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Injectors in π-Separable Groups

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Abstract. Let π be a set of primes. We show that π-separable groups have a conjugacy class of $\mathfrak{F}$-injectors for suitable Fitting classes $\mathfrak{F}$, which coincide with the usual ones when specializing to soluble groups.

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1. Introduction and Preliminaries

All groups considered are finite.

One of the fundamentals facts in the theory of finite soluble groups is the theorem of B. Fischer, W. Gaschütz and B. Hartley, which states the existence and conjugacy of $\mathfrak{F}$-injectors in finite soluble groups for Fitting classes $\mathfrak{F}$ ([6], [5, VIII. Theorem (2.9), IX. Theorem (1.4)]). An important stream of research has considered the extent of the validity of this result to all finite groups. We refer to [4,5] for background on classes of groups and for accounts about the development of this topic; we shall adhere to their notations. We recall that if $\mathfrak{F}$ is a class of groups, an $\mathfrak{F}$-injector of a group G is a subgroup V of G with the property that $V \cap K$ is an $\mathfrak{F}$-maximal subgroup of K (i.e. maximal as a subgroup of K in $\mathfrak{F}$) for all subnormal subgroups K of G. The existence of $\mathfrak{F}$-injectors in all groups implies that $\mathfrak{F}$ is a Fitting class, which is defined as a non-empty class of groups closed under taking normal subgroups and products of normal subgroups. The existence and properties of Carter subgroups, i.e. self-normalizing nilpotent subgroups, in soluble groups are the cornerstone in the proof of the above-mentioned theorem of Fischer, Gaschütz and Hartley. In a previous paper [1] we initiated the study of an extension of the theory of soluble groups to the universe of π-separable groups, π a set of primes. We analyzed the reach of π-separability further from soluble groups, by means of complement and Sylow bases and Hall systems, based on the remarkable property that π-separable groups have Hall π-subgroups, and every π-subgroup is contained in a conjugate of any Hall π-subgroup.

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We also proved that π-separable groups have a conjugacy class of subgroups which specialize in Carter subgroups within the universe of soluble groups. The main results in the present paper, namely Lemma 2.9, Theorem 2.10 and Corollary 2.11 below, show that these Carter-like subgroups enable an extension of the existence and conjugacy of injectors to π-separable groups.

If π is a set of primes, let us recall that a group G is π-separable if every composition factor of G is either a π-group or a π'-group, where π' stands for the complement of π in the set $\mathbb{P}$ of all prime numbers. Clearly, π-separability is equivalent to π'-separability, so that by the Feit-Thompson theorem, every π-separable group is either π-soluble or π'-soluble, where for any set of primes ρ, a group is ρ-soluble if it is ρ-separable with every ρ-composition factor a p-group for some prime $p \in \rho$. Also Burnside's $p^a q^b$-theorem implies that π-separable groups are π-soluble if $|\pi| \leq 2$. We refer to [7] for basic results on π-separable groups.

These remarks about ρ-solubility clarify the reach of our main results, which apply to π'-soluble groups once the set of primes π is fixed.

More precisely, our study of π-separable groups relies on convenient extensions of the class $\mathfrak{N}$ of nilpotent groups as well as of normal subgroups, according to the set of primes π, as follows.

Let π be a set of primes. Let

$$\mathfrak{N}^\pi = \mathfrak{E}_\pi \times \mathfrak{N}_{\pi'} = (G = H \times K \mid H \in \mathfrak{E}_\pi,\ K \in \mathfrak{N}_{\pi'}),$$

where $\mathfrak{E}_\pi$ denotes the class of all π-groups and $\mathfrak{N}_{\pi'}$ the class of all nilpotent π'-groups. In the particular cases when either $\pi = \emptyset$ or $\pi = \{p\}$, p a prime, ($|\pi| \leq 1$), then $\mathfrak{N}^\pi = \mathfrak{N}$ is the class of all nilpotent groups.

We observe that $\mathfrak{N}^\pi$ is a saturated formation and appeal to the concept of $\mathfrak{N}^\pi$-Dnormal subgroup. We refer to [1,2] for the concept of $\mathfrak{G}$-Dnormal subgroup for general saturated formations $\mathfrak{G}$, and specialize this definition to our particular saturated formation $\mathfrak{N}^\pi$ next. For notation, if ρ is a set of primes and G is a group, $\mathrm{Hall}_\rho(G)$ denotes the set of all Hall ρ-subgroups of G. If p is a prime, then $\mathrm{Syl}_p(G)$ stands for the set of all Sylow p-subgroups of G. If $G_\rho \in \mathrm{Hall}_\rho(G)$ and $H \leq G$, we write $G_\rho \searrow H$ to mean that G_ρ reduces into H, i.e. $G_\rho \cap H \in \mathrm{Hall}_\rho(H)$.

Definition 1.1 ([2, Definition 3.1], [1]). A subgroup H of a group G is said to be $\mathfrak{N}^\pi$-*Dnormal* in G if it satisfies the following conditions:

1. whenever $p \in \pi'$ and $G_p \in \mathrm{Syl}_p(G)$, $G_p \searrow H$, then $G_p \leq N_G(H)$;
2. whenever $p \in \pi$ and $G_p \in \mathrm{Syl}_p(G)$, $G_p \searrow H$, then
   - $G_p \leq N_G(H)$ if $\pi = \{p\}$, or
   - $G_p \leq N_G(O^\pi(H))$ if $|\pi| \geq 2$.

Hence, for $\mathfrak{N}^\pi = \mathfrak{N}$, $\mathfrak{N}$-Dnormal subgroups are exactly normal subgroups. Note also that normal subgroups are $\mathfrak{N}^\pi$-Dnormal for any set of primes π.

$\mathfrak{N}^\pi$-Dnormal subgroups are nicely characterized as follows.

Proposition 1.2 [1, Proposition 3.7]. *Let H be a subgroup of a group G. Then:*

1. *Assume that $|\pi| \leq 1$. Then $\mathfrak{N}^\pi = \mathfrak{N}$ and H is $\mathfrak{N}$-Dnormal in G if and only if H is normal in G.*
2. *Assume that $|\pi| \geq 2$. Then the following statements are equivalent:*
   (i) *H is $\mathfrak{N}^\pi$-Dnormal in G;*
   (ii) *$O^\pi(H) \trianglelefteq G$ and $O^\pi(G) \leq N_G(H)$.*

For our study of Fitting classes it is useful to introduce a corresponding extension of subnormality.

Definition 1.3. A subgroup S of a group G is said to be $\mathfrak{N}^\pi$-*Dsubnormal* in G if there is a chain of subgroups

$$S = S_0 \leq S_1 \leq \cdots \leq S_k = G,$$

such that S_i is $\mathfrak{N}^\pi$-Dnormal in S_{i+1} if $0 \leq i \leq k-1$.

As for $\mathfrak{N}^\pi$-Dnormal subgroups, if $\mathfrak{N}^\pi = \mathfrak{N}$, then $\mathfrak{N}$-Dsubnormal subgroups are exactly subnormal subgroups; also subnormal subgroups are $\mathfrak{N}^\pi$-Dsubnormal for any set of primes π.

To prove our main results we will need some properties of $\mathfrak{N}^\pi$-Dnormal and $\mathfrak{N}^\pi$-Dsubnormal subgroups that we gather in the next lemmas.

Lemma 1.4. *Let H be a subgroup of a group G, $N \trianglelefteq G$ and $g \in G$. Then:*

1. *If H is $\mathfrak{N}^\pi$-Dnormal in G, then H^g is $\mathfrak{N}^\pi$-Dnormal in G.*
2. *If H is $\mathfrak{N}^\pi$-Dnormal in G and $H \leq L \leq G$, then H is $\mathfrak{N}^\pi$-Dnormal in L.*
3. *If H is $\mathfrak{N}^\pi$-Dnormal in G, then HN/N is $\mathfrak{N}^\pi$-Dnormal in G/N.*
4. *If $N \leq H$ and H/N is $\mathfrak{N}^\pi$-Dnormal in G/N, then H is $\mathfrak{N}^\pi$-Dnormal in G.*
5. *If $N \leq H$ and H/N is $\mathfrak{N}^\pi$-Dsubnormal in G/N, then H is $\mathfrak{N}^\pi$-Dsubnormal in G.*
6. *If $G \in \mathfrak{N}^\pi$, then H is $\mathfrak{N}^\pi$-Dsubnormal in G.*

Proof. If $|\pi| \leq 1$, the result refers to normal and subnormal subgroups and it is clear. Assume that $|\pi| \geq 2$. Then:

(1) and (2) are easily proven.

(3) By Proposition 1.2, assuming that $O^\pi(H) \trianglelefteq G$ and $O^\pi(G) \leq N_G(H)$, we need to prove that $O^\pi(HN/N) \trianglelefteq G/N$ and $O^\pi(G/N) \leq N_G(HN/N)$. But this is clear since for any $L \leq G$, $O^\pi(LN/N) = O^\pi(L)N/N$.

(4) Again by Proposition 1.2, we assume that $O^\pi(H/N) = O^\pi(H)N/N \trianglelefteq G/N$ and $O^\pi(G/N) = O^\pi(G)N/N \leq N_G(H/N)$, and need to prove that $O^\pi(H) \trianglelefteq G$ and $O^\pi(G) \leq N_G(H)$. The second property follows clearly, and also that $[G, O^\pi(H)] \leq O^\pi(H)N \leq H$. Then $[G, O^\pi(H)] \leq H$, which is equivalent to $[G, O^\pi(H)] \leq O^\pi(H)$ (see [1, Remark 3.6]). Therefore, $G \leq N_G(O^\pi(H))$, and we are done.

(5) It follows from the definition of $\mathfrak{N}^\pi$-Dsubnormal subgroup together with part 4.

(6) If $G \in \mathfrak{N}^\pi$ then $G = G_\pi \times G_{\pi'}$, $G_\pi = O_\pi(G)$, and $G_{\pi'} = O_{\pi'}(G) \in \mathfrak{N}$. Also $H \in \mathfrak{N}^\pi$ so that $H = H_\pi \times H_{\pi'}$, $H_\pi = O_\pi(H)$, and $H_{\pi'} = O_{\pi'}(H) \in \mathfrak{N}$. Then we deduce easily that H is $\mathfrak{N}^\pi$-Dnormal in $G_\pi \times H_{\pi'}$. Since $G_{\pi'}$ is nilpotent, $H_{\pi'}$ is subnormal in $G_{\pi'}$. Hence $G_\pi \times H_{\pi'}$ is subnormal in G, and so also $\mathfrak{N}^\pi$-Dsubnormal. Therefore, H is $\mathfrak{N}^\pi$-Dsubnormal in G, as desired. □

Lemma 1.5. *Let H be an $\mathfrak{N}^\pi$-Dnormal subgroup of a group G. Then:*

1. $H/O^\pi(H) \leq O_\pi(G/O^\pi(H))$.
2. *Let $C = \mathrm{Core}_G(H)$ and $\langle H^G\rangle$ be the normal closure of H in G. Then $\langle H^G\rangle/C \leq O_\pi(G/C)$; equivalently, $\langle H^G\rangle/C \in \mathfrak{E}_\pi$.*
3. *If $V \leq G$, then $H \cap V$ is $\mathfrak{N}^\pi$-Dnormal in V.*

Proof. 1. By Proposition 1.2 we know that $O^\pi(H) \trianglelefteq G$, and we aim to prove that $H/O^\pi(H) \leq O_\pi(G/O^\pi(H))$. We argue by induction on $|G|$. If $O^\pi(H) \neq 1$, the result is clear by Lemma 1.4(3) and inductive hypothesis. We may then assume that $O^\pi(H) = 1$, i.e. H is a π-group, and we need to prove that $H \leq O_\pi(G)$. If $O^\pi(G) = 1$, then $H \leq G = O_\pi(G)$, and we are done. Consider now the case when $O^\pi(G) \neq 1$. Note that $O^\pi(G) \leq N_G(H)$ by Proposition 1.2, because H is $\mathfrak{N}^\pi$-Dnormal in G. Then $H \cap O^\pi(G) \leq O_\pi(O^\pi(G)) \leq O_\pi(G)$. If $H \cap O^\pi(G) \neq 1$, again by Lemma 1.4(3) and inductive hypotesis it follows that $HO_\pi(G)/O_\pi(G) \leq O_\pi(G/O_\pi(G)) = O_\pi(G)/O_\pi(G)$, and so $H \leq O_\pi(G)$ as desired. If $H \cap O^\pi(G) = 1$, then $[H, O^\pi(G)] = H \cap O^\pi(G) = 1$ and so $H \leq C_G(O^\pi(G))$. In the case when $C_G(O^\pi(G)) < G$, by Lemma 1.4(2) and inductive hypothesis we deduce that $H \leq O_\pi(C_G(O^\pi(G))) \leq O_\pi(G)$. Assume finally that $C_G(O^\pi(G)) = G$. Then $O^\pi(G) = O_{\pi'}(G)$ and, by the Schur-Zassenhaus theorem and the fact that $O^\pi(G) \leq Z(G)$, there is a unique Hall π-subgroup of G, which is $O_\pi(G)$, and also $H \leq O_\pi(G)$, which concludes the proof.

2. This is clear from part 1.

3. If $|\pi| \leq 1$, then $H \trianglelefteq G$ and the result is clear. Assume that $|\pi| \geq 2$. Then, by Proposition 1.2, assuming that $O^\pi(G) \leq N_G(H)$ and $O^\pi(H) \trianglelefteq G$, we need to prove that $O^\pi(V) \leq N_V(V \cap H)$ and $O^\pi(V \cap H) \trianglelefteq V$. Since $O^\pi(V) \leq O^\pi(G)$, it is clear that $O^\pi(V) \leq N_V(V \cap H)$. We consider $O^\pi(V \cap H) = \langle X \mid X\ \pi'\text{-subgroup of } V \cap H\rangle$. Let $x \in V$. If X is a π'-subgroup of $V \cap H$, then $X^x \leq V \cap O^\pi(H)^x = V \cap O^\pi(H) \leq V \cap H$, which implies that $X^x \leq O^\pi(V \cap H)$. Hence, $O^\pi(V \cap H)^x \leq O^\pi(V \cap H)$, and so $O^\pi(V \cap H) \trianglelefteq V$, which concludes the proof. □

Lemma 1.6. *If M is a maximal $\mathfrak{N}^\pi$-Dnormal proper subgroup of a π'-soluble group G, then $G^{\mathfrak{N}^\pi} \leq M$, where $G^{\mathfrak{N}^\pi}$ denotes the $\mathfrak{N}^\pi$-residual of G, i.e. the smallest normal subgroup in G with quotient group an $\mathfrak{N}^\pi$-group.*

Proof. By Lemma 1.5(2) we know that $\langle M^G\rangle/\,\mathrm{Core}_G(M) \in \mathfrak{E}_\pi$. Since M is a maximal $\mathfrak{N}^\pi$-Dnormal proper subgroup of G we deduce that either $\langle M^G\rangle = M$, i.e. $M \trianglelefteq G$, or $\langle M^G\rangle = G$. In the first case, since G is π'-soluble, either

$O^{\pi}(G) \leq M$ or $O^{p}(G) \leq M$ for some $p \in \pi'$, which imply that $G^{\mathfrak{N}^{\pi}} \leq M$. If $\langle M^{G} \rangle = G$, then $G/\operatorname{Core}_{G}(M) \in \mathfrak{E}_{\pi}$, and so also $G^{\mathfrak{N}^{\pi}} \leq M$. □

2. Injectors in π-Separable Groups

This section is devoted to proving our main results. We fix a set of primes π and introduce suitable Fitting classes, which we will call $\mathfrak{N}^{\pi}$-Fitting classes, as defined below in this section. They appear to be Fitting classes with stronger closure properties involving $\mathfrak{N}^{\pi}$-Dnormal subgroups. Then we prove the existence and conjugacy of associated injectors in π'-soluble groups. (See Theorem 2.10, Corollary 2.11, below).

Our treatment adheres to the approach in [5, Chapter VIII], and we present our study within the framework of Fitting sets, instead of general Fitting classes. As mentioned in [5, VIII.1], it does not cause any additional difficulty and has important advantages, both in terms of scope of results and working techniques.

We extend the concept of Fitting set [5, VIII. Definition (2.1)] to $\mathfrak{N}^{\pi}$-Fitting set as follows, by replacing the terms "normal subgroup" and "subnormal subgroup" by "$\mathfrak{N}^{\pi}$-Dnormal subgroup" and "$\mathfrak{N}^{\pi}$-Dsubnormal subgroup", respectively, in the original definition.

Definition 2.1. A non-empty set $\mathcal{F}$ of subgroups of a group G is called an $\mathfrak{N}^{\pi}$*-Fitting set* of G if the following conditions are satisfied:

- **FS1:** If T is an $\mathfrak{N}^{\pi}$-Dsubnormal subgroup of $S \in \mathcal{F}$, then $T \in \mathcal{F}$;
- **FS2:** If $S, T \in \mathcal{F}$ and S, T are $\mathfrak{N}^{\pi}$-Dnormal subgroups in $\langle S, T \rangle$, then $\langle S, T \rangle \in \mathcal{F}$;
- **FS3:** If $S \in \mathcal{F}$ and $x \in G$, then $S^{x} \in \mathcal{F}$.

Since normal subgroups are $\mathfrak{N}^{\pi}$-Dnormal, so subnormal subgroups are $\mathfrak{N}^{\pi}$-Dsubnormal, it is clear that an $\mathfrak{N}^{\pi}$-Fitting set is a Fitting set. Also, Fitting sets are exactly $\mathfrak{N}$-Fitting sets, for $\mathfrak{N}^{\pi} = \mathfrak{N}$ with $|\pi| \leq 1$.

For the basic results about Fitting sets, we refer to [5, VIII.2].

We recall in particular that for a Fitting set $\mathcal{F}$ of a group G, the $\mathcal{F}$-radical of G, denoted $G_{\mathcal{F}}$ is the join of all normal $\mathcal{F}$-subgroups of G; or equivalently, the join of all subnormal $\mathcal{F}$-subgroups of G. For a subgroup H of G, we set $H_{\mathcal{F}}$ for the radical of H associated to its Fitting set $\mathcal{F}_{H} = \{S \leq H \mid S \in \mathcal{F}\}$ which we shall denote simply as $\mathcal{F}$. (See [5, VIII. Definitions (2.3), Proposition (2.4)].)

The following property is often useful:

Lemma 2.2 [5, VIII. Proposition (2.4)(b)]. *Let $\mathcal{F}$ be a Fitting set of a group G, and let $H \leq G$ and $x \in G$. Then $(H_{\mathcal{F}})^{x} = (H^{x})_{\mathcal{F}}$. In particular, $N_{G}(H) \leq N_{G}(H_{\mathcal{F}})$.*

Our first aim is to prove that if $\mathcal{F}$ is an $\mathfrak{N}^{\pi}$-Fitting set, the $\mathcal{F}$-radical is equally described as the join of all $\mathfrak{N}^{\pi}$-Dnormal $\mathcal{F}$-subgroups, and also as the join of all $\mathfrak{N}^{\pi}$-Dsubnormal $\mathcal{F}$-subgroups. (Proposition 2.5 below.) The next lemma supplies a basic fact about the join of $\mathfrak{N}^{\pi}$-Dnormal subgroups.

Lemma 2.3. *If H, K are $\mathfrak{N}^\pi$-Dnormal subgroups of a group G, then $\langle H, K\rangle$ is $\mathfrak{N}^\pi$-Dnormal in G.*

Proof. If $|\pi| \le 1$, then $\mathfrak{N}^\pi$-Dnormal subgroups are exactly normal subgroups and the result is clear. Assume that $|\pi| \ge 2$. We argue by induction on $|G|$. By Proposition 1.2 we have that $O^\pi(H)$ and $O^\pi(K)$ are normal subgroups of G. If either $O^\pi(H) \ne 1$ or $O^\pi(K) \ne 1$, then the result is easily deduced by Lemma 1.4, parts (3), (4), and inductive hypothesis. Assume that $O^\pi(H) = O^\pi(K) = 1$. By Lemma 1.5(1), $H, K \le O_\pi(G)$. Consequently, $\langle H, K\rangle$ is a π-group, and $O^\pi(\langle H, K\rangle) = 1$ is a normal subgroup of G. On the other hand, by Proposition 1.2, $O^\pi(G) \le N_G(H) \cap N_G(K)$, and so $O^\pi(G) \le N_G(\langle H, K\rangle)$. Again Proposition 1.2 implies finally that $\langle H, K\rangle$ is $\mathfrak{N}^\pi$-Dnormal in G. □

Lemma 2.4. *Let $\mathcal{F}$ be an $\mathfrak{N}^\pi$-Fitting set of a group G. If H is an $\mathfrak{N}^\pi$-Dnormal subgroup of G, then $H_\mathcal{F}$ is $\mathfrak{N}^\pi$-Dnormal in G.*

Proof. If $|\pi| \le 1$, then again $\mathfrak{N}^\pi$-Dnormal subgroups are exactly normal subgroups and the result follows from Lemma 2.2. Assume that $|\pi| \ge 2$. By Proposition 1.2 we need to prove that $O^\pi(G) \le N_G(H_\mathcal{F})$ and $O^\pi(H_\mathcal{F}) \trianglelefteq G$.

Since H is $\mathfrak{N}^\pi$-Dnormal in G, again Proposition 1.2 and Lemma 2.2 imply that $O^\pi(G) \le N_G(H_\mathcal{F})$. We claim now that $O^\pi(H_\mathcal{F}) = O^\pi(O^\pi(H)_\mathcal{F})$. This will imply that $O^\pi(H_\mathcal{F}) \trianglelefteq G$ by Proposition 1.2 and Lemma 2.2, as above, and we will be done.

It is not difficult to check that $O^\pi(H_\mathcal{F}) \le O^\pi(H)_\mathcal{F} \le H_\mathcal{F}$. Then $O^\pi(O^\pi(H)_\mathcal{F}) \le O^\pi(H_\mathcal{F})$, $O^\pi(O^\pi(H)_\mathcal{F}) \trianglelefteq H_\mathcal{F}$ and $H_\mathcal{F}/O^\pi(O^\pi(H)_\mathcal{F})$ is a π-group. Hence $O^\pi(H_\mathcal{F}) = O^\pi(O^\pi(H)_\mathcal{F})$ and the claim is proven. □

Proposition 2.5. *If $\mathcal{F}$ is an $\mathfrak{N}^\pi$-Fitting set of a group G, then*

$$\begin{aligned} G_\mathcal{F} &= \langle H \le G \mid H \trianglelefteq G,\ H \in \mathcal{F}\rangle = \langle H \le G \mid H \mathrel{\triangleleft\triangleleft} G,\ H \in \mathcal{F}\rangle \\ &= \langle H \le G \mid H\ \mathfrak{N}^\pi\text{-}\mathit{Dnormal\ in}\ G,\ H \in \mathcal{F}\rangle \\ &= \langle H \le G \mid H\ \mathfrak{N}^\pi\text{-}\mathit{Dsubnormal\ in}\ G,\ H \in \mathcal{F}\rangle. \end{aligned}$$

Proof. Set $R = \langle H \le G \mid H\ \mathfrak{N}^\pi\text{-Dnormal in } G,\ H \in \mathcal{F}\rangle$. Since normal subgroups are $\mathfrak{N}^\pi$-Dnormal, it is clear that $G_\mathcal{F} \le R$. Moreover, it follows from the definition of $\mathfrak{N}^\pi$-Fitting set and Lemmas 1.4(1) and 2.3, that R is a normal subgroup of G in $\mathcal{F}$, which implies that $R \le G_\mathcal{F}$. Consequently, $G_\mathcal{F} = R$.

Let now $S = \langle H \le G \mid H\ \mathfrak{N}^\pi\text{-Dsubnormal in } G,\ H \in \mathcal{F}\rangle$. It is clear that $G_\mathcal{F} \le S$. Let H be an $\mathfrak{N}^\pi$-Dsubnormal subgroup of G. We claim that $H_\mathcal{F} \le G_\mathcal{F}$. Hence, if in addition $H \in \mathcal{F}$, then $H \le G_\mathcal{F}$. It will follow that $S \le G_\mathcal{F}$, which will conclude the proof.

If H is an $\mathfrak{N}^\pi$-Dsubnormal subgroup of G, there is a chain of subgroups $H = H_0 \le H_1 \le \cdots \le H_k = G$, such that H_i is $\mathfrak{N}^\pi$-Dnormal in H_{i+1} if $0 \le i \le k-1$. For each $i = 0, \dots, k-1$, Lemma 2.4 implies that $(H_i)_\mathcal{F}$ is $\mathfrak{N}^\pi$-Dnormal in H_{i+1}, and then $(H_i)_\mathcal{F} \le (H_{i+1})_\mathcal{F}$. Therefore, $H_\mathcal{F} = (H_0)_\mathcal{F} \le (H_1)_\mathcal{F} \le \cdots \le (H_k)_\mathcal{F} = G_\mathcal{F}$, which proves the claim. □

Remark. As it might be expected, as a consequence of Proposition 2.5, in the definition of $\mathfrak{N}^\pi$-Fitting set (Definition 2.1), $\mathfrak{N}^\pi$-Dnormal subgroups can be equivalently replaced by $\mathfrak{N}^\pi$-Dsubnormal subgroups in condition (FS2).

- Let $\mathcal{F}$ be an $\mathfrak{N}^\pi$-Fitting set of a group G. Then:
  If $S, T \in \mathcal{F}$ and S, T are $\mathfrak{N}^\pi$-Dsubnormal subgroups in $\langle S, T\rangle$, then $\langle S, T\rangle \in \mathcal{F}$

Proof. By Proposition 2.5 we deduce that $S, T \leq \langle S, T\rangle_{\mathcal{F}}$, and then $\langle S, T\rangle = \langle S, T\rangle_{\mathcal{F}} \in \mathcal{F}$. □

We recall also the concept of $\mathcal{X}$-injector of a group for a set of subgroups $\mathcal{X}$ of the group.

Definition 2.6 [5, VIII. Definition (2.5)(b)]. Let $\mathcal{X}$ be a set of subgroups of a group G. An $\mathcal{X}$*-injector* of G is a subgroup V of G with the property that $V \cap K$ is an $\mathcal{X}$-maximal subgroup of K (i.e. maximal as subgroup of K in $\mathcal{X}$) for every subnormal subgroup K of G. We shall denote the (possibly empty) set of $\mathcal{X}$-injectors of G by $\mathrm{Inj}_{\mathcal{X}}(G)$.

The following results about the existence and properties of $\mathfrak{N}^\pi$-projectors in π'-soluble groups are essential for our purposes.

Let us recall that given a class of groups $\mathfrak{X}$, a subgroup U of a group G is called an $\mathfrak{X}$-projector of G if UK/K is an $\mathfrak{X}$-maximal subgroup of G/K for all $K \trianglelefteq G$. The (possibly empty) set of $\mathfrak{X}$-projectors of G will be denoted by $\mathrm{Proj}_{\mathfrak{X}}(G)$.

In addition, an $\mathfrak{X}$-covering subgroup of G is a subgroup E of G with the property that $E \in \mathrm{Proj}_{\mathfrak{X}}(H)$ whenever $E \leq H \leq G$. The set of $\mathfrak{X}$-covering subgroups of G will be denoted by $\mathrm{Cov}_{\mathfrak{X}}(G)$.

We refer to [4,5] for convenient background about projectors and covering subgroups.

As in [1, Definition 4.7], we say that a subgroup H of a group G is *self-*$\mathfrak{N}^\pi$*-Dnormalizing* in G if whenever $H \leq K \leq G$ and H is $\mathfrak{N}^\pi$-Dnormal in K, then $H = K$.

Lemma 2.7 ([1, Theorem 4.5]). *For all* π'*-soluble groups* G, $\emptyset \neq \mathrm{Proj}_{\mathfrak{N}^\pi}(G) = \mathrm{Cov}_{\mathfrak{N}^\pi}(G)$ *and it is a conjugacy class of* G.

Lemma 2.8 ([1, Theorem 4.14]). *For a subgroup* H *of a* π'*-soluble group* G *the following statements are pairwise equivalent:*

1. H *is an* $\mathfrak{N}^\pi$*-projector of* G.
2. H *is an* $\mathfrak{N}^\pi$*-covering subgroup of* G.
3. $H \in \mathfrak{N}^\pi$ *is a self-*$\mathfrak{N}^\pi$*-Dnormalizing subgroup of* G *and* H *satisfies the following property:*

$$\text{If } H \leq X \leq G, \text{ then } H \cap X^{\mathfrak{N}^\pi} \leq (X^{\mathfrak{N}^\pi})'. \qquad (*)$$

The next result extends Hartley's result [5, VIII. Lemma (2.8)] for $\mathfrak{N}^\pi$-Fitting sets and π'-soluble groups.

Lemma 2.9. *Let* $\mathcal{F}$ *be an* $\mathfrak{N}^\pi$*-Fitting set of a* π'*-soluble group* G*. Let* K *be a normal subgroup of* G *containing the* $\mathfrak{N}^\pi$*-residual* $G^{\mathfrak{N}^\pi}$ *of* G*, let* W *be an* $\mathcal{F}$*-maximal subgroup of* K*, and let* V *and* V_1 *be* $\mathcal{F}$*-maximal subgroups of* G *which contain* W*.*

(a) *If* $W \trianglelefteq K$*, then* $V = (WP)_{\mathcal{F}}$*, where* P *is a suitable* $\mathfrak{N}^\pi$*-projector of* G*.*

(b) *In any case V and V_1 are conjugate in $\langle V, V_1\rangle$. More precisely, there exists $x \in \langle V, V_1\rangle^{\mathfrak{N}^\pi}$ such that $V_1^x = V$.*

Proof. We mimic the proof of [5, VIII. Lemma (2.8)]. The arguments get a bit more involved mainly by the fact that $\mathfrak{N}^\pi$-projectors are not characterized as self-$\mathfrak{N}^\pi$-Dnormalizing $\mathfrak{N}^\pi$-subgroups, in order to play the role of Carter subgroups in the original proof, but the characterization of $\mathfrak{N}^\pi$-projectors in Lemma 2.8 can be instead used to prove the result.

(a) We argue by induction on $|G|$. If $W \trianglelefteq K$, then $W = K_{\mathcal{F}}$, and so also $W \trianglelefteq G$ by Lemma 2.2. We gather the following facts which will be useful in the proof, where L denotes any subgroup of G containing W and U denotes an $\mathcal{F}$-maximal subgroup of L containing W:

1. L satisfies the hypotheses of the statement by considering $K \cap L$ and U playing the role of K and V, respectively.

   It is clear that W is an $\mathcal{F}$-maximal subgroup of $K \cap L \trianglelefteq L$. Moreover, since $\mathfrak{N}^\pi$ is closed under taking subgroups, $L^{\mathfrak{N}^\pi} \leq G^{\mathfrak{N}^\pi} \cap L \leq K \cap L$.
2. If $W \leq X \leq G$, $X \in \mathcal{F}$, then $X \cap K = W$.

   We have that $W \leq X \cap K \leq K$, and $X \cap K \in \mathcal{F}$, because $X \cap K \trianglelefteq X \in \mathcal{F}$. Since W is $\mathcal{F}$-maximal in K we deduce that $W = X \cap K$.
3. Whenever U/W is $\mathfrak{N}^\pi$-Dsubnormal in $R^*/W \leq L/W$, then $U = R^*_{\mathcal{F}}$ and $R^* \leq N_L(U)$. In particular, this holds if $U/W \leq R^*/W \in \mathfrak{N}^\pi$, $R^*/W \leq L/W$. Also, if $H/W \leq L/W$ and R^*/W is $\mathfrak{N}^\pi$-Dnormal in H/W, then $H \leq N_L(U)$. Moreover, $N_G(R^*) \leq N_G(U)$.

   By Lemma 1.4(5) we deduce that U is $\mathfrak{N}^\pi$-Dsubnormal in $R^* \leq L$. Since U is $\mathcal{F}$-maximal in L, Proposition 2.5 implies that $U = R^*_{\mathcal{F}}$. If $U/W \leq R^*/W \in \mathfrak{N}^\pi$, Lemma 1.4(6) implies that U/W is $\mathfrak{N}^\pi$-Dsubnormal in R^*/W. Obviously, if R^*/W is $\mathfrak{N}^\pi$-Dnormal in H/W, then U/W is $\mathfrak{N}^\pi$-Dsubnormal in H/W and $H \leq N_L(U)$, as above. Moreover, by Lemma 2.2 it follows that $N_G(R^*) \leq N_G(R^*_{\mathcal{F}}) = N_G(U)$.
4. The following statements are pairwise equivalent:
   (i) There exists $R \in \mathrm{Proj}_{\mathfrak{N}^\pi}(L)$ such that $U = (RW)_{\mathcal{F}}$.
   (ii) There exists $W \leq R^* \leq L$ such that $R^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(L/W)$ and $U = R^*_{\mathcal{F}}$.
   (iii) There exists $W \leq R^* \leq L$ such that $U/W \leq R^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(L/W)$.

   In this case, $\mathrm{Proj}_{\mathfrak{N}^\pi}(N_L(U)/W) \subseteq \mathrm{Proj}_{\mathfrak{N}^\pi}(L/W)$.

   The equivalence (i) $\Leftrightarrow$ (ii) is clear by [5, III. Proposition (3.7)]. The equivalence (ii) $\Leftrightarrow$ (iii) is a consequence of the fact 3. Finally, if we assume (ii), $R^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(L/W)$ and $U = R^*_{\mathcal{F}}$, then $R^* \leq N_L(U)$. By Lemma 2.7, $R^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(N_L(U)/W)$, and also $\mathrm{Proj}_{\mathfrak{N}^\pi}(N_L(U)/W) = \{(R^*/W)^x \mid x \in N_L(U)\} \subseteq \mathrm{Proj}_{\mathfrak{N}^\pi}(L/W)$.

By fact 2, $V \cap K = W$. We claim that $V/W \leq Z_{\mathfrak{N}^\pi}(N_G(V)/W)$ the $\mathfrak{N}^\pi$-hypercentre of $N_G(V)/W$ (see [5, IV. Notation and Definitions (6.8)]). Set $N = N_G(V)$. Since $N/(N \cap K) \cong NK/K \leq G/K \in \mathfrak{N}^\pi$, we have that $N/(N \cap K) \in \mathfrak{N}^\pi$, since $\mathfrak{N}^\pi$ is closed under taking subgroups, and we can deduce that N acts $\mathfrak{N}^\pi$-hypercentrally on $N/(N \cap K)$. Then N acts $\mathfrak{N}^\pi$-hypercentrally on $V(N \cap K)/(N \cap K)$ which is N-isomorphic to $V/(V \cap N \cap K) = V/(K \cap$

$V) = V/W$. It follows that $V/W \leq Z_{\mathfrak{N}^\pi}(N/W)$, which proves the claim. Consequently, $V/W \leq P^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(N_G(V)/W)$ (see [5, IV. Theorem (6.14)]).

We aim to prove that $V/W \leq P^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(G/W)$, which will conclude the proof by fact 4.

We prove next that P^*/W is self-$\mathfrak{N}^\pi$-Dnormalizing in G/W; in particular we will have that P^*/W is an $\mathfrak{N}^\pi$-maximal subgroup of G/W by [1, Corollary 4.11]. Assume that P^*/W is $\mathfrak{N}^\pi$-Dnormal in $H/W \leq G/W$. By fact 3, $H \leq N_G(V)$. Since $P^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(N_G(V)/W)$, Lemma 2.8 implies that $P^*/W = H/W$, and P^*/W is self-$\mathfrak{N}^\pi$-Dnormalizing in G/W.

For any $H \leq G$, we denote $\overline{H} = HW/W$. We distinguish next the following cases:

Case 1: $\overline{G} = \overline{G}^{\mathfrak{N}^\pi}\, \overline{N_G(V)}$.

Case 2: $\overline{G}^{\mathfrak{N}^\pi}\, \overline{N_G(V)} < \overline{G}$.

Case 1: $\overline{G} = \overline{G}^{\mathfrak{N}^\pi}\, \overline{N_G(V)}$.

In this case, $\overline{G} = \overline{G}^{\mathfrak{N}^\pi}\, \overline{N_G(V)}^{\mathfrak{N}^\pi}\, \overline{P^*} = \overline{G}^{\mathfrak{N}^\pi}\, \overline{P^*}$, because $\mathfrak{N}^\pi$ is closed under taking subgroups and so $\overline{N_G(V)}^{\mathfrak{N}^\pi} \leq \overline{G}^{\mathfrak{N}^\pi}$. Then

$$\begin{aligned}\overline{G}/\left(\overline{G}^{\mathfrak{N}^\pi}\right)' &= \left(\overline{G}^{\mathfrak{N}^\pi}/\left(\overline{G}^{\mathfrak{N}^\pi}\right)'\right)\left(\overline{P^*}\left(\overline{G}^{\mathfrak{N}^\pi}\right)'/\left(\overline{G}^{\mathfrak{N}^\pi}\right)'\right)\\ &= \left(\overline{G}/\left(\overline{G}^{\mathfrak{N}^\pi}\right)'\right)^{\mathfrak{N}^\pi}\left(\overline{P^*}\left(\overline{G}^{\mathfrak{N}^\pi}\right)'/\left(\overline{G}^{\mathfrak{N}^\pi}\right)'\right).\end{aligned}$$

Let $\overline{Q}\,/\left(\overline{G}^{\mathfrak{N}^\pi}\right)'$ be an $\mathfrak{N}^\pi$-maximal subgroup of $\overline{G}/\left(\overline{G}^{\mathfrak{N}^\pi}\right)'$ such that

$$\overline{P^*}\left(\overline{G}^{\mathfrak{N}^\pi}\right)'/\left(\overline{G}^{\mathfrak{N}^\pi}\right)' \leq \overline{Q}\,/\left(\overline{G}^{\mathfrak{N}^\pi}\right)'.$$

Then $\overline{Q}\,/\left(\overline{G}^{\mathfrak{N}^\pi}\right)' \in \mathrm{Proj}_{\mathfrak{N}^\pi}\left(\overline{G}/\left(\overline{G}^{\mathfrak{N}^\pi}\right)'\right)$ by [5, III. Lemma (3.14)].

We consider the following two possibilities for $\overline{Q}$:

<u>Case 1.1:</u> $\overline{Q} < \overline{G}$.

Let $W \leq Q \leq G$ such that $\overline{Q} = Q/W$. Then $W \leq V \leq P^* \leq Q < G$. In particular, $P^*/W \leq N_Q(V)/W \leq N_G(V)/W$, which implies that $P^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(N_Q(V)/W)$ because $P^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(N_G(V)/W) = \mathrm{Cov}_{\mathfrak{N}^\pi}(N_G(V)/W)$ by Lemma 2.7. By inductive hypothesis (fact 1) and fact 4, we deduce that $\overline{P^*} = P^*/W \in \mathrm{Proj}_{\mathfrak{N}^\pi}(Q/W) = \mathrm{Proj}_{\mathfrak{N}^\pi}(\overline{Q})$. But $\overline{Q}\,/\left(\overline{G}^{\mathfrak{N}^\pi}\right)' \in \mathrm{Proj}_{\mathfrak{N}^\pi}\left(\overline{G}/\left(\overline{G}^{\mathfrak{N}^\pi}\right)'\right)$, which implies that $\overline{P^*} \in \mathrm{Proj}_{\mathfrak{N}^\pi}(\overline{G})$, by [5, III. Proposition (3.7)], and we are done.

Case 1.2: $\overline{Q} = \overline{G}$.

If $\overline{Q} = \overline{G}$, then $\overline{G}/\left(\overline{G}^{\mathfrak{N}^\pi}\right)' \in \mathfrak{N}^\pi$ and so $\overline{G}^{\mathfrak{N}^\pi} = \left(\overline{G}^{\mathfrak{N}^\pi}\right)'$.

Consequently,

$$\overline{P^*} \cap \overline{G}^{\mathfrak{N}^\pi} = \overline{P^*} \cap \left(\overline{G}^{\mathfrak{N}^\pi}\right)' \leq \left(\overline{G}^{\mathfrak{N}^\pi}\right)'.$$

Assume that $\overline{P^*} \leq \overline{X} < \overline{G}$. We may argue as above, in Case 1.1, to deduce by inductive hypothesis that $\overline{P^*} \in \mathrm{Proj}_{\mathfrak{N}^\pi}(\overline{X})$. Therefore, by Lemma 2.8, it follows that

$$\overline{P^*} \cap \overline{X}^{\mathfrak{N}^\pi} \leq \left(\overline{X}^{\mathfrak{N}^\pi}\right)'.$$

Since $\overline{P^*}$ is self-$\mathfrak{N}^\pi$-Dnormalizing in $\overline{G}$, again Lemma 2.8 implies that $\overline{P^*} \in \mathrm{Proj}_{\mathfrak{N}^\pi}(\overline{G})$, which concludes the proof.

It remains to consider Case 2.

Case 2: $\overline{G}^{\mathfrak{N}^\pi}\, \overline{N_G(V)} < \overline{G}$.

Since $\overline{V} \in \mathfrak{N}^\pi$, we can write $\overline{V} = \overline{V}_\pi \times \overline{V}_{\pi'}$ where $\overline{V}_\pi = O_\pi(\overline{V})$ and $\overline{V}_{\pi'} = O_{\pi'}(\overline{V}) \in \mathfrak{N}$. Moreover, if $\overline{G}_\pi$ is a Hall π-subgroup of $\overline{G}$, then $\overline{G}^{\mathfrak{N}^\pi}\overline{G}_\pi \trianglelefteq \overline{G}$, and we can form the subgroup $\overline{G}^{\mathfrak{N}^\pi}\overline{G}_\pi\, \overline{V}_{\pi'}$, which clearly contains $\overline{V}$. Assume that $\overline{G}^{\mathfrak{N}^\pi}\overline{G}_\pi\, \overline{V}_{\pi'} < \overline{G}$, and let $\overline{L}$ be a maximal subgroup of $\overline{G}$ containing $\overline{G}^{\mathfrak{N}^\pi}\overline{G}_\pi\, \overline{V}_{\pi'}$. Since $\overline{G}/\overline{G}^{\mathfrak{N}^\pi}\overline{G}_\pi \in \mathfrak{N}$, we have that $\overline{L} \trianglelefteq \overline{G}$. By inductive hypothesis, $\overline{V} \leq \overline{R^*}$ for some $\overline{R^*} \in \mathrm{Proj}_{\mathfrak{N}^\pi}(\overline{L})$. Hence, by the Frattini Argument (see [5, A.(5.13)]) and fact 3, $\overline{G} = \overline{L}\, N_{\overline{G}}(\overline{R^*}) = \overline{L}^{\mathfrak{N}^\pi}\, \overline{R^*}\, \overline{N_G(V)} = \overline{G}^{\mathfrak{N}^\pi}\, \overline{N_G(V)}$, which is not the considered case. Therefore, $\overline{G}^{\mathfrak{N}^\pi}\overline{G}_\pi\, \overline{V}_{\pi'} = \overline{G}$. In particular, $O^\pi(\overline{G}) = \overline{G}^{\mathfrak{N}^\pi}\, \overline{V}_{\pi'} \leq \overline{G}^{\mathfrak{N}^\pi}\, \overline{N_G(V)} < \overline{G}$, and $O^\pi(G)W = G^{\mathfrak{N}^\pi} V_{\pi'} W < G$, where $V_{\pi'} \leq V$ such that $\overline{V}_{\pi'} = V_{\pi'}W/W$.

Denote $L = O^\pi(G)W = G^{\mathfrak{N}^\pi} V_{\pi'} W < G$. We prove next that $V_{\pi'}W$ is an $\mathcal{F}$-maximal subgroup of L. Since $V_{\pi'}W \trianglelefteq V \in \mathcal{F}$, we have that $V_{\pi'}W \in \mathcal{F}$. Assume that $V_{\pi'}W \leq X \leq L$ with $X \in \mathcal{F}$. By fact 2, $W = X \cap K$; in particular, $X \cap G^{\mathfrak{N}^\pi} \leq W$. Hence $X = V_{\pi'}W(G^{\mathfrak{N}^\pi} \cap X) = V_{\pi'}W$. We deduce by inductive hypothesis (fact 1) and fact 3 that $\overline{V}_{\pi'} \leq \overline{R^*}$ for some $\overline{R^*} \in \mathrm{Proj}_{\mathfrak{N}^\pi}(\overline{L})$, and $N_{\overline{G}}(\overline{R^*}) \leq N_{\overline{G}}(\overline{V}_{\pi'})$. Consequently, $\overline{G} = \overline{L}\, N_{\overline{G}}(\overline{R^*}) = \overline{G}^{\mathfrak{N}^\pi} N_{\overline{G}}(\overline{V}_{\pi'})$.

If $N_{\overline{G}}(\overline{V}_{\pi'}) < \overline{G}$, since $\overline{V} \leq N_{\overline{G}}(\overline{V}_{\pi'})$, by inductive hypothesis (fact 1) and fact 3 we have that $\overline{V} \leq \overline{R^*} \in \mathrm{Proj}_{\mathfrak{N}^\pi}(N_{\overline{G}}(\overline{V}_{\pi'}))$ and $\overline{R^*} \leq \overline{N_G(V)}$. Then $\overline{G} = \overline{G}^{\mathfrak{N}^\pi} N_{\overline{G}}(\overline{V}_{\pi'}) = \overline{G}^{\mathfrak{N}^\pi}\, \overline{R^*} = \overline{G}^{\mathfrak{N}^\pi}\, \overline{N_G(V)}$, which is not the case.

Assume finally that $\overline{V}_{\pi'} \trianglelefteq \overline{G}$. Then $\overline{V}_\pi \cong \overline{V} / \overline{V}_{\pi'} \leq \overline{G}_\pi\, \overline{V}_{\pi'} / \overline{V}_{\pi'}$ for some Hall π-subgroup G_π of G. But then $\overline{V} / \overline{V}_{\pi'}$ is $\mathfrak{N}^\pi$-Dnormal in $\overline{G}_\pi\, \overline{V}_{\pi'} / \overline{V}_{\pi'}$, which implies by Lemma 1.4(4) that $\overline{V}$ is $\mathfrak{N}^\pi$-Dnormal in $\overline{G}_\pi\, \overline{V}_{\pi'}$, and so $V = (G_\pi V_{\pi'} W)_{\mathcal{F}}$ by fact 3. Consequently, $\overline{V} \trianglelefteq \overline{G}_\pi\, \overline{V}_{\pi'}$, and so $\overline{G} = \overline{G}^{\mathfrak{N}^\pi}\, \overline{N_G(V)}$, the final contradiction.

(b) Let $G^* = \langle V, V_1 \rangle$, $K^* = K \cap G^* \trianglelefteq G^*$. Hence, $G^*/K^* \cong G^*K/K \leq G/K \in \mathfrak{N}^\pi$, which implies that $G^*/K^* \in \mathfrak{N}^\pi$, since $\mathfrak{N}^\pi$ is closed under taking subgroups, and so $(G^*)^{\mathfrak{N}^\pi} \leq K^*$. Also, W is $\mathcal{F}$-maximal in K^*, because $W \leq K \cap G^* = K^*$ and W is $\mathcal{F}$-maximal in K. As in Part (a), fact 2, we can deduce that $K^* \cap V = K^* \cap V_1 = W$. Consequently, $W \trianglelefteq \langle V, V_1 \rangle = G^*$. By Part (a) there exist $\mathfrak{N}^\pi$-projectors P and Q of G^* such that $V = (WP)_\mathcal{F}$ and $V_1 = (WQ)_\mathcal{F}$. By Lemma 2.7 there exists $x \in G^*$ such that $Q^x = P$. Moreover, note that $G^* = (G^*)^{\mathfrak{N}^\pi} Q$, so that we can take $x \in (G^*)^{\mathfrak{N}^\pi}$. Consequently, by Lemma 2.2, it follows that

$$V_1^x = ((WQ)_\mathcal{F})^x = ((WQ)^x)_\mathcal{F} = (WQ^x)_\mathcal{F} = (WP)_\mathcal{F} = V,$$

with $x \in (G^*)^{\mathfrak{N}^\pi}$. □

Theorem 2.10. *If $\mathcal{F}$ is an $\mathfrak{N}^\pi$-Fitting set of a π'-soluble group G, then G possesses exactly one conjugacy class of $\mathcal{F}$-injectors. Moreover, if V and V^* are $\mathcal{F}$-injectors of G, there exists $g \in G^{\mathfrak{N}^\pi}$ such that $(V^*)^g = V$.*

In addition, if I is an $\mathcal{F}$-injector of G and N is an $\mathfrak{N}^\pi$-Dnormal subgroup of G, then $I \cap N$ is an $\mathcal{F}$-injector of N.

Proof. We argue by induction on $|G|$. We may assume that $|G| \neq 1$ and that the result holds for all proper subgroups of G. Since G is π'-soluble, $K = G^{\mathfrak{N}^\pi}$ is a normal proper subgroup of G. Let $W \in \mathrm{Inj}_\mathcal{F}(K)$ and V be an $\mathcal{F}$-maximal subgroup of G containing W. We aim to prove first that $V \cap N \in \mathrm{Inj}_\mathcal{F}(N)$ whenever N is an $\mathfrak{N}^\pi$-Dnormal subgroup of G; in particular, $V \in \mathrm{Inj}_\mathcal{F}(G)$ and V satisfies the property stated in the last part of the statement. Let M be a maximal $\mathfrak{N}^\pi$-Dnormal proper subgroup of G. It is enough to prove that $V \cap M \in \mathrm{Inj}_\mathcal{F}(M)$. Note that $K = G^{\mathfrak{N}^\pi} \leq M$, by Lemma 1.6. Let $V_0 \in \mathrm{Inj}_\mathcal{F}(M)$. Then $V_0 \cap K \in \mathrm{Inj}_\mathcal{F}(K)$ and by inductive hypothesis we deduce that $W = (V_0 \cap K)^g = V_0^g \cap K$ for some $g \in K$. We may replace V_0 by V_0^g, if necessary, and suppose that $V_0 \cap K = W$. Let V_1 be an $\mathcal{F}$-maximal subgroup of G such that $V_0 \leq V_1$. By Lemma 2.9(b) and taking into account that $\mathfrak{N}^\pi$ is closed under taking subgroups, there exists $x \in \langle V, V_1 \rangle^{\mathfrak{N}^\pi} \leq G^{\mathfrak{N}^\pi} = K \leq M$ such that $V_1^x = V$. Consequently, $V_0^x = V_0^x \cap M \leq V_1^x \cap M = V \cap M$. By Lemma 1.5(3), $V \cap M$ is $\mathfrak{N}^\pi$-Dnormal in $V \in \mathcal{F}$, so that $V \cap M \in \mathcal{F}$. But also $V_0^x \in \mathrm{Inj}_\mathcal{F}(M^x) = \mathrm{Inj}_\mathcal{F}(M)$, which implies that $V_0^x = V \cap M \in \mathrm{Inj}_\mathcal{F}(M)$ as claimed.

We prove finally the conjugacy of $\mathcal{F}$-injectors, i.e. assume that $V^* \in \mathrm{Inj}_\mathcal{F}(G)$ and prove that there exists $g \in G^{\mathfrak{N}^\pi} = K$ such that $(V^*)^g = V$. It holds that $V^* \cap K \in \mathrm{Inj}_\mathcal{F}(K)$. Then the inductive hypothesis implies that $(V^* \cap K)^k = W$ for some $k \in K^{\mathfrak{N}^\pi} \leq K$. We consider now V and $(V^*)^k$, which are $\mathcal{F}$-maximal subgroups of G containing W. By Lemma 2.9(b), there exists $t \in \langle V, (V^*)^k \rangle^{\mathfrak{N}^\pi} \leq G^{\mathfrak{N}^\pi}$ such that $(V^*)^{kt} = V$, with $kt \in G^{\mathfrak{N}^\pi}$, which concludes the proof. □

As in [5, IX.1] we state now corresponding concepts and results for general Fitting classes from the theory of Fitting sets.

In a natural way, we define a non-empty class $\mathfrak{F}$ to be an $\mathfrak{N}^\pi$-Fitting class if the following conditions are satisfied:

(i) If $G \in \mathfrak{F}$ and N is an $\mathfrak{N}^\pi$-Dnormal subgroup of G, then $N \in \mathfrak{F}$.
(ii) If M, N are $\mathfrak{N}^\pi$-Dnormal subgroups of $G = \langle M, N\rangle$ with $M, N \in \mathfrak{F}$, then $G \in \mathfrak{F}$.

Now we have that an $\mathfrak{N}^\pi$-Fitting class is a Fitting class, and also Fitting classes are exactly $\mathfrak{N}$-Fitting classes, for $\mathfrak{N}^\pi = \mathfrak{N}$ with $|\pi| \leq 1$. As we show in the remarks below, if $|\pi| \geq 2$, not every Fitting class is an $\mathfrak{N}^\pi$-Fitting class, and $\mathfrak{N}^\pi$ is the smallest $\mathfrak{N}^\pi$-Fitting class of full characteristic (cf. [5, IX. Theorem (1.9)]).

If $\mathfrak{F}$ is an $\mathfrak{N}^\pi$-Fitting class and G a group, then the trace of $\mathfrak{F}$ in G, that is the set $\mathrm{Tr}_{\mathfrak{F}}(G) = \{H \leq G \mid H \in \mathfrak{F}\}$, is a $\mathfrak{N}^\pi$-Fitting set of G, and $\mathfrak{F}$-injectors of G are exactly $\mathrm{Tr}_{\mathfrak{F}}(G)$-injectors.

From Theorem 2.10 we can derive now the following result for $\mathfrak{N}^\pi$-Fitting classes and π'-soluble groups.

Corollary 2.11. *Let $\mathfrak{F}$ be an $\mathfrak{N}^\pi$-Fitting class and G be a π'-soluble group, then G possesses exactly one conjugacy class of $\mathfrak{F}$-injectors. Moreover, if V and V^* are $\mathfrak{F}$-injectors of G, there exists $g \in G^{\mathfrak{N}^\pi}$ such that $(V^*)^g = V$.*

In addition, if I is an $\mathfrak{F}$-injector of G and N is an $\mathfrak{N}^\pi$-Dnormal subgroup of G, then $I \cap N$ is an $\mathfrak{F}$-injector of N.

Remarks. 1. The class $\mathfrak{N}^\pi$ is a particular case of the so-called lattice formations, which are classes of groups whose elements are direct product of Hall subgroups corresponding to pairwise disjoint sets of primes. With the same flavour as $\mathfrak{N}^\pi$-Fitting classes, though within the universe of finite soluble groups, $\mathfrak{L}$-Fitting classes, for general lattice formations $\mathfrak{L}$ of soluble groups, were already defined in [3].

2. If $\mathfrak{F}$ is an $\mathfrak{N}^\pi$-Fitting class of characteristic τ, then $\mathfrak{N}^\pi \cap \mathfrak{E}_\tau \subseteq \mathfrak{F}$. In particular, if $|\pi| \geq 2$, then $\mathfrak{N}$ or also $\mathfrak{N}^m$, for any integer $m > 1$, are never $\mathfrak{N}^\pi$-Fitting classes.

Proof. ([3, Proposition 3.6]) Assume that G is a group of minimal order in $\mathfrak{N}^\pi \cap \mathfrak{E}_\tau \setminus \mathfrak{F}$. By Lemma 1.4(6), a maximal subgroup of G is $\mathfrak{N}^\pi$-Dnormal in G. By the choice of G, there is a unique maximal subgroup of G, which implies that G is a cyclic p-group for some prime $p \in \tau$. Then $G \in \mathfrak{F}$ by [5, IX. Lemma (1.8)]), a contradiction. □

3. $\mathfrak{N}^\pi$ is an $\mathfrak{N}^\pi$-Fitting class.

Proof. Since $\mathfrak{N}^\pi$ is closed under taking subgroups, condition (i) of the definition of $\mathfrak{N}^\pi$-Fitting class is satisfied.

Assume now that M, N are $\mathfrak{N}^\pi$-Dnormal subgroups of $G = \langle M, N\rangle$ with $M, N \in \mathfrak{N}^\pi$, and we aim to prove that $G \in \mathfrak{N}^\pi$. For any $X \in \{M, N\}$, let $X = X_\pi \times X_{\pi'}$ with $X_\pi = O_\pi(X)$, $X_{\pi'} = O_{\pi'}(X) = O^\pi(X) \in \mathfrak{N}$. Since X is $\mathfrak{N}^\pi$-Dnormal in G, by Proposition 1.2 and Lemma 1.5(1), we have that $X_{\pi'} \trianglelefteq G$, $M_{\pi'}N_{\pi'} \leq O^\pi(G) \leq N_G(X) \leq N_G(X_\pi)$ and $\langle X^G\rangle/X_{\pi'}$ is a π-group. Hence $G = M_{\pi'}N_{\pi'}\langle M_\pi, N_\pi\rangle$ with $M_{\pi'}N_{\pi'}$ a normal nilpotent π'-subgroup of G and $[M_{\pi'}N_{\pi'}, \langle M_\pi, N_\pi\rangle] = 1$. Moreover $\langle M^G\rangle N_{\pi'}/M_{\pi'}N_{\pi'}$ and $\langle N^G\rangle M_{\pi'}/M_{\pi'}N_{\pi'}$ are π-groups, which implies that $G/M_{\pi'}N_{\pi'}$ is a π-group, and so $G = M_{\pi'}N_{\pi'}G_\pi$ with G_π a Hall π-subgroup of G, by the Schur-Zassenhaus theorem. Moreover, there exist $x, y \in M_{\pi'}N_{\pi'}$ such that $M_\pi \leq G_\pi^x$ and $N_\pi \leq G_\pi^y$, which implies that $M_\pi = M_\pi^{x^{-1}} \leq G_\pi$ and also $N_\pi = N_\pi^{y^{-1}} \leq G_\pi$. Therefore, $\langle M_\pi, N_\pi\rangle \leq G_\pi$, and then $\langle M_\pi, N_\pi\rangle = G_\pi$. It follows that $G = M_{\pi'}N_{\pi'} \times G_\pi \in \mathfrak{N}^\pi$, which concludes the proof. $\square$

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