end{pmatrix}}_{\widetilde{G}_R}$$

where both factors depend on the symbol $R \in \mathcal{R}$ (or the point λ if $R = z-\lambda$), but the roles of A_-, A_+ (i.e., ϕ and ψ) and θ are separated.

Since $H_R^- \in \mathcal{G}(\overline{H^\infty} + \mathcal{R})^{4\times 4}$, it follows that G_R admits a WH-factorization in L^2 [9] (also known as generalised factorisation [30] or Φ-factorisation [28]), whose existence is equivalent to T_{G_R} being Fredholm [9], if and only if $\widetilde{G}_R$ admits such a factorization (Theorem 3.10 in [30]). Since $\widetilde{G}_R$ does not depend on A_+ or A_-, and for $A_+ = 0$ or $A_- = 0$ the operator W defined in Theorem 2.3 is triangular, taking into account this theorem we conclude the following:

Theorem 6.14. *T_R^M is Fredholm if and only if A_R^θ is Fredholm.*

The Fredholmness of truncated Toeplitz operators with rational symbols $R \in \mathcal{R}$ was studied in Section 5 of [13] in the equivalent setting of the real line. For the case $R = z - \lambda$, taking into account also Corollary 6.13, we have:

Corollary 6.15. *The essential spectrum $\sigma_e(T_z^M)$ is contained in $\mathbb{T}$ and does not depend of A_+ or A_-, but only on the point λ and the inner function θ. We have*

$$\sigma_e(T_z^M) = \sigma_e(A_z^\theta) = \sigma(\theta) = \left\{ \lambda \in \mathbb{T} : \liminf_{z\to\lambda, z\in\mathbb{D}} |\theta(z)| = 0 \right\}.$$

Remark 6.16. Since, by Theorems 4.7 and 6.8 , the dimensions of $\ker T^M_{z-\lambda}$ and $\ker(T^M_{z-\lambda})^*$ are equal and finite, we see that if $\lambda \in \sigma(\theta)$ the range of $T^M_{z-\lambda}$ is not closed.

6.3. Invertibility, Spectrum and Resolvent Operators

From Corollary 6.15 and from the description of $\ker T^M_{z-\lambda}$ obtained in Section 6.1, taking moreover into account Corollary 4.8, we easily get the spectrum of the dual-band shift.

Theorem 6.17.

(*i*) $\lambda \in \sigma(T_z^M) \iff (\lambda \in \mathbb{D}, \Delta_\lambda = 0) \vee (\lambda \in \mathbb{D}^-, \widetilde{\Delta_\lambda} = 0) \vee (\lambda \in \mathbb{T} \cap \sigma(\theta))$
$\vee (\lambda \in \mathbb{T}_{ADC} \setminus \sigma(\theta), \Delta_\lambda = 0)$;

(ii)

$$\sigma_p(T_z^M) = \{\lambda \in \mathbb{D}, \Delta_\lambda = 0\} \cup \{\lambda \in \mathbb{D}^-, \widetilde{\Delta_\lambda} = 0\} \cup \{\lambda \in \mathbb{T}_{ADC}, \Delta_\lambda = 0\},$$
$$\sigma_e(T_z^M) = \mathbb{T} \backslash \mathbb{T}_{ADC} \cup \{\lambda \in \mathbb{T}_{ADC} \cap \sigma(\theta), \Delta_\lambda \neq 0\}$$
$$\sigma_r(T_z^M) = \emptyset$$

where Δ_λ and $\widetilde{\Delta_\lambda}$ were defined in Propositions 6.1 and 6.2.

From Theorem 6.17 we see in particular that unlike the essential spectrum, $\sigma(T_z^M)$ is in general clearly different from $\sigma(A_z^\theta)$.

For $\lambda \notin \sigma(T_z^M)$, i.e., $\lambda \in \mathbb{D}$, $\Delta_\lambda \neq 0$, or $\lambda \in \mathbb{D}^-$, $\widetilde{\Delta_\lambda} \neq 0$, or $\lambda \in \mathbb{T}_{\mathrm{ADC}} \setminus \sigma(\theta)$, $\Delta_\lambda \neq 0$, we can explicitly define the resolvent operator by using Corollary 4.6 and a bounded canonical Wiener–Hopf factorization of $\mathcal{G}_\lambda$, of the form $\mathcal{G}_\lambda = \mathcal{G}_{\lambda -}\mathcal{G}_{\lambda +}$, since we have, in that case, $(T_{\mathcal{G}_\lambda})^{-1} = (\mathcal{G}_{\lambda+})^{-1}P^+(\mathcal{G}_{\lambda-})^{-1}P^+_{|(H^2)^4}$ [9]. That canonical factorization will be given below for $\lambda \in \mathbb{D} \cup \mathbb{T}_{\mathrm{ADC}} \backslash \sigma(\theta)$ with $\Delta_\lambda \neq 0$; for $\lambda \in \mathbb{D}^-$, $\widetilde{\Delta_\lambda} \neq 0$, the canonical factorization can be obtained analogously. Those factorizations, which were obtained by solving a Riemann–Hilbert problem of the form $\mathcal{G}_\lambda \phi_+ = \phi_-$ with $\phi_+ \in (H^2)^4$ and $\phi_- \in (\overline{H^2})^4$ for each of the column factors as in [8], can be checked directly by multiplication of the matricial factors.

Note that if $\lambda \in \mathbb{T} \backslash \sigma(\theta)$ then θ has an analytic continuation to a neighbourhood of λ and there is clearly an ADC for θ there with $\frac{\theta - \theta(\lambda)}{z - \lambda} \in H^\infty$. If $\lambda \in \mathbb{T}_{\mathrm{ADC}} \cap \sigma(\theta)$ then $\frac{\theta - \theta(\lambda)}{z - \lambda} \in H^2 \backslash H^\infty$ (page 505 of [35]); in this case $\mathcal{G}_\lambda$ has an L^2-factorization [28], but it is not bounded nor a WH factorization, although the factors are given by the same expressions as in the theorem below.

Theorem 6.18. (i) *If $\lambda \in \mathbb{D}$ or $\lambda \in \mathbb{T}_{ADC} \backslash \sigma(\theta)$ and $\Delta_\lambda \neq 0$ then $\mathcal{G}_\lambda$ admits a bounded canonical factorization of the form $\mathcal{G}_\lambda = \mathcal{G}_{\lambda-}\mathcal{G}_{\lambda+}$ where*

(i) if $\Delta = 1 - \overline{A_+(0)}\,\overline{\overline{A_-}(0)}\,\overline{\theta(0)}^2 \neq 0$,

$$\mathcal{G}_{\lambda+}^{-1} = \begin{pmatrix} \theta + \frac{\overline{A_+(0)}\,\overline{\overline{A_-}(0)}\,\overline{\theta(0)}}{\Delta} & \frac{\theta - \theta(\lambda)}{z - \lambda} & \frac{-\overline{A_+(0)}}{\Delta} & 0 \\ -\frac{\overline{\overline{A_-}(0)}}{\Delta} & 0 & \theta + \frac{\overline{A_+(0)}\,\overline{\overline{A_-}(0)}\,\overline{\theta(0)}}{\Delta} & \frac{\theta - \theta(\lambda)}{z - \lambda} \\ -(z - \lambda) & -1 & 0 & 0 \\ 0 & 0 & -(z - \lambda) & -1 \end{pmatrix},$$

and

$$\mathcal{G}_{\lambda-} = \begin{pmatrix} 1 + \frac{\overline{A_+(0)}\,\overline{\overline{A_-}(0)}\,\overline{\theta(0)}}{\Delta}\overline{\theta} & \frac{1 - \theta(\lambda)\overline{\theta}}{z - \lambda} & -\frac{\overline{A_+(0)}}{\Delta}\overline{\theta} & 0 \\ -\frac{\overline{\overline{A_-}(0)}}{\Delta}\overline{\theta} & 0 & 1 + \frac{\overline{A_+(0)}\,\overline{\overline{A_-}(0)}\,\overline{\theta(0)}}{\Delta}\overline{\theta} & \frac{1 - \theta(\lambda)\overline{\theta}}{z - \lambda} \\ g_{31}^{-} & -\theta(\lambda) & g_{33}^{-} & \overline{A_+}(1 - \theta(\lambda)\overline{\theta}) \\ g_{41}^{-} & A_-(1 - \theta(\lambda)\overline{\theta}) & g_{43}^{-} & -\theta(\lambda) \end{pmatrix},$$

with

$$
\begin{aligned}
g_{31}^{-} &= -\frac{\overline{\overline{A_-}(0)}}{\Delta}(z-\lambda)(\overline{A_+\theta} - \overline{A_+(0)\theta(0)}),\\
g_{41}^{-} &= \frac{z-\lambda}{\Delta}(A_- - \overline{\overline{A_-}(0)} + \overline{A_+(0)}\,\overline{\overline{A_-}(0)}\overline{\theta(0)}\,A_-\,(\overline{\theta} - \overline{\theta(0)}))\\
g_{33}^{-} &= \frac{z-\lambda}{\Delta}(\overline{A_+} - \overline{A_+(0)} + \overline{A_+(0)}\,\overline{\overline{A_-}(0)}\,\overline{\theta(0)}\,\overline{A_+}\,(\overline{\theta} - \overline{\theta(0)})),\\
g_{43}^{-} &= -\frac{\overline{A_+(0)}}{\Delta}(z-\lambda)(A_-\overline{\theta} - \overline{\overline{A_-}(0)}\overline{\theta(0)}).
\end{aligned}
$$

Note that in this case $\det \mathcal{G}_{\lambda+}^{-1} = \det \mathcal{G}_{\lambda-} = -\Delta_\lambda/\Delta \in \mathbb{C}\setminus\{0\}$.

(ii) *If* $\Delta = 0$, *in which case* $A_+(0)$, $\overline{A_-}(0)$, $\theta(0) \neq 0$ *and* $A_+(0)\theta(0) = \dfrac{1}{\overline{A_-}(0)\,\theta(0)}$, *we have*

$$
\mathcal{G}_{\lambda+}^{-1} = \begin{pmatrix}
-\overline{A_+(0)}(1-\theta\overline{\theta(0)}) & \frac{\theta-\theta(\lambda)}{z-\lambda} & -\overline{A_+(0)\theta(0)} & 0\\
\theta & 0 & 1 & \frac{\theta-\theta(\lambda)}{z-\lambda}\\
-\overline{\theta(0)A_+(0)}(z-\lambda) & -1 & 0 & 0\\
-(z-\lambda) & 0 & 0 & -1
\end{pmatrix},
$$

and

$$
\mathcal{G}_{\lambda-} = \begin{pmatrix}
-\overline{A_+(0)}(\overline{\theta} - \overline{\theta(0)}) & \frac{1-\theta(\lambda)\overline{\theta}}{z-\lambda} & -\overline{A_+(0)}\,\overline{\theta(0)}\,\overline{\theta} & 0\\
1 & 0 & \overline{\theta} & \frac{1-\theta(\lambda)\overline{\theta}}{z-\lambda}\\
(\overline{A_+} - \overline{A_+(0)})(z-\lambda) & -\theta(\lambda) & (\overline{A_+\theta} - \overline{A_+(0)\theta(0)})(z-\lambda) & \overline{A_+}(1-\theta(\lambda)\overline{\theta})\\
-\overline{A_+(0)}A_-(\overline{\theta} - \overline{\theta(0)})(z-\lambda) & A_-(1-\theta(\lambda)\overline{\theta}) & 1 - \overline{A_+(0)\theta(0)}A_-\overline{\theta}(z-\lambda) & -\theta(\lambda)
\end{pmatrix}
$$

with

$$
\det(\mathcal{G}_{\lambda+})^{-1} = \det \mathcal{G}_{\lambda-} = \overline{A_+(0)}(1 - 2\overline{\theta(0)}\theta(\lambda)) = -\frac{\Delta_\lambda}{\overline{\overline{A_-}(0)}} \in \mathbb{C}\setminus\{0\}.
$$

7. L^2-Factorization and Angular Derivatives

In Sect. 6.2 the Fredholmness of $T^M_{z-\lambda}$, or $T_{\mathcal{G}_\lambda}$, was studied from a factorization of $\mathcal{G}_\lambda$, which we repeat here for convenience:

$$
\mathcal{G}_\lambda = H_- \widetilde{\mathcal{G}}_\lambda \tag{7.1}
$$

with

$$
H_-^{\pm} \in (\overline{H^\infty} + \mathcal{R})^{4\times 4} \text{ and } \widetilde{\mathcal{G}}_\lambda = \begin{pmatrix} \overline{\theta} I_{2\times 2} & 0_{2\times 2}\\ (z-\lambda)(1-\overline{\theta}) I_{2\times 2} & \theta I_{2\times 2}\end{pmatrix}, \tag{7.2}
$$

using the fact that $\mathcal{G}_\lambda$ admits a WH factorization if and only if $\widetilde{\mathcal{G}}_\lambda$ admits a factorization of the same type. Indeed it is well known that the existence of a WH factorization for $G \in (L^\infty)^{n\times n}$ is equivalent to the Fredholmness of the Toeplitz operator T_G (see, e.g. [9, Thm. 1.1] or [28,30]).

It may happen that $\mathcal{G}_\lambda$ admits an L^2-factorization which does not satisfy the condition of boundedness for the densely defined operator $\mathcal{G}_{\lambda+}^{-1}P^+\mathcal{G}_{\lambda_-}^{-1}$: $\mathcal{R}^4 \to (L^2)^4$ (see [9,28]) Although in that case one cannot conclude that $T_{\mathcal{G}_\lambda}$

is Fredholm, it is still possible to use it to characterize several important properties of the operator, such that as injectivity [28]. This leads us to the question of existence of such a L^2-factorization for $\widetilde{\mathcal{G}}_\lambda$. Somewhat surprisingly, we obtain a necessary and sufficient condition for existence of an ADC of θ at $\lambda \in \mathbb{T}$ in terms of an L^2-factorization, a relation which appears here for the first time.

Theorem 7.1. *Let $\lambda \in \mathbb{T}$. The inner function θ has an ADC at λ if and only if $\widetilde{\mathcal{G}}_\lambda$ has an L^2 factorization.*

Proof. (i) Assume that $\widetilde{\mathcal{G}}_\lambda = \widetilde{\mathcal{G}}_{\lambda-} D \widetilde{\mathcal{G}}_{\lambda+}$ with $D = \operatorname{diag}\{z^{k_j}\}$ with $k_j \in \mathbb{Z}$, $j = 1, 2, 3, 4$. Since $\det \widetilde{\mathcal{G}}_\lambda = 1$, we must have $\sum_{j=1}^4 k_j = 0$. If, for some j, we have $k_j = -n < 0$, then there exists a non-zero solution to

$$\begin{pmatrix} \overline{\theta} I_{2\times 2} & 0_{2\times 2} \\ (z-\lambda)(1-\overline{\theta}) I_{2\times 2} & \theta I_{2\times 2} \end{pmatrix} \begin{pmatrix} \psi_{1+} \\ \psi_{2+} \end{pmatrix} = \overline{z}^n \begin{pmatrix} \psi_{1-} \\ \psi_{2-} \end{pmatrix} \tag{7.3}$$

with $\psi_{1+}, \psi_{2+} \in (H^2)^2$ and $\psi_{1-}, \psi_{2-} \in (\overline{H^2})^2$, given by the j'th column of the factors. So we have, with $\overline{\theta}\psi_{1+} = \overline{z}^n \psi_{1-}$,

$$\begin{aligned} &(z-\lambda)(1-\overline{\theta})\psi_{1+} + \theta\psi_{2+} = \overline{z}^n \psi_{2-} \\ &\quad \Longrightarrow (z-\lambda)\psi_{1+} - (z-\lambda)\overline{z}^n \psi_{1-} + \theta\psi_{2+} = \overline{z}^n \psi_{2-} \\ &\quad \Longrightarrow (z-\lambda)\psi_{1+} + \theta\psi_{2+} = \overline{z}^n \psi_{2-} + (z-\lambda)\overline{z}^n \psi_{1-} = C \in \mathbb{C}^2. \end{aligned} \tag{7.4}$$

If $C = 0$ (which necessarily happens if $n > 1$) then $\psi_{1+} = \psi_{2+} = 0$ and there is no nontrivial solution to (7.3). So we must have $n \leqslant 1$ and $C \neq 0$. For $n \geqslant 1$, from (7.4),

$$\overline{z}^n (z-\lambda)\psi_{1-} - \overline{\theta} C = -\psi_{2+} = A(z-\lambda) + B, \text{ with } A, B \in \mathbb{C}^2.$$

Now we must also have $B \neq 0$ because, otherwise we would have $\psi_{1-} - A = \overline{\theta} C \frac{z^n}{z-\lambda} \in \overline{H^2}$ with $C \neq 0$, which is impossible because $\lambda \in \mathbb{T}$. Going back to (7.4), then,

$$\begin{aligned} &(z-\lambda)\psi_{1+} + \theta(-A(z-\lambda) - B) = C \\ &\quad \Longleftrightarrow (z-\lambda)(\psi_{1+} - \theta A) = C + \theta B; \end{aligned}$$

by Lemma 6.3, it follows that $\theta(\lambda)$ exists and, since $\psi_{1+} - \theta A \in (H^2)^2$ we get that $\frac{\theta - \theta(\lambda)}{z-\lambda} \in H^2$ and θ has an ADC at λ by Lemma 6.4.

(ii) Conversely, suppose that θ has an ADC at λ. Then $\widetilde{\mathcal{G}}_\lambda$ admits the following L^2 factorization (obtained as before by solving a Riemann-Hilbert problem):

- if $\theta(\lambda) \neq \frac{-1}{1-\overline{\theta(0)}}$, $\widetilde{\mathcal{G}}_\lambda = \widetilde{\mathcal{G}}_{\lambda-}\widetilde{\mathcal{G}}_{\lambda+}$ with

$$\widetilde{\mathcal{G}}_{\lambda-} = \begin{pmatrix} \frac{\overline{\theta}}{1-\overline{\theta(0)}}+1 & \frac{1-\theta(\lambda)\overline{\theta}}{z-\lambda} & 0 & 0 \\ 0 & 0 & \frac{\overline{\theta}}{1-\overline{\theta(0)}}+1 & \frac{1-\theta(\lambda)\overline{\theta}}{z-\lambda} \\ -\frac{\overline{\theta}-\overline{\theta(0)}}{1-\overline{\theta(0)}}(z-\lambda) & -1+\theta(\lambda)(\overline{\theta}-1) & 0 & 0 \\ 0 & 0 & -\frac{\overline{\theta}-\overline{\theta(0)}}{1-\overline{\theta(0)}}(z-\lambda) & -1+\theta(\lambda)(\overline{\theta}-1) \end{pmatrix}$$

$$\widetilde{\mathcal{G}}_{\lambda+}^{-1} = \begin{pmatrix} \frac{1}{1-\overline{\theta(0)}}+\theta & \frac{\theta-\theta(\lambda)}{z-\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{1-\overline{\theta(0)}}+\theta & \frac{\theta-\theta(\lambda)}{z-\lambda} \\ -(z-\lambda) & -1 & 0 & 0 \\ 0 & 0 & -(z-\lambda) & -1 \end{pmatrix},$$

-if $\theta(\lambda) = \frac{-1}{1-\overline{\theta(0)}}$, $\widetilde{\mathcal{G}}_\lambda = \widetilde{\mathcal{G}}_{\lambda-}\,\mathrm{diag}\,(\overline{z},\overline{z},z,z)\widetilde{\mathcal{G}}_{\lambda+}$ with

$$\widetilde{\mathcal{G}}_{\lambda-} = \begin{pmatrix} z\frac{1-\theta(\lambda)\overline{\theta}}{z-\lambda} & 0 & \overline{z} & 0 \\ 0 & z\frac{1-\theta(\lambda)\overline{\theta}}{z-\lambda} & 0 & \overline{z} \\ z(\theta(\lambda)\overline{\theta}-1-\theta(\lambda)) & 0 & -\frac{z-\lambda}{z} & 0 \\ 0 & z(\theta(\lambda)\overline{\theta}-1-\theta(\lambda)) & 0 & -\frac{z-\lambda}{z} \end{pmatrix},$$

$$\widetilde{\mathcal{G}}_{\lambda+}^{-1} = \begin{pmatrix} \frac{\theta-\theta(\lambda)}{z-\lambda} & 0 & \theta & 0 \\ 0 & \frac{\theta-\theta(\lambda)}{z-\lambda} & 0 & \theta \\ -1 & 0 & -(z-\lambda) & 0 \\ 0 & -1 & 0 & -(z-\lambda) \end{pmatrix}.$$

□

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Received: July 1, 2021.
Accepted: May 24, 2022.