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New Oscillation Criteria of Nonlinear Second Order Delay Difference Equations

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Abstract. In the paper, we study some new criteria for the oscillation of the nonlinear second-order delay difference equations of the form $\Delta(r(t)(\Delta x(t))^{\alpha}) + q(t)\, x^{\beta}(t-m+1) = 0$, via comparison with a second-order linear difference equation or a first-order linear delay difference equation whose oscillatory behavior is discussed intensively in the literature. The presented results essentially improve existing ones.

Mathematics Subject Classification. 34K11, 34C10.

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1. Introduction

In this paper, we shall study the oscillatory behavior of the solutions of nonlinear second-order delay difference equations of the form

$$\Delta(r(t)(\Delta x(t))^{\alpha}) + q(t)\, x^{\beta}(t-m+1) = 0. \tag{1.1}$$

We shall assume that

(i) $\{q(t)\}$ and $\{r(t)\}$ are positive real sequences,
(ii) α and β are ratios of positive odd integers,
(iii) $m \geq 1$ is a positive integer.

Moreover, it is assumed that

$$R(t, t_0) = \sum_{s=t_0}^{t-1} r^{-\frac{1}{\alpha}}(s) \to \infty \text{ as } t \to \infty. \tag{1.2}$$

Recall that a solution of (1.1) is a nontrivial real-valued sequence $\{x(t)\}$ satisfying (1.1) for $t \geq t_0 - m + 1$.

Solutions vanishing identically in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory.

The problem of investigating oscillation criteria for various types of difference equations has been a very active research area over the past several decades. A large number of papers and monographs have been devoted to this problem; for a few examples, see [1–11,13–15] and the references contained therein.

The main goal of this paper was to provide some new oscillation criteria for Eq. (1.1) via comparison with a second-order linear difference equation or a first-order linear delay difference equation whose oscillatory behavior is discussed intensively in the literature. We will demonstrate the usefulness of our main results via some applications to neutral difference equations and some examples.

2. Comparison Theorems

To obtain our result, we need the following two lemmas:

Lemma 2.1. *Let* $\{q(t)\}$ *be a sequence of positive real numbers;* m *is a positive real number and* f*:* $R\to R$ *is a continuous nondecreasing function, and* $x\,f(x)>0$ *for* $x\neq 0$*. If the first-order delay differential inequality*

$$\Delta y(t)+q(t)f(y(t-m+1))\leq 0$$

has an eventually positive solution, so does the delay equation

$$\Delta y(t)+q(t)f(y(t-m+1))=0.$$

This Lemma is an extension of the discrete analogue of known results. See Lemma 6.2.2 in [2] and also in [11]. The proof is immediate.

Lemma 2.2. *Let* $\{x(t)\}$ *be an eventually increasing solution of Eq. (1.1). Then* $x^{\beta-\alpha}(t)\geq\varphi(t)$*, where* $\varphi\,(t)$ *is given by*

$$\varphi\,(t)\;=\begin{cases}1 \quad if\,\alpha=\beta\\ a \quad if\,\alpha<\beta\\ \mathrm{b}R^{\beta-\alpha}\,(t,\,t_1)\;if\,\alpha>\beta,\end{cases}\tag{2.1}$$

where a *and* b *are positive constants and all large* $t\geq t_1\geq t_0$*.*

Proof. Since $\{x(t)\}$ is a positive increasing solution of Eq. (1.1), there exists a constant $c>0$ such that x(t) $\geq c\,for\,all\,t\geq t_1\,for\,some\,t_1\geq t_0$. Now, one can easily find that

$$x(t)\geq\sum_{s=t_1}^{t-1}r^{-1/\alpha}(s)\left(r^{1/\alpha}(s)\Delta x(s)\right)\geq R(t,t_1)\left(r^{1/\alpha}(t)\Delta x(t)\right).\tag{2.2}$$

Since $r(t)\,(\Delta x(t))^{\alpha}$ is positive and non-increasing on $[t_1,\infty)$, there exists a constant $C>0$ such that

$$r(t)\,(\Delta x(t))^{\alpha}<C\quad\text{for}\quad t\geq t_1.$$

Summing this inequality from t_1 to t-1, we have

$$x(t) \leq CR(t, t_1) \text{ for t } \geq t_1$$

and for some constant $C > 0$ and so,

$$x^{\beta-\alpha}(t) \geq \varphi(t) = \begin{cases} 1 & if\, \alpha = \beta \\ a & if\, \alpha < \beta \\ \mathrm{b}R^{\beta-\alpha}(t, \mathrm{t}_1) & if\, \alpha > \beta, \end{cases}$$

where $a = \quad c^{\beta-\alpha}\, and\, b = C^{\beta-\alpha}$. This proves the Lemma.

For $t \geq t_1 \geq t_0$, we let

$$Q_1(t, t_1) = \frac{1}{\alpha} q(t) \left(\frac{R^{\alpha}(t - m + 1, \mathrm{t}_1)}{R(t + 1, \mathrm{t}_1)} \right) \varphi(t - m + 1)$$

and

$$Q_2(t, t_1) = \frac{1}{\alpha} \frac{R^{\alpha}(t - m + 1, \mathrm{t}_1)}{r(t - m + 1)} q(t) \varphi(t - m + 1).$$

Now, we present our first oscillation result for Eq. (1.1) via comparison with second-order linear difference equation.

Theorem 2.1. *Let* $\alpha \geq 1$*,the conditions (i)–(iii) and (1.2) hold. If the second-order linear difference equation*

$$\Delta\left(r^{\frac{1}{\alpha}}(t) \Delta x(t)\right) + Q_1(t, t_1) x(t + 1) = 0 \tag{2.3}$$

is oscillatory for all large $t \geq t_{1,}$ *then Eq. (1.1) is oscillatory.*

Proof. Let $\{x(t)\}$ be a nonoscillatory solution of Eq. (1.1), say $x(t) > 0$, and $x(t - m + 1)) > 0$ for $t \geq t_1 \,\mathrm{for\, some}\, \mathrm{t}_1 \geq t_0$. The proof if x(t) is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. Then, it follows from Eq. (1.1) that

$$\Delta(r(t)(\Delta x(t))^{\alpha}) = -q(t)\, x^{\beta-\alpha}(t - m + 1)\, x^{\alpha}(t - m + 1) < 0, \tag{2.4}$$

It is easy to see that there exists a $\mathrm{t}_2 \geq t_1$ such that

$$x(t) > 0, \Delta x(t) > 0 \quad \text{and} \quad \Delta(r(t)(\Delta x(t))^{\alpha}) < 0, \text{ for } t \geq \mathrm{t}_2.$$

By using (2.1) in (2.4) we have

$$\Delta(r(t)(\Delta x(t))^{\alpha}) \leq -q(t)\, \varphi(t - m + 1) x^{\alpha}(t - m + 1)$$

Inequality (2.4) can be written in the following form:

$$\Delta\left(r^{\frac{1}{\alpha}}(t) \Delta x(t)\right)^{\alpha} + q(t)\, \varphi(t - m + 1)\, x^{\alpha}(t - m + 1) \leq 0.$$

$\Delta-$derivative yields

$$\begin{aligned} \Delta(r(t)(\Delta x(t))^{\alpha}) &= \Delta\left(r^{\frac{1}{\alpha}}(t)\, \Delta x(t)\right)^{\alpha} \\ &\geq \alpha \left(r^{\frac{1}{\alpha}}(t)\, \Delta x(t)\right)^{\alpha-1} \Delta\left(r^{\frac{1}{\alpha}}(t)\, \Delta x(t)\right), \end{aligned}$$

or,

$$\Delta\left(r^{\frac{1}{\alpha}}(t)\Delta x\left(t\right)\right)+\frac{1}{\alpha}\left(r^{\frac{1}{\alpha}}(t)\Delta x\left(t\right)\right)^{1-\alpha}q\left(t\right)\,\varphi\left(t-m+1\right)x^{\alpha}\left(t-m+1\right)\leq 0. \tag{2.5}$$

Using (2.2), there exists $t_3 \geq t_2$ such that

$$x(t-m+1)\geq R\left(t-m+1,t_1\right)\,r^{\frac{1}{\alpha}}\left(t-m+1\right)\Delta x(t-m+1) \quad \text{for } t\geq t_3, \tag{2.6}$$

using the fact that $r^{\frac{1}{\alpha}}\left(t\right)\Delta x\left(t\right)$is a nonincreasing sequence, we see that

$$r^{\frac{1}{\alpha}}\left(t\right)\Delta x\left(t\right)\leq r^{\frac{1}{\alpha}}\left(t-m+1\right)\Delta x(t-m+1) \quad \text{for } t\geq t_3, \tag{2.7}$$

and using (2.6) in (2.7) we get

$$\begin{aligned}r^{\frac{1}{\alpha}}\left(t\right)\Delta x\left(t\right)&\leq r^{\frac{1}{\alpha}}\left(t-m+1\right)\Delta x\,\left(t-m+1\right)\\&\leq R^{-1}\left(t-m+1,t_1\right)x(t-m+1).\end{aligned}$$

Substituting this inequality in (2.5) and using the fact that $\alpha \geq 1$, we see that

$$\begin{aligned}&\Delta\left(r^{\frac{1}{\alpha}}(t)\Delta x\left(t\right)\right)+\frac{1}{\alpha}\left(R^{-1}\left(t-m+1,t_1\right)x(t-m+1)\right)^{1-\alpha}\\&\quad\times q\left(t\right)\,\varphi\left(t-m+1\right)x^{\alpha}\left(t-m+1\right)\leq 0,\end{aligned}$$

or,

$$\begin{aligned}&\Delta\left(r^{\frac{1}{\alpha}}(t)\Delta x\left(t\right)\right)+\frac{1}{\alpha}R^{\alpha-1}\left(t-m+1,t_1\right)\\&\quad\times q\left(t\right)\,\varphi\left(t-m+1\right)x\left(t-m+1\right)\leq 0.\end{aligned}$$

From (2.2) It follows that

$$\Delta\left(\frac{\mathrm{x(t)}}{R(t,\mathrm{t}_1)}\right)=\frac{R(t,\mathrm{t}_1)\Delta x(\mathrm{t})-\frac{\mathrm{x(t)}}{\mathrm{r}^{1/\alpha}(\mathrm{t})}}{R(t+1,\mathrm{t}_1)R(t,\mathrm{t}_1)}\leq 0,$$

i.e. $\frac{x(t)}{R(t,t_1)}$ is eventually nonincreasing for $t \geq t_2$. Thus, we have

$$\frac{x(t-m+1))}{R(t-m+1,\mathrm{t}_1)} \geq \frac{x(t+1))}{R(t+1,\mathrm{t}_1)},$$

and so,

$$\begin{aligned}&\Delta\left(r^{\frac{1}{\alpha}}(t)\Delta x\left(t\right)\right)+\frac{1}{\alpha}R^{\alpha-1}\left(t-m+1,t_1\right)\\&\quad\times q\left(t\right)\,\left(\frac{R\left(t-m+1,\mathrm{t}_1\right)}{R\left(t+1,\mathrm{t}_1\right)}\right)\varphi\left(t-m+1\right)x\left(t+1\right)\leq 0,\end{aligned}$$

or,

$$\Delta\left(r^{\frac{1}{\alpha}}(t)\Delta x\left(t\right)\right)+\,Q_1\left(t,t_1\right)x\left(t+1\right)\leq 0.$$

But by Lemma 1 of [14], the corresponding Eq. (2.3) has a positive solution. We derive a contradiction which completes the proof.

By Applying Theorem 3.5 in [8] to Eq. (2.3), we have the following oscillation result;

Corollary 2.1. *Let* $\alpha \geq 1$, *the, conditions (i)–(iii) and (1.2) hold. If there exists a nondecreasing positive sequence* $\{\pi(t)\}$ *such that for any* $t \geq t_0$

$$\lim_{t\to\infty} sup \sum_{t_0}^{t-1}\left[\pi(s)\,Q_1(t,t_1) - \frac{r^{\frac{1}{\alpha}}(s)}{4}\left(\frac{\Delta\pi(s)}{\pi(s)}\right)^2\right] = \infty, \tag{2.8}$$

then Eq. (1.1) is oscillatory.

Next, we present our second oscillation result for Eq. (1.1) via comparison with first-order delay difference equation.

Theorem 2.2. *Let* $0 < \alpha \leq 1$, *the conditions (i)–(iii) and (1.2) hold. If the first-order linear delay difference equation*

$$\Delta w(t) + Q_2(t,t_1)\,w(t-m+1) = 0, \tag{2.9}$$

is oscillatory for all large $t \geq t_1$ *, then Eq. (1.1) is oscillatory.*

Proof. Let x(t) be a nonoscillatory solution of Eq. (1.1), say $x(t) > 0$ and x $(\tau(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. The proof if x(t) is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. Proceeding as in the proof of Theorem 2.1, we obtain the inequalities (2.5)–(2.7). Using (2.6) into (2.5), we have

$$\begin{aligned} &\Delta\left(r^{\frac{1}{\alpha}}(t)\Delta x(t)\right) + \quad \tfrac{1}{\alpha}\left(r^{\frac{1}{\alpha}}(t)\Delta x(t)\right)^{1-\alpha} q(t)\varphi(t-m+1)\\ &\times\left(R(t-m+1,t_1)\; r^{\frac{1}{\alpha}}(t-m+1)\Delta x(t-m+1)\right)^{\alpha} \leq 0, \quad \text{for } t \geq t_3, \end{aligned}$$

or,

$$\begin{aligned} &\Delta\left(r^{\frac{1}{\alpha}}(t)\Delta x(t)\right) + \frac{1}{\alpha}\left(r^{\frac{1}{\alpha}}(t-m+1)\Delta x(t-m+1)\right)^{1-\alpha}\\ &\times \frac{\left(R(t-m+1,t_1)\right)^{\alpha}}{r(t-m+1)} q(t)\varphi(t-m+1)\\ &\times\left(r^{\frac{1}{\alpha}}(t-m+1)\Delta x(t-m+1)\right)^{\alpha} \leq 0, \end{aligned}$$

or,

$$\Delta w(t) + Q_2(t,t_1)\; w(t-m+1) \leq 0, \tag{2.10}$$

where, $w(t) = r^{\frac{1}{\alpha}}(t)\Delta x(t) > 0$. Summing inequality (2.10) from $t \geq t_3$to u and letting u $\rightarrow \quad \infty$, we obtain

$$w(t) \geq \sum_{s=t}^{\infty} Q_2(t,t_1)\; w(s-m+1) \quad \text{for } t \geq t_3.$$

The function w(t) is strictly decreasing on $[t_3, \infty)$. It follows from Lemma 2.1 that the corresponding difference Eq. (2.9) also has a positive solution. We arrive at a contradiction which completes the proof.

By summing Eq. (2.9) from t - m+1 to t -1, we have the following result:

Corollary 2.2. *Let* $0 < \alpha \leq 1$*,the conditions (i)–(iii), and (1.2) hold. If for all large* $t \geq t_1$

$$\lim_{t\to\infty} \inf \sum_{s=t-m+1}^{t-1} Q_2(t, t_1) = \infty, \tag{2.11}$$

then Eq. (1.1) is oscillatory.

Example 2.1. Consider the second order difference equation

$$\Delta\left(\Delta x(t)^3\right) + q(t)\, x^\beta (t-m+1) = 0, \tag{2.12}$$

Here, $\alpha = 3$, and $\beta > 0$ and $q(t)$ is a positive of real sequence, r(t) =*1*, $R(t, t_0) = t - t_0$.

If the second-order linear difference equation

$$\Delta^2 y(t) + \frac{C}{\alpha}\left(\frac{t-m+1-t_1}{t+1-t_1}\right)(t-m+1-t_0)^2\, q(t)\, \varphi(t-m+1)\, y(t+1) = 0$$

is oscillatory for all large $t \geq t_1$ *and any constant* $C \in (0, 1]$ .All condition of Theorem 2.1 are satisfied and hence we see that Eq. (2.12) is oscillatory.

3. Applications

In this section we apply our previous results to neutral second-order difference equations of the form

$$\Delta(r(t)(\Delta y(t))^\alpha) + q(t)\, x^\beta (t-m+1) = 0, \tag{3.1}$$

where $(I) y(t) = x(t) + p_1(t)\, x^\gamma (t-k_1) + p_2(t)\, x^\delta (t-k_2)$ or, (II) $y(t) = x(t) + p(t)\, x^\gamma (t-k)$,

γ and δ are ratios of positive odd integers with $0 < \gamma \leq 1$ and $\delta \geq 1$, k, k_1 and k_2 are positive integers and $\{p(t)\}$, $\{p_1(t)\}$ and $\{p_2(t)\}$ are positive sequences of real numbers.

To obtain our results we need the following lemma:

Lemma 3.1. *[12]. If X and Y are nonnegative, then*

$$X^\lambda - (1-\lambda)\, Y^\lambda - \lambda X Y^{\lambda-1} \leq \lambda < 1, \tag{3.2}$$

where equality holds if and only if $X = Y$.
Now, we present our oscillation result for Eq. (3.1) with (I), i.e., second-order equation with sublinear and superlinear neutral terms.

Theorem 3.1. *Let the conditions (i)–(iii), and (1.2) hold and let*

$$\lim_{t\to\infty} R^{\delta-1}(t, t_0)\, p_2(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} p_1(t) = 0. \tag{3.3}$$

Equation (3.1) is oscillatory if one of the following conditions holds for all large:

$$t \geq t_1 \text{ and any constant } C \in (0, 1]:$$

(I) The second-order linear difference equation

$$\Delta\left(r^{\frac{1}{\alpha}}(t)\Delta x\,(t)\right)+C\,Q_1\,(t,t_1)\,x\,(t+1)=0, \tag{3.4}$$

with $\alpha\geq 1$, *is oscillatory.*

(II) Let $\alpha\geq 1$ *and assume that there exists a nondecreasing positive sequence* $\{\pi(t)\}$ *such that for any* $t\geq t_1\geq t_0$

$$\lim_{t\to\infty}\sup\sum_{t_0}^{t-1}\left[C\pi\,(s)\;Q_1\,(t,t_1)\;-\frac{r^{\frac{1}{\alpha}}\,(s)}{4}\left(\frac{\Delta\pi\,(s)}{\pi\,(s)}\right)^2\right]=\infty, \tag{3.5}$$

(III) The first-order linear delay difference equation

$$\Delta\,w(t)+CQ_2\,(t,t_1)\;w\,(t-m+1)=0 \tag{3.6}$$

with $\alpha>0$ *is oscillatory.*

(IV) Let $0<\alpha\leq 1$ *and condition (2.11).*

Proof. Let $\{x(t)\}$ be a nonoscillatory solution of Eq. (1.1), say $x(t)>0, x(\mathrm{t}-k_1)>0, x(\mathrm{t}-k_2)>0$, and $x(t-m+1)>0$ and $y(t)>0$ for $t\geq t_1$ for some $\mathrm{t}_1\geq t_0$. The proof if x(t) is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. Then, it follows from (3.1) that

$$\Delta\left(r(t)\left(\Delta y(t)\right)^{\alpha}\right)\leq -\,q\,(t)\,x^{\beta}\,(t-m+1)<0, \tag{3.7}$$

Since $\mathrm{x(t)}\leq \mathrm{y(t)}$, it follows from the definition of y(t) that

$$\begin{aligned} x\,(t)&=y\,(t)-\;p_1\,(t)\,x^{\gamma}\,(\mathrm{t}-k_1)-p_2\,(t)\,x^{\delta}\,(\mathrm{t}-k_2)\\ &\geq\;y\,(t)-p_1\,(t)\,y^{\gamma}\,(\mathrm{t}-k_1)-p_2\,(t)\,y^{\delta}\,(\mathrm{t}-k_2)\\ &\geq y\,(t)-p_1\,(t)\,y^{\gamma}\,(t)-p_2\,(t)\,y^{\delta}\,(t)\\ &\geq y\,(t)\;-p_2\,(t)\,\frac{y\,(t)}{y^{1-\delta}\,(t)}-p_1\,(t)\left[y^{\gamma}\,(t)-y\,(t)\right]-p_1(t)y(t) \end{aligned} \tag{3.8}$$

By applying (3.2) with

$$\lambda=\gamma\,,X=y\quad\text{and}\quad Y=(\frac{1}{\gamma})^{\frac{1}{\gamma-1}},$$

we obtain

$$y^{\gamma}\,(t)-y\,(t)\leq(1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}}\quad\text{for } t\geq t_1. \tag{3.9}$$

Substituting (3.9) into (3.8) we find

$$\begin{aligned} x(t)\;&\geq y(t)-p_2(t)\frac{y(t)}{y^{1-\delta}(t)}-p_1(t)y(t)\,-p_1(t)\,(1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}}\\ &\geq y(t)-p_2(t)\frac{y(t)}{y^{1-\delta}(t)}-p_1(t)y(t))\\ &-\mathrm{p}_1(t)\,(1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}}\\ &\geq y(t)\left[1-p_2\,(t)\,\frac{1}{y^{1-\delta}\,(t)}-p_1\,(t)-\frac{p_1(t)\,(1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}}}{y(t)}\right] \end{aligned} \tag{3.10}$$

Since $y(t) > 0$ and $\Delta y(t) > 0$ on $[\mathrm{t}_2, \infty)$, there exists a constant $c_1 > 0$ such that

$$y(t) \geq c_1 \text{ for t} \geq \mathrm{t}_2. \tag{3.11}$$

Since $r(t)\left(\Delta y(t)\right)^{\alpha}$ is positive and non-increasing on $[t_1, \infty)$, there exist a constant $C > 0$ and a $\mathrm{t}_3 \geq t_2$such that

$$r(t)\left(\Delta y(t)\right)^{\alpha} < C \quad \text{for } t \geq t_3. \tag{3.12}$$

Summing the inequality (3.12) from t_3 to t-1 we have

$$y\left(t\right) \leq CR\left(t, t_3\right) \text{ for t } \geq t_4, \tag{3.13}$$

for some $t_4 \geq t_3$ and for some constant $C > 0$.

Using (3.11) and (3.13) in (3.10) gives

$$x(t) \geq y(t)\left[1 - p_2\left(t\right)\left(CR\left(t, t_3\right)\right)^{\delta-1} - p_1\left(t\right)\left(1 + \frac{\left(1-\gamma\right)\gamma^{\frac{\gamma}{1-\gamma}}}{c_1}\right)\right]$$

$$x(t) \geq y(t)\left[1 - \mathrm{B}\left[p_2\left(t\right)\left(R\left(t, t_3\right)\right)^{\delta-1} + p_1\left(t\right)\right]\right] \text{ for t } \geq t_4 \tag{3.14}$$

where $\mathrm{B} = \max\left\{1 + \frac{(1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}}{c_1}, C^{\delta-1}\right\}$.

Now, in view of (3.3), for any $\rho \in (0, 1)$ there exists $t_\rho \geq t_4$ such that

$$x(t) \geq \rho y(t) \quad \text{for } t \geq t_\rho. \tag{3.15}$$

Fix $\rho \in (0, 1)$ and choose $t_\rho$by (3.15). Since $\lim_{t\to\infty} \tau(t) = \infty$, we can choose $t_5 \geq t_\rho$ such that τ (t) $\geq t_\rho$ for all t $\geq t_5$. Thus, from (3.15) we have

$$x(\tau(t)) \geq \rho y(\tau(t)) \quad \text{for } t \geq t_5. \tag{3.16}$$

Using this inequality in Eq. (3.1) we find

$$\Delta\left(r(t)\left(\Delta y(t)\right)^{\alpha}\right) + q(t)\rho^{\beta} y^{\beta}(t - m + 1) \leq 0. \tag{3.17}$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted.

Example 3.1. Consider the second-order neutral difference equation

$$\Delta\left(\left(\Delta\left[\mathrm{x}\left(\mathrm{t}\right) + \frac{1}{t}\mathrm{x}^{\frac{1}{3}}\left(\mathrm{t}-k_1\right) + \frac{1}{t^3}\mathrm{x}^3\left(\mathrm{t}-k_2\right)\right]\right)^3\right) + q\left(t\right)x^{\beta}\left(t - m + 1\right) = 0, \tag{3.18}$$

and the second-order difference equation with a sublinear neutral term of the form

$$\Delta\left(\left(\Delta\left[\mathrm{x}\left(\mathrm{t}\right) + \frac{1}{t}\mathrm{x}^{\frac{1}{3}}\left(\mathrm{t}-k_1\right)\right]\right)^3\right) + q\left(t\right)x^{\beta}\left(t - m + 1\right) = 0. \tag{3.19}$$

Here, $\alpha = 3, \gamma = \frac{1}{3}, \delta = 3$ and $\beta > 0$, k, k_1 and k_2 are positive integers and p(t)=1/t= p_1 (t) ,

$\mathrm{p}_2\left(\mathrm{t}\right) = \frac{1}{t^3}$, $q\left(t\right)$ are positive sequences of real numbers. $r(t) = 1, R(t, t_0) = t - 1 - t_0$.

If the second-order linear difference equation

$$\Delta^2 y(t) + \frac{C}{\alpha}\left(\frac{t - m + 1 - t_1}{t + 1 - \mathrm{t}_1}\right)\left(t - m + 1 - t_0\right)^2 q\left(t\right)\,\varphi\left(t - m + 1\right) y\left(t+1\right) = 0,$$

is oscillatory for all large $t \geq t_1$ *and any constant* $C \in (0\,, 1]$ .Applying Theorems 2.1 and 3.2 we see that both Eqs. (3.18) and (3.19) are oscillatory.

Next, we present the following oscillation result for Eq. (3.1) with (II), i.e., second-order equation with a sublinear neutral term.

Theorem 3.2. *Let the conditions (i)–(iii), and (1.2) hold and let* $\lim_{t\to\infty} p_1 (t) = 0$.

Then the conclusions of Theorem 3.1 hold.

Proof. Let $\{x(t)\}$ be a nonoscillatory solution of Eq. (1.1), say $x(t) > 0$, $x(\mathrm{t}-\mathrm{k})) > 0$ and $x(t-m+1) > 0$ and $y(t) > 0$ for $t \geq t_1$ for some $\mathrm{t}_1 \geq t_0$. The proof if x(t) is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. Then, it follows from (3.1) that

$$\Delta\left(r(t)\left(\Delta y(t)\right)^{\alpha}\right) \leq -\,q\,(t)\,x^{\beta}\,(t-m+1) < 0, \tag{3.20}$$

Since x(t) $\leq$ y(t), it follows from the definition of y(t) with y(t) is a nondecreasing sequence that

$$\begin{gathered} x\,(t) = y\,(t) -\; p\,(t)\,x^{\gamma}\,(\mathrm{t}-\mathrm{k}) \\ x\,(t) \geq y\,(t) -\; p\,(t)\,y^{\gamma}(t) \\ x\,(t) \geq y\,(t)\left(1-\frac{p\,(t)}{y^{1-\gamma}\,(t)}\right). \end{gathered}$$

Using (3.11), there exists a constant b $\in (0\,, 1]$ such that

$$x(t) \geq b\,y(t) \quad \text{for t} \geq t_1.$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted.

Example 3.1. Consider the second-order neutral difference equation

$$\Delta\left(\left(\Delta\left[\mathrm{x}\,(\mathrm{t}) + \frac{1}{t}\mathrm{x}^{\frac{1}{3}}\,(\mathrm{t}-k_1) + \frac{1}{t^3}\mathrm{x}^3\,(\mathrm{t}-k_2)\right]\right)^{1/3}\right) +\, q\,(t)\,x^{\beta}\,(t-m+1) = 0, \tag{3.21}$$

and the second-order difference equation with a sublinear neutral term of the form

$$\Delta\left(\left(\Delta\left[\mathrm{x}\,(\mathrm{t}) + \frac{1}{t}\mathrm{x}^{\frac{1}{3}}\,(\mathrm{t}-k_1)\right]\right)^{1/3}\right) +\, q\,(t)\,x^{\beta}\,(t-m+1) = 0. \tag{3.22}$$

Here, $\alpha = 1/3, \gamma = \frac{1}{3}$, $\delta = 3$ and $\beta > 0$, k, k_1 and k_2 are positive integers and $\mathrm{p(t)}=1/\mathrm{t}=\mathrm{p}_1\,(\mathrm{t})$,

$\mathrm{p}_2\,(\mathrm{t}) = \frac{1}{t^3}$, $q\,(t)$ are positive of real numbers. r(t) =1, $\mathrm{R}(\mathrm{t}, t_0) = \mathrm{t}\text{-}1\text{-}t_0$.

If the first-order linear delay difference equation

$$\Delta w(t) + \frac{C}{\alpha}\,(t-m+1-t_1)^{\frac{1}{3}}q\,(t)\;\varphi\,(t-m+1)\,w\,(t-m\,+1) = 0,$$

is oscillatory for all large $t \geq t_2$ *and any constant* $C \in (0\,, 1]$, then Eqs. (3.21) and (3.22) are oscillatory.

General Remarks

The results of this paper are presented in a new form and of high degree of generality.

Our main task here is to reduce the oscillation of half-linear delay difference equations and/or nonlinear delay difference equations to the oscillation of linear or first-order difference equations whose oscillatory behavior is known and literature is filled with all types of criteria.

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