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General Reiteration Theorems for $\mathcal{R}$ and $\mathcal{L}$ Classes: Case of Right $\mathcal{R}$-Spaces and Left $\mathcal{L}$-Spaces

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Abstract. Given E_0, E_1, E, F rearrangement invariant spaces, $a, \mathrm{b}, \mathrm{b}_0, \mathrm{b}_1$ slowly varying functions and $0 \leq \theta_0 < \theta_1 \leq 1$, we characterize the interpolation spaces

$$(\overline{X}_{\theta_0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{\theta,\mathrm{b},E} \quad \text{and} \quad (\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1})_{\theta,\mathrm{b},E},$$

for all possible values of $\theta \in [0,1]$. Applications to interpolation identities for grand and small Lebesgue spaces, Gamma spaces and A and B-type spaces are given.

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1. Introduction

Reiteration theorems are important results in Interpolation Theory. These results not only assure the interpolation process is stable under reiteration, but also they are very useful identifying interpolation spaces. The classical results can be found in the monographs [8,9,11,51]. Additionally, there is an extensive literature concerning explicit reiteration formulae in various special cases, see, e.g., [3,5,6,15,17,21,33,39,41].

This paper is the second of a series in which we study reiteration results for couples formed by the spaces

$$\overline{X}_{\theta,\mathrm{b},E}, \ \overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}, \ \overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}. \tag{1.1}$$

Here, $0 \leq \theta \leq 1$, a and b are slowly varying functions and, finally, E and F are rearrangement invariant (r.i.) function spaces. In fact, given a couple $\overline{X} = (X_0, X_1)$, these spaces are defined as

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$$\overline{X}_{\theta,\mathrm{b},E} = \Big\{ f \in X_0 + X_1 \ :\ \big\| t^{-\theta} \mathrm{b}(t) K(t,f) \big\|_{\widetilde{E}} < \infty \Big\},$$
$$\overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F} = \Big\{ f \in X_0 + X_1 \ :\ \big\| \mathrm{b}(t) \| s^{-\theta} a(s) K(s,f) \|_{\widetilde{F}(0,t)} \big\|_{\widetilde{E}} < \infty \Big\}$$

and

$$\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F} = \Big\{ f \in X_0 + X_1 \ :\ \big\| \mathrm{b}(t) \| s^{-\theta} a(s) K(s,f) \|_{\widetilde{F}(t,\infty)} \big\|_{\widetilde{E}} < \infty \Big\}.$$

In the previous paper [26], we identified the interpolation spaces

$$\Big(\overline{X}^{\mathcal{R}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1} \Big)_{\theta,\mathrm{b},E} \quad \text{and} \quad \Big(\overline{X}_{\theta_0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{L}}_{\theta_1,\mathrm{b}_1,E_1,a,F} \Big)_{\theta,\mathrm{b},E}. \tag{1.2}$$

This time we shall focus in the "dual" situation

$$\Big(\overline{X}_{\theta_0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F} \Big)_{\theta,\mathrm{b},E} \quad \text{and} \quad \Big(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1} \Big)_{\theta,\mathrm{b},E}, \tag{1.3}$$

when $0 \le \theta_0 < \theta_1 \le 1$, $0 \le \theta \le 1$, $a, \mathrm{b}, \mathrm{b}_0, \mathrm{b}_1$ are slowly varying functions and E_0, E_1, E, F are r.i. spaces.

We remark that the identities in (1.3) do not follow from those in (1.2) by a usual "symmetry" argument (i.e., interchanging X_0 and X_1), since such argument would not preserve the condition $0 \le \theta_0 < \theta_1 \le 1$, which is crucial in the identification of the spaces in (1.2). Moreover, in the limiting cases $\theta = 0,\ 1$, the reiteration spaces obtained in (1.3) will no longer belong to the scales in (1.1). In fact, new interpolation functors

$$\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G} \quad \text{and} \quad \overline{X}^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G} \tag{1.4}$$

will be needed; see Definition 2.11.

As in [26], a motivation for this study arises from various recent applications (see [1,2,30]) to the so-called *grand* and *small Lebesgue* spaces

$$L^{p),\alpha} \quad \text{and} \quad L^{(p,\alpha}, \quad 1 < p < \infty,\ \alpha > 0;$$

see Definition 5.2 below. As observed in [28,42], one can write

$$L^{p),\alpha} = (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p},\ell^{-\frac{\alpha}{p}}(t),L_\infty,1,L_p} \quad \text{and}$$
$$L^{(p,\alpha} = (L_1, L_\infty)^{\mathcal{L}}_{1-\frac{1}{p},\ell^{-\frac{\alpha}{p}+\alpha-1}(t),L_1,1,L_p},$$

with $\ell(t) = 1 + |\log(t)|$, $t \in (0,1)$. So, in particular (1.3) allows to identify the interpolation spaces

$$\big(L_{p_0}, L^{p_1),\alpha} \big)_{\theta,\mathrm{b},E} \quad \text{and} \quad \big(L^{(p_0,\alpha}, L_{p_1} \big)_{\theta,\mathrm{b},E},$$

for $1 \le p_0 < p_1 \le \infty$, $\alpha > 0$, and $0 \le \theta \le 1$. Moreover, using additionally reiteration results from [24] or using limiting cases for θ_0 and θ_1 in (1.3), one computes the pairs

$$\big(L^{(p_0,\alpha}, L^{p_1),\beta} \big)_{\theta,\mathrm{b},E}, \quad \big(L \log L, L^{p_1),\beta} \big)_{\theta,\mathrm{b},E} \quad \text{and} \quad \big(L^{(p_0,\alpha}, L_{\exp} \big)_{\theta,\mathrm{b},E};$$

see Theorem 5.7 and Corollaries 5.8, 5.11 below.

Some of these special cases are contained in the recent papers [1,2,30], together with other interpolation formulae for pairs involving grand or small

Lebesgue (Lorentz) spaces. Our goal here is to present a unified study for such identities in the setting of general couples $\overline{X} = (X_0, X_1)$ of (quasi-) Banach spaces, and arbitrary parameters b and E. This point of view, besides being more general, also produces new formulae compared to [1,2,30], and allows to apply the results to other situations, such as *Gamma* spaces and A and B-type spaces; see Sect. 5 below.

As in [26], the proofs use a direct approach, which closely follows the classical methods of reiteration. The main point is to obtain Holmstedt type formulae for the interpolation couples described above; these can be seen as quantitative forms of the reiteration theorems. We also make use of standard techniques that already appeared in [26], such as Hardy-type inequalities in the context of r.i. spaces and slowly varying functions (see Sect. 4.1), and an estimate that is specific of this situation (Lemma 4.4).

The paper is organized as follows. In Sect. 2, we recall basic concepts regarding rearrangement invariant spaces and slowly varying functions. We also describe the interpolation methods we shall work with, namely $\overline{X}_{\theta,\mathrm{b},E}$, the $\mathcal{R}$ and $\mathcal{L}$-spaces, $\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$, $\overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$, and the new constructions in (1.4). Generalized Holmstedt-type formulae for the K-functional of the couples involved can be found in Sect. 3. The reiteration results appear in Sect. 4 and finally Sect. 5 is devoted to applications.

2. Preliminaries

We refer to the monographs [8,9,11,37,51] for the basic concepts and facts on Interpolation Theory and Banach function spaces. A Banach function space E on $(0, \infty)$ is called *rearrangement invariant* (r.i.) if, for any two measurable functions f, g,

$$g \in E \text{ and } f^* \leq g^* \Longrightarrow f \in E \text{ and } \|f\|_E \leq \|g\|_E,$$

where f^* and g^* stand for the non-increasing rearrangements of f and g. Following [8], we assume that every Banach function space E enjoys the *Fatou property*. Under this assumption every r.i. space E can be obtained by applying an exact interpolation method to the couple (L_1, L_∞).

Along this paper, we will handle two different measures on $(0, \infty)$; the usual Lebesgue measure and the homogeneous measure $\nu(A) = \int_0^\infty \chi_A(t)\frac{dt}{t}$. We use a tilde to denote rearrangement invariant spaces with respect to the second measure. For example,

$$\|f\|_{\widetilde{L}_1} = \int_0^\infty |f(t)|\frac{dt}{t} \quad \text{and} \quad \|f\|_{\widetilde{L}_\infty} = \|f\|_{L_\infty}.$$

If E is an r.i. space obtained by applying the interpolation functor $\mathcal{F}$ to the couple (L_1, L_∞), $E = \mathcal{F}(L_1, L_\infty)$, we will denote by $\widetilde{E}$ the space generated by $\mathcal{F}$ acting on the couple $(\widetilde{L}_1, L_\infty)$, $\widetilde{E} = \mathcal{F}(\widetilde{L}_1, L_\infty)$.

Sometimes we will need to restrict the space to some partial interval $(a, b) \subset (0, \infty)$. Then, we will use the notation $E(a, b)$ and $\widetilde{E}(a, b)$; the norm being $\|f\|_{E(a,b)} = \|f(t)\chi_{(a,b)}(t)\|_E$.

For two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $Y \subset X$ and the natural embedding is continuous. The symbol $X = Y$ means that $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

Let A and B be two non-negative quantities depending on certain parameters. We write $A \lesssim B$ if there is a constant $c > 0$, independent of the parameters involved in A and B, such that $A \leq cB$. If $A \lesssim B$ and $B \lesssim A$, we say that A and B are equivalent and write $A \sim B$.

2.1. Slowly Varying Functions

In this subsection, we recall the definition and basic properties of *slowly varying functions*. See [10,38].

Definition 2.1. A positive Lebesgue measurable function b, $0 \not\equiv \mathrm{b} \not\equiv \infty$, is said to be *slowly varying* on $(0,\infty)$ (notation $\mathrm{b} \in SV$) if, for each $\varepsilon > 0$, the function $t \rightsquigarrow t^{\varepsilon}\mathrm{b}(t)$ is equivalent to a non-decreasing function on $(0,\infty)$ and $t \rightsquigarrow t^{-\varepsilon}\mathrm{b}(t)$ is equivalent to a non-increasing function on $(0,\infty)$.

Examples of SV-functions include powers of logarithms,

$$\ell^{\alpha}(t) = (1 + |\log t|)^{\alpha}, \quad t > 0, \quad \alpha \in \mathbb{R},$$

"broken" logarithmic functions defined as

$$\ell^{(\alpha,\beta)}(t) = \begin{cases} \ell^{\alpha}(t), \, 0 < t \leq 1 \\ \ell^{\beta}(t), \, t > 1 \end{cases}, \quad (\alpha,\beta) \in \mathbb{R}^2, \tag{2.1}$$

reiterated logarithms $(\ell \circ \ldots \circ \ell)^{\alpha}(t),\ \alpha \in \mathbb{R},\ t > 0$ and also the family of functions $\exp(|\log t|^{\alpha}),\ \alpha \in (0,1),\ t > 0$.

In the following lemmas, we summarize some of the basic properties of slowly varying functions.

Lemma 2.2. *Let* $\mathrm{b}, \mathrm{b}_1, \mathrm{b}_2 \in SV$*.*

(i) Then $\mathrm{b}_1\mathrm{b}_2 \in SV$*,* $\mathrm{b}(1/t) \in SV$ *and* $\mathrm{b}^r \in SV$ *for all* $r \in \mathbb{R}$*.*
(ii) If $\alpha > 0$*, then* $\mathrm{b}(t^{\alpha}\mathrm{b}_1(t)) \in SV$*.*
(iii) If $\epsilon, s > 0$ *then there are positive constants* c_ϵ *and* C_ϵ *such that*

$$c_\epsilon \min\{s^{-\epsilon}, s^{\epsilon}\}\mathrm{b}(t) \leq \mathrm{b}(st) \leq C_\epsilon \max\{s^{\epsilon}, s^{-\epsilon}\}\mathrm{b}(t) \quad \textit{for every } t > 0.$$

Lemma 2.3. *Let* E *be an r.i. space on* $(0,\infty)$ *and* $\mathrm{b} \in SV$*.*

(i) If $\alpha > 0$*, then, for all* $t > 0$*,*

$$\|s^{\alpha}\mathrm{b}(s)\|_{\widetilde{E}(0,t)} \sim t^{\alpha}\mathrm{b}(t) \quad \textit{and} \quad \|s^{-\alpha}\mathrm{b}(s)\|_{\widetilde{E}(t,\infty)} \sim t^{-\alpha}\mathrm{b}(t).$$

(ii) The following functions belong to SV

$$B_0(t) := \|\mathrm{b}\|_{\widetilde{E}(0,t)} \quad \textit{and} \quad B_\infty(t) := \|\mathrm{b}\|_{\widetilde{E}(t,\infty)}, \quad t > 0.$$

(iii) For all $t > 0$*,*

$$\mathrm{b}(t) \lesssim \|\mathrm{b}\|_{\widetilde{E}(0,t)} \quad \textit{and} \quad \mathrm{b}(t) \lesssim \|\mathrm{b}\|_{\widetilde{E}(t,\infty)}.$$

We refer to [22,33] for the proof of Lemma 2.2 and 2.3 respectively.

Remark 2.4. The property (iii) of Lemma 2.2 implies that if $\mathrm{b} \in SV$ is such that $\mathrm{b}(t_0) = 0$ ($\mathrm{b}(t_0) = \infty$) for some $t_0 > 0$, then $\mathrm{b} \equiv 0$ ($\mathrm{b} \equiv \infty$). Thus, by Lemma 2.3 (ii), if $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$ then $\|\mathrm{b}\|_{\widetilde{E}(0,t)} < \infty$ for all $t > 0$, and if $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$ then $\|\mathrm{b}\|_{\widetilde{E}(t,\infty)} < \infty$ for all $t > 0$.

Moreover, if $f \sim g$ then, using Definition 2.1 and Lemma 2.2 (iii), one can show that $\mathrm{b} \circ f \sim \mathrm{b} \circ g$ for any $b \in SV$.

2.2. Interpolation Methods

In what follows $\overline{X} = (X_0, X_1)$ will be a *compatible (quasi-) Banach couple*, that is, two (quasi-) Banach spaces continuously embedded in a Hausdorff topological vector space. The *Peetre K-functional* $K(t, f) \equiv K(t, f; X_0, X_1)$ is defined for $f \in X_0 + X_1$ and $t > 0$ by

$$K(t, f; X_0, X_1) = \inf \Big\{ \|f_0\|_{X_0} + t\|f_1\|_{X_1} : \ f = f_0 + f_1, \ f_i \in X_i, \ i = 0, 1 \Big\}.$$

It is known that the function $t \rightsquigarrow K(t, f)$ is non-decreasing, while $t \rightsquigarrow t^{-1}K(t, f)$, $t > 0$, is non-increasing. Other important property of the K-functional is the fact that

$$K(t, f; X_0, X_1) = tK(t^{-1}, f; X_1, X_0) \quad \text{for all } t > 0, \tag{2.2}$$

(see [8, Chap. 5, Proposition 1.2]).

Now, we recall the definition and some properties of the real interpolation method $\overline{X}_{\theta,\mathrm{b},E}$ and of the limiting constructions $\mathcal{L}$ and $\mathcal{R}$. See [22] for the proof of the results of this subsection.

Definition 2.5. Let E be an r.i. space, $\mathrm{b} \in SV$ and $0 \le \theta \le 1$. The real interpolation space $\overline{X}_{\theta,\mathrm{b},E} \equiv (X_0, X_1)_{\theta,\mathrm{b},E}$ consists of all f in $X_0 + X_1$ that satisfy

$$\|f\|_{\theta,\mathrm{b},E} := \left\| t^{-\theta}\mathrm{b}(t)K(t, f) \right\|_{\widetilde{E}} < \infty.$$

The space $\overline{X}_{\theta,\mathrm{b},E}$ is a (quasi-) Banach space, and is an intermediate space for the couple $\overline{X}$, that is,

$$X_0 \cap X_1 \hookrightarrow \overline{X}_{\theta,\mathrm{b},E} \hookrightarrow X_0 + X_1,$$

provided that one of the following conditions holds

$$\begin{cases} 0 < \theta < 1 & \\ \theta = 0, & \|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty \\ \theta = 1, & \|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty. \end{cases}$$

If none of the above conditions holds, then $\overline{X}_{\theta,\mathrm{b},E} = \{0\}$.

When $E = L_q$ and $\mathrm{b} \equiv 1$, then $\overline{X}_{\theta,\mathrm{b},E}$ coincides with the classical real interpolation space $\overline{X}_{\theta,q}$. We emphasize, however, that the spaces $\overline{X}_{\theta,b,E}$ are well defined even for the extremal values of the parameter $\theta = 0$ and $\theta = 1$. Thus, this scale contains extrapolation spaces in the sense of Milman [39] and Gómez and Milman [34]. The interpolation spaces $\overline{X}_{\theta,\mathrm{b},L_q}$ have been studied in detail by Gogatishvili, Opic and Trebels in [33], while the special cases $\overline{X}_{\theta,\ell^{(\alpha,\beta)}(t),L_q}$, see (2.1), were considered earlier by Evans, Opic and Pick in

[21] and by Evans and Opic in [20]. See also [14,17,35,45,50], among other references.

In the next remark, we collect some elementary estimates that will be used in the rest of the paper.

Remark 2.6. Using Lemma 2.3 (i) and the monotonicity of $t \rightsquigarrow t^{-1}K(t,\cdot)$, it is easy to check that for $t>0$ and $f \in X_0+X_1$

$$t^{-\theta}\mathrm{b}(t)K(t,f) \lesssim \left\| s^{-\theta}\mathrm{b}(s)K(s,f)\right\|_{\widetilde{E}(0,t)}, \qquad 0 \le \theta < 1, \tag{2.3}$$

and

$$t^{-1}\|\mathrm{b}\|_{\widetilde{E}(0,t)}K(t,f) \lesssim \left\| s^{-1}\mathrm{b}(s)K(s,f)\right\|_{\widetilde{E}(0,t)}. \tag{2.4}$$

Similarly,

$$t^{-\theta}\mathrm{b}(t)K(t,f) \lesssim \left\| s^{-\theta}\mathrm{b}(s)K(s,f)\right\|_{\widetilde{E}(t,\infty)}, \qquad 0 < \theta \le 1, \tag{2.5}$$

and for $\theta = 0$

$$\|\mathrm{b}\|_{\widetilde{E}(t,\infty)}K(t,f) \lesssim \left\| \mathrm{b}(s)K(s,f)\right\|_{\widetilde{E}(t,\infty)}. \tag{2.6}$$

It is worth remarking that by (2.3) and (2.5), we have

$$K(t,f) \lesssim \frac{t^{\theta}}{\mathrm{b}(t)}\|f\|_{\theta,\mathrm{b},E}, \qquad 0<\theta<1, \tag{2.7}$$

for all $t>0$ and $f \in \overline{X}_{\theta,\mathrm{b},E}$. In the cases $\theta = 0,1$ the estimate (2.7) is also true if we replace $\mathrm{b}(t)$ by $\|\mathrm{b}\|_{\widetilde{E}(t,\infty)}$ or $\|\mathrm{b}\|_{\widetilde{E}(0,t)}$, respectively.

Definition 2.7. Let E, F be two r.i. spaces, $a, \mathrm{b} \in SV$ and $0 \le \theta \le 1$. The space $\overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F} \equiv (X_0,X_1)^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$ consists of all $f \in X_0+X_1$ for which

$$\|f\|_{\mathcal{L};\theta,\mathrm{b},E,a,F} := \left\| \mathrm{b}(t)\|s^{-\theta}a(s)K(s,f)\|_{\widetilde{F}(0,t)}\right\|_{\widetilde{E}} < \infty.$$

This is a (quasi-) Banach intermediate space for the couple $\overline{X}$,

$$X_0 \cap X_1 \hookrightarrow \overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F} \hookrightarrow X_0+X_1,$$

provided that

1. $0<\theta<1$ and $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, or
2. $\theta = 0$, $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, $\left\|\mathrm{b}(t)\|a\|_{\widetilde{F}(1,t)}\right\|_{\widetilde{E}(1,\infty)} < \infty$ and $\|a\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$ or
3. $\theta = 1$, $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$ and $\left\|\mathrm{b}(t)\|a\|_{\widetilde{F}(0,t)}\right\|_{\widetilde{E}(0,1)} < \infty$.

If none of these conditions holds, then $\overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F} = \{0\}$.

Definition 2.8. Let E, F be two r.i. spaces, $a, \mathrm{b} \in SV$ and $0 \le \theta \le 1$. The space $\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F} \equiv (X_0,X_1)^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$ consists of all $f \in X_0+X_1$ for which

$$\|f\|_{\mathcal{R};\theta,\mathrm{b},E,a,F} := \left\| \mathrm{b}(t)\|s^{-\theta}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\right\|_{\widetilde{E}} < \infty.$$

The space $\mathcal{R}$ is a (quasi)-Banach intermediate space for the couple $\overline{X}$, that is,

$$X_0 \cap X_1 \hookrightarrow \overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F} \hookrightarrow X_0+X_1,$$

provided that

1. $0 < \theta < 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$ or
2. $\theta = 0$, $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$ and $\big\|\mathrm{b}(t)\|a\|_{\widetilde{F}(t,\infty)}\big\|_{\widetilde{E}(1,\infty)} < \infty$
3. $\theta = 1$, $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, $\big\|\mathrm{b}(t)\|a\|_{\widetilde{F}(t,1)}\big\|_{\widetilde{E}(0,1)} < \infty$ and $\|a\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$.

If none of these conditions holds, then $\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$ is a trivial space.

The above definitions generalize a previous notion by Evans and Opic [20] in which a, b are broken logarithms while E, F remain within the classes L_q. Earlier versions of these spaces appeared in a paper by Doktorskii [17], with a, b powers of logarithms and $\overline{X}$ an ordered couple. These spaces also appear in the work of Gogatishvili, Opic and Trebels [33] and Ahmed *et al.* [3]. In all these cases, the spaces E and F remain within the L_q classes.

Some inclusions between the three constructions can be obtained using inequalities (2.3)–(2.6). In fact,

$$\begin{aligned} \overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F} &\hookrightarrow \overline{X}_{\theta,\mathrm{b}a,E} && \text{if} \quad 0 \le \theta < 1, \\ \overline{X}^{\mathcal{L}}_{1,\mathrm{b},E,a,F} &\hookrightarrow \overline{X}_{1,\mathrm{b}\|a\|_{\widetilde{F}(0,t)},E} && \text{if } \theta = 1, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F} &\hookrightarrow \overline{X}_{\theta,\mathrm{b}a,E} && \text{if } 0 < \theta \le 1, \\ \overline{X}^{\mathcal{R}}_{0,\mathrm{b},E,a,F} &\hookrightarrow \overline{X}_{0,\mathrm{b}\|a\|_{\widetilde{F}(t,\infty)},E} && \text{if } \theta = 0. \end{aligned} \tag{2.9}$$

See [22, Lemma 6.7] for the identities in the case $E = F = L_q$.

The spaces $(X_0, X_1)_{\theta,\mathrm{b},E}$, $(X_0, X_1)^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$ and $(X_0, X_1)^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$ are interpolation spaces and satisfy the following symmetry property.

Lemma 2.9. *Let E, F be r.i. spaces, a, $\mathrm{b} \in SV$ and $0 \le \theta \le 1$. Then*

$$(X_0, X_1)_{\theta,\mathrm{b},E} = (X_1, X_0)_{1-\theta,\overline{\mathrm{b}},E}, \quad (X_0, X_1)^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F} = (X_1, X_0)^{\mathcal{R}}_{1-\theta,\overline{\mathrm{b}},E,\overline{a},F},$$

where $\overline{a}(t) = a(1/t)$ and $\overline{\mathrm{b}}(t) = \mathrm{b}(1/t)$, $t > 0$.

We conclude this subsection with some inequalities that will be used later.

Lemma 2.10. *Let E, F be r.i. spaces, $a, \mathrm{b} \in SV$ and $0 \le \theta \le 1$. Then, for all $f \in X_0 + X_1$ and $u > 0$*

$$u^{-\theta} a(u)\|\mathrm{b}\|_{\widetilde{E}(0,u)} K(u, f) \lesssim \big\|\mathrm{b}(t)\|s^{-\theta} a(s) K(s, f)\|_{\widetilde{F}(t,u)}\big\|_{\widetilde{E}(0,u)} \tag{2.10}$$

and

$$u^{-\theta} a(u)\mathrm{b}(u) K(u, f) \lesssim \big\|\mathrm{b}(t)\|s^{-\theta} a(s) K(s, f)\|_{\widetilde{F}(t,\infty)}\big\|_{\widetilde{E}(u,\infty)}. \tag{2.11}$$

Proof. We refer to [26, Lemma 2.12] for the proof of (2.10). Inequality (2.11) is an easy consequence of Lemma 2.3 (i) and the monotonicity of the K-functional. Indeed,

$$\begin{aligned} \Big\|\mathrm{b}(t)\|s^{-\theta} a(s) K(s, f)\|_{\widetilde{F}(t,\infty)}\Big\|_{\widetilde{E}(u,\infty)} &\gtrsim K(u, f)\Big\|\mathrm{b}(t)\|s^{-\theta} a(s)\|_{\widetilde{F}(t,\infty)}\Big\|_{\widetilde{E}(u,\infty)} \\ &\sim K(u, f)\|t^{-\theta}\mathrm{b}(t) a(t)\|_{\widetilde{E}(u,\infty)} \\ &\sim u^{-\theta} a(u)\mathrm{b}(u) K(u, f). \end{aligned}$$

For $\theta = 0$, we need to use Lemma 2.3 (iii) instead of 2.3 (i). □

Consequently, for $0 \le \theta \le 1$ and $f \in \overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$, we have

$$K(u,f) \lesssim \frac{u^\theta}{a(u)\|\mathrm{b}\|_{\widetilde{E}(0,u)}} \|f\|_{\mathcal{R};\theta,\mathrm{b},E,a,F}, \quad u>0. \tag{2.12}$$

2.3. New Extremal Spaces

Our next definition introduces two new types of extremal interpolation spaces that will appear in relation with the extreme reiteration results that will be studied in §4.

Definition 2.11. Let E, F, G be r.i. spaces, $a, \mathrm{b}, c \in SV$ and $0 \le \theta \le 1$. The space $\overline{X}^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G} \equiv (X_0, X_1)^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G}$ is the set of all $f \in X_0 + X_1$ for which

$$\|f\|_{\mathcal{L},\mathcal{L};\theta,c,E,\mathrm{b},F,a,G} := \left\| c(u) \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s,f)\|_{\widetilde{G}(0,t)} \right\|_{\widetilde{F}(0,u)} \right\|_{\widetilde{E}} < \infty.$$

Similarly, the space $\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G} \equiv (X_0, X_1)^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G}$ is the set of all $f \in X_0 + X_1$ such that

$$\|f\|_{\mathcal{R},\mathcal{R};\theta,c,E,\mathrm{b},F,a,G} := \left\| c(u) \left\| \mathrm{b}(t) \|s^{-\theta} a(s) K(s,f)\|_{\widetilde{G}(t,\infty)} \right\|_{\widetilde{F}(u,\infty)} \right\|_{\widetilde{E}} < \infty.$$

A standard reasoning (see [8, 20]) shows that the above two classes are (quasi-) Banach spaces, provided X_0 and X_1 are so. Moreover, using estimates (2.3)–(2.6), one can obtain the following inclusions

$$\begin{aligned} \overline{X}^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G} &\hookrightarrow \overline{X}^{\mathcal{L}}_{\theta,c,E,\mathrm{b}a,F} && \text{if} \quad 0 \le \theta < 1, \\ \overline{X}^{\mathcal{L},\mathcal{L}}_{1,c,E,\mathrm{b},F,a,G} &\hookrightarrow \overline{X}^{\mathcal{L}}_{1,c,E,\mathrm{b}\|a\|_{\widetilde{G}(0,t)},F} && \text{if } \theta = 1, \end{aligned}$$

and

$$\begin{aligned} \overline{X}^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G} &\hookrightarrow \overline{X}^{\mathcal{R}}_{\theta,c,E,\mathrm{b}a,F} && \text{if } 0 < \theta \le 1, \\ \overline{X}^{\mathcal{R},\mathcal{R}}_{0,c,E,\mathrm{b},F,a,G} &\hookrightarrow \overline{X}^{\mathcal{R}}_{0,c,E,\mathrm{b}\|a\|_{\widetilde{G}(t,\infty)},F} && \text{if } \theta = 0. \end{aligned}$$

It is easy to show that the spaces $\mathcal{L},\mathcal{L}$ and $\mathcal{R},\mathcal{R}$ are also related by the following symmetry property.

Lemma 2.12. *Let E, F, G be r.i. spaces, a, b, $c \in SV$ and $0 \le \theta \le 1$. Then*

$$(X_0, X_1)^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G} = (X_1, X_0)^{\mathcal{R},\mathcal{R}}_{1-\theta,\overline{c},E,\overline{\mathrm{b}},F,\overline{a},G}$$

where $\overline{a}(t) = a(1/t)$, $\overline{\mathrm{b}}(t) = \mathrm{b}(1/t)$ and $\overline{c}(t) = c(1/t)$, $t > 0$.

3. Generalized Holmstedt-Type Formulae

For parameters $0 \le \theta_0 < \theta_1 \le 1$, a, b_0, $\mathrm{b}_1 \in SV$ and E_0, E_1 r.i. spaces, the couples

$$(\overline{X}_{\theta_0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}) \quad \text{and} \quad (\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1}),$$

are compatible (quasi-) Banach couples (assuming the usual conditions on b_0 and b_1 in the extreme cases $\theta_0 = 0$, $\theta_1 = 1$, respectively). In this section, we relate the K-functionals of these couples with the K-functional of the original one, $\overline{X}$, by means of some generalized Holmstedt type formulae.

3.1. The K-Functional of the Couple $(\overline{X}_{\theta_0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})$, $0 \le \theta_0 < \theta_1 < 1$

Theorem 3.1. *Let* $0 < \theta_0 < \theta_1 < 1$. *Let* E_0, E_1, F *be r.i. spaces and* a, b_0, $\mathrm{b}_1 \in SV$ *such that* $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$.

a) Then, for every $f \in \overline{X}_{\theta_0,\mathrm{b}_0,E_0} + \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}$ *and all* $u > 0$

$$\begin{aligned} K\big(\rho(u), f; \overline{X}_{\theta_0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}\big) &\sim \|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)} \\ &+ \rho(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)} \\ &+ \rho(u)\big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\big\|_{\widetilde{E}_1(u,\infty)}, \end{aligned}$$

where

$$\rho(u) = u^{\theta_1-\theta_0}\frac{\mathrm{b}_0(u)}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u > 0. \tag{3.1}$$

b) If $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$, *then, for every* $f \in \overline{X}_{0,\mathrm{b}_0,E_0} + \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}$ *and all* $u > 0$

$$\begin{aligned} K\big(\rho(u), f; \overline{X}_{0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}\big) &\sim \|\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)} \\ &+ \rho(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)} \\ &+ \rho(u)\big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\big\|_{\widetilde{E}_1(u,\infty)}, \end{aligned}$$

where $\rho(u) = u^{\theta_1}\frac{\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}$, $u > 0$

c) Moreover, for every $f \in X_0 + \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}$ *and all* $u > 0$

$$\begin{aligned} K\big(\rho(u), f; X_0, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}\big) &\sim \rho(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)} \\ &+\rho(u)\big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\big\|_{\widetilde{E}_1(u,\infty)}, \end{aligned} \tag{3.2}$$

where $\rho(u) = u^{\theta_1}\frac{1}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}$, $u > 0$.

Proof. We prove only a), the proofs of b) and c) are similar. Given $f \in X_0 + X_1$ and $u > 0$ we consider the (quasi-) norms

$$\begin{aligned}(P_0 f)(u) &= \|t^{-\theta_0} \mathrm{b}_0(t) K(t,f)\|_{\widetilde{E}_0(0,u)},\\ (Q_0 f)(u) &= \|t^{-\theta_0} \mathrm{b}_0(t) K(t,f)\|_{\widetilde{E}_0(u,\infty)},\\ (P_1 f)(u) &= \Big\| \mathrm{b}_1(t) \|s^{-\theta_1} a(s) K(s,f)\|_{\widetilde{F}(t,u)} \Big\|_{\widetilde{E}_1(0,u)},\\ (R_1 f)(u) &= \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|t^{-\theta_1} a(t) K(t,f)\|_{\widetilde{F}(u,\infty)},\\ (Q_1 f)(u) &= \Big\| \mathrm{b}_1(t) \|s^{-\theta_1} a(s) K(s,f)\|_{\widetilde{F}(t,\infty)} \Big\|_{\widetilde{E}_1(u,\infty)},\end{aligned}$$

and we denote $Y_0 = \overline{X}_{\theta_0,\mathrm{b}_0,E_0}$ and $Y_1 = \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}$. With this notation, what we pursue to show is the equivalence

$$K(\rho(u), f; Y_0, Y_1) \sim (P_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)], \tag{3.3}$$

for all $f \in Y_0 + Y_1$ and $u > 0$, where ρ is defined by (3.1).

We first prove the upper estimate $\lesssim$ of (3.3) for all $f \in X_0 + X_1$ and any positive function $\rho : (0,\infty) \to (0,\infty)$.

Suppose that $f \in X_0 + X_1$ and fix $u > 0$. We may assume with no loss of generality that $(P_0 f)(u)$, $(R_1 f)(u)$ and $(Q_1 f)(u)$ are finite, otherwise the upper estimate of (3.3) holds trivially. As usual (see for example [8] or [21]) we choose a decomposition $f = g + h$ such that

$$\|g\|_{X_0} + u\|h\|_{X_1} \le 2K(u,f)$$

and

$$K(t,g) \le 2K(u,f) \quad \text{and} \quad \frac{K(t,h)}{t} \le 2\frac{K(u,f)}{u} \tag{3.4}$$

for all $t > 0$. So, to obtain the upper estimate of (3.3), it suffices to prove that

$$\|g\|_{Y_0} + \rho(u)\|h\|_{Y_1} \lesssim (P_0 f)(u) + \rho(u)[(R_1 f(u) + (Q_1 f)(u)].$$

We start by showing that $\|g\|_{Y_0} \lesssim (P_0 f)(u)$. The triangle inequality and the (quasi-) subadditivity of the K-functional establish that

$$\|g\|_{Y_0} \le (P_0 g)(u) + (Q_0 g)(u) \lesssim (P_0 f)(u) + (P_0 h)(u) + (Q_0 g)(u).$$

Using (3.4), Lemma 2.3 (i) and (2.3), we obtain

$$\begin{aligned}(P_0 h)(u) &= \big\|t^{-\theta_0} \mathrm{b}_0(t) K(t,h)\big\|_{\widetilde{E}_0(0,u)} \lesssim \frac{K(u,f)}{u} \|t^{1-\theta_0} \mathrm{b}_0(t)\|_{\widetilde{E}_0(0,u)}\\ &\sim u^{-\theta_0} \mathrm{b}_0(u) K(u,f) \lesssim (P_0 f)(u)\end{aligned}$$

and

$$\begin{aligned}(Q_0 g)(u) &= \|t^{-\theta_0} \mathrm{b}_0(t) K(t,g)\|_{\widetilde{E}_0(u,\infty)} \lesssim K(u,f) \|t^{-\theta_0} \mathrm{b}_0(t)\|_{\widetilde{E}_0(u,\infty)}\\ &\sim u^{-\theta_0} \mathrm{b}_0(u) K(u,f) \lesssim (P_0 f)(u).\end{aligned}$$

These give

$$\|g\|_{Y_0} \lesssim (P_0 f)(u) < \infty$$

and therefore $g \in Y_0$. Now we proceed with $\|h\|_{Y_1}$. Again by triangle inequality and the (quasi-) subadditivity of the K-functional, we have

$$\begin{aligned}\|h\|_{Y_1} &\leq (P_1 h)(u) + (R_1 h)(u) + (Q_1 h)(u) \\ &\lesssim (P_1 h)(u) + (R_1 f)(u) + (R_1 g)(u) + (Q_1 f)(u) + (Q_1 g)(u).\end{aligned}$$

Using (3.4), Lemma 2.3 (i) and (2.5), we obtain

$$\begin{aligned}(P_1 h)(u) &= \left\| \mathrm{b}_1(t) \| s^{-\theta_1} a(s) K(s,h) \|_{\widetilde{F}(t,u)} \right\|_{\widetilde{E}_1(0,u)} \\ &\lesssim \frac{K(u,f)}{u} \left\| \mathrm{b}_1(t) \| s^{1-\theta_1} a(s) \|_{\widetilde{F}(t,u)} \right\|_{\widetilde{E}_1(0,u)} \\ &\leq \frac{K(u,f)}{u} \left\| \mathrm{b}_1(t) \| s^{1-\theta_1} a(s) \|_{\widetilde{F}(0,u)} \right\|_{\widetilde{E}_1(0,u)} \\ &\sim u^{-\theta_1} a(u) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} K(u,f) \lesssim (R_1 f)(u)\end{aligned}$$

and

$$\begin{aligned}(R_1 g)(u) &= \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|t^{-\theta_1} a(t) K(t,g)\|_{\widetilde{F}(u,\infty)} \\ &\lesssim K(u,f) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|t^{-\theta_1} a(t)\|_{\widetilde{F}(u,\infty)} \\ &\sim u^{-\theta_1} a(u) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} K(u,f) \lesssim (R_1 f)(u).\end{aligned}$$

Similarly, using also (2.11), we estimate $(Q_1 g)(u)$ from above

$$\begin{aligned}(Q_1 g)(u) &= \left\| \mathrm{b}_1(t) \| s^{-\theta_1} a(s) K(s,g) \|_{\widetilde{F}(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \\ &\lesssim K(u,f) \left\| \mathrm{b}_1(t) \| s^{-\theta_1} a(s) \|_{\widetilde{F}(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \\ &\sim u^{-\theta_1} \mathrm{b}_1(u) a(u) K(u,f) \lesssim (Q_1 f)(u).\end{aligned}$$

Thus,

$$\|h\|_{Y_1} \lesssim (R_1 f)(u) + (Q_1 f)(u) < \infty,$$

and we obtain that $h \in Y_1$. Summing up, we deduce that

$$\begin{aligned}K(\rho(u), f; Y_0, Y_1) &\leq \|g\|_{Y_0} + \rho(u)\|h\|_{Y_1} \\ &\lesssim (P_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)],\end{aligned}$$

which is the upper estimate of (3.3).

Let us prove the lower estimate of (3.3). More precisely,

$$(P_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)] \lesssim K(\rho(u), f; Y_0, Y_1), \tag{3.5}$$

for all $f \in Y_0 + Y_1$, $u > 0$ and ρ defined by (3.1).

Choose $f = g + h$ any decomposition of f with $g \in Y_0$ and $h \in Y_1$, and fix again $u > 0$. Using the (quasi-) subadditivity of the K-functional and the definition of the norm in Y_0 and Y_1, we have

$$\begin{aligned}(P_0 f)(u) &\lesssim (P_0 g)(u) + (P_0 h)(u) \leq \|g\|_{Y_0} + (P_0 h)(u), \\ (R_1 f)(u) &\lesssim (R_1 g)(u) + (R_1 h)(u) \leq (R_1 g)(u) + \|h\|_{Y_1}, \\ (Q_1 f)(u) &\lesssim (Q_1 g)(u) + (Q_1 h)(u) \leq (Q_1 g)(u) + \|h\|_{Y_1}.\end{aligned}$$

Then,

$$(P_0 f)(u) + \rho(u)[(R_1 f(u) + (Q_1 f)(u)] \\ \lesssim \|g\|_{Y_0} + (P_0 h)(u) + \rho(u)[(R_1 g)(u) + (Q_1 g)(u) + \|h\|_{Y_1}].$$

Thus, it is enough to verify that $(P_0 h)(u)$, $\rho(u)(R_1 g)(u)$ and $\rho(u)(Q_1 g)(u)$ are bounded by $\|g\|_{Y_0} + \rho(u)\|h\|_{Y_1}$. We begin with $(P_0 h)(u)$. Estimate (2.12) with $f = h$ and Lemma 2.3 (i) imply that

$$(P_0 h)(u) \lesssim \|h\|_{Y_1} \left\| t^{\theta_1 - \theta_0} \frac{\mathrm{b}_0(t)}{a(t)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,t)}} \right\|_{\widetilde{E}_0(0,u)} \sim \rho(u)\|h\|_{Y_1}.$$

Observe that by hypothesis $0 < \theta_0 < \theta_1 < 1$ and then $\theta_1 - \theta_0 > 0$.

Similarly, using (2.7) with $f = g$ and Lemma 2.3 (i), we have the estimates

$$(R_1 g)(u) \lesssim \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \left\| t^{\theta_0 - \theta_1} \frac{a(t)}{\mathrm{b}_0(t)} \right\|_{\widetilde{F}(u,\infty)} \|g\|_{Y_0} \\ \sim u^{\theta_0 - \theta_1} \frac{a(u)}{\mathrm{b}_0(u)} \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|g\|_{Y_0} = \frac{1}{\rho(u)} \|g\|_{Y_0}$$

and

$$(Q_1 g)(u) \leq \|g\|_{Y_0} \left\| \mathrm{b}_1(t) \left\| s^{\theta_0 - \theta_1} \frac{a(s)}{\mathrm{b}_0(s)} \right\|_{\widetilde{F}(t,\infty)} \right\|_{\widetilde{E}_1(u,\infty)} \\ \sim u^{\theta_0 - \theta_1} \frac{a(u)\mathrm{b}_1(u)}{\mathrm{b}_0(u)} \|g\|_{Y_0} \lesssim \frac{1}{\rho(u)} \|g\|_{Y_0},$$

where the last equivalence follows from Lemma 2.3 (iii).

Putting together the previous estimates we obtain that

$$(P_0 f)(u) + \rho(u)[(R_1 f)(u) + (Q_1 f)(u)] \lesssim \|g\|_{Y_0} + \rho(u)\|h\|_{Y_1}.$$

Finally, taking infimum over all possible decomposition of $f = g + h$, with $g \in Y_0$ and $h \in Y_1$, we obtain (3.5) and the proof of a) is finished. □

3.2. The K-Functional of the Couple $\left(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1}\right)$, $0 < \theta_0 < \theta_1 \leq 1$

Next theorem can be proved as Theorem 3.1, although we shall make use of a symmetry argument.

Theorem 3.2. *Let* $0 < \theta_0 < \theta_1 < 1$. *Let* E_0, E_1, F *be r.i. spaces and* a, b_0, $\mathrm{b}_1 \in SV$ *such that* $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$.

a) Then, for every $f \in \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F} + \overline{X}_{\theta_1,\mathrm{b}_1,E_1}$ *and all* $u > 0$

$$K\left(\rho(u), f; \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1}\right) \sim \left\| \mathrm{b}_0(t) \|s^{-\theta_0} a(s) K(s,f)\|_{\widetilde{F}(0,t)} \right\|_{\widetilde{E}_0(0,u)} \\ + \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)} \|t^{-\theta_0} a(t) K(t,f)\|_{\widetilde{F}(0,u)} \\ + \rho(u) \|t^{-\theta_1} \mathrm{b}_1(t) K(t,f)\|_{\widetilde{E}_1(u,\infty)},$$

where $\rho(u) = u^{\theta_1 - \theta_0} \frac{a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{\mathrm{b}_1(u)}$, $u > 0$.

b) If $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$, *then, for every* $f \in \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F} + \overline{X}_{1,\mathrm{b}_1,E_1}$ *and all* $u > 0$

$$K\big(\rho(u), f; \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{1,\mathrm{b}_1,E_1}\big) \sim \Big\|\mathrm{b}_0(t)\|s^{-\theta_0}a(s)K(s,f)\|_{\widetilde{F}(0,t)}\Big\|_{\widetilde{E}_0(0,u)} + \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\|t^{-\theta_0}a(t)K(t,f)\|_{\widetilde{F}(0,u)} + \rho(u)\|t^{-1}\mathrm{b}_1(t)K(t,f)\|_{\widetilde{E}_1(u,\infty)},$$

where $\rho(u) = u^{1-\theta_0}\frac{a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}$, $u > 0$.

c) Moreover, for every $f \in \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F} + X_1$ *and all* $u > 0$

$$K\big(\rho(u), f; \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, X_1\big) \sim \big\|\mathrm{b}_0(t)\|s^{-\theta_0}a(s)K(s,f)\|_{\widetilde{F}(0,t)}\big\|_{\widetilde{E}_0(0,u)} + \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\|t^{-\theta_0}a(t)K(t,f)\|_{\widetilde{F}(0,u)}, \quad (3.6)$$

where $\rho(u) = u^{1-\theta_0}a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}$, $u > 0$.

Proof. We prove only b), the proofs of a) and c) are similar. We consider the slowly varying functions $\overline{\mathrm{b}}_i(t) = \mathrm{b}_i(1/t)$, $i = 0, 1$, and $\overline{a}(t) = a(1/t)$. Notice that $\|f\|_{\widetilde{E}(\frac{1}{t},\infty)} = \|f(1/s)\|_{\widetilde{E}(0,t)}$ and hence

$$\frac{1}{\rho(u)} = \Big(\frac{1}{u}\Big)^{1-\theta_0}\frac{\|\overline{\mathrm{b}}_1\|_{\widetilde{E}_1(\frac{1}{u},\infty)}}{\overline{a}(\frac{1}{u})\|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,\frac{1}{u})}}, \quad u > 0.$$

By (2.2) and Lemma 2.9, we have that

$$K\big(\rho(u), f; \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{1,\mathrm{b}_1,E_1}\big) = \rho(u)K\Big(\frac{1}{\rho(u)}, f; (X_1,X_0)_{0,\overline{\mathrm{b}}_1,E_1}, (X_1,X_0)^{\mathcal{R}}_{1-\theta_0,\overline{\mathrm{b}}_0,E_0,\overline{a},F}\Big).$$

Now applying Theorem 3.1 b), we obtain the estimate

$$K\big(\rho(u), f; \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{1,\mathrm{b}_1,E_1}\big) \sim \rho(u)\|\overline{\mathrm{b}}_1(t)K(t,f;X_1,X_0)\|_{\widetilde{E}_1(0,\frac{1}{u})} + \|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,\frac{1}{u})}\|t^{\theta_0-1}\overline{a}(t)K(t,f;X_1,X_0)\|_{\widetilde{F}(\frac{1}{u},\infty)} + \Big\|\overline{\mathrm{b}}_0(t)\|s^{\theta_0-1}\overline{a}(s)K(s,f;X_1,X_0)\|_{\widetilde{F}(t,\infty)}\Big\|_{\widetilde{E}_0(\frac{1}{u},\infty)}.$$

Finally, the relations $\|f\|_{\widetilde{E}(\frac{1}{t},\infty)} = \|f(1/s)\|_{\widetilde{E}(0,t)}$ and (2.2) give the desired equivalence. □

4. Reiteration Formulae for $\mathcal{R}$- and $\mathcal{L}$-Spaces.

The aim of this section is to identify the spaces

$$(\overline{X}_{\theta_0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{\theta,\mathrm{b},E} \quad \text{and} \quad (\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1})_{\theta,\mathrm{b},E}$$

for all possible values of $\theta \in [0,1]$. In that process, the lemmas that we collect in the next subsection play a key role.

4.1. Lemmas

Next three lemmas can be found in [22,26].

Lemma 4.1. *[26, Lemma 4.1] Let E be an r.i. space, a, $\mathrm{b} \in SV$, $0 \le \theta \le 1$, $0 < \alpha < 1$ and consider the function $\rho(u) = u^\alpha a(u)$, $u > 0$. Then, the equivalence*

$$\left\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))K(\rho(u), f)\right\|_{\widetilde{E}} \sim \left\|u^{-\theta}\mathrm{b}(u)K(u, f)\right\|_{\widetilde{E}}$$

hold for all $f \in X_0 + X_1$, with equivalent constant independent of f.

Lemma 4.2. *[22, Lemma 2.4] Let $\mathrm{b} \in SV$, φ a quasi-concave function and $\alpha \in \mathbb{R}$. Then, for any r.i. E and any $t > 0$,*

$$\|s^\alpha \mathrm{b}(s)\varphi(s)\|_{\widetilde{E}(0,t)} \lesssim \int_0^t s^\alpha \mathrm{b}(s)\varphi(s)\,\frac{ds}{s} \tag{4.1}$$

and

$$\|s^\alpha \mathrm{b}(s)\varphi(s)\|_{\widetilde{E}(t,\infty)} \lesssim \int_t^\infty s^\alpha \mathrm{b}(s)\varphi(s)\,\frac{ds}{s}. \tag{4.2}$$

Lemma 4.3. *[22, Lemma 2.5] Let $\mathrm{b} \in SV$ and $\alpha > 0$. Then, for any r.i. E, the inequalities*

$$\left\|t^{-\alpha}\mathrm{b}(t)\int_0^t f(s)\,ds\right\|_{\widetilde{E}} \lesssim \|t^{1-\alpha}\mathrm{b}(t)f(t)\|_{\widetilde{E}} \tag{4.3}$$

and

$$\left\|t^{\alpha}\mathrm{b}(t)\int_t^\infty f(s)\,ds\right\|_{\widetilde{E}} \lesssim \|t^{1+\alpha}\mathrm{b}(t)f(t)\|_{\widetilde{E}} \tag{4.4}$$

hold for all positive measurable functions f on $(0,\infty)$.

Reiteration results of next subsections do not follow from Theorems 4.4, 4.5, 4.7 of [26] and the usual symmetry argument, since the order of the parameters $\theta_0 < \theta_1$ is crucial. However, similar proofs can be carried out now using additionally the following lemma from [23].

Lemma 4.4. *[23, Theorem 3.6] Let E, F be r.i. spaces, $a, b \in SV$ and $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$. Then, the equivalence*

$$\left\|t^\beta \mathrm{b}(t)\|s^\alpha a(s)f(s)\|_{\widetilde{F}(t,\infty)}\right\|_{\widetilde{E}} \sim \|t^{\alpha+\beta}a(t)\mathrm{b}(t)f(t)\|_{\widetilde{E}}$$

holds for all positive and non-increasing measurable function f on $(0,\infty)$.

4.2. The Space $(\overline{X}_{\theta_0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{\theta,\mathrm{b},E}$, $0 \le \theta_0 < \theta_1 < 1$ and $0 \le \theta \le 1$

Theorem 4.5. *Let $0 < \theta_0 < \theta_1 < 1$ and let E, E_0, E_1, F be r.i. spaces. Let a, b, b_0, $\mathrm{b}_1 \in SV$ with b_1 satisfying $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$ and consider the function*

$$\rho(u) = u^{\theta_1-\theta_0}\frac{\mathrm{b}_0(u)}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u > 0.$$

a) If $0<\theta<1$, *then*

$$(\overline{X}_{\theta_0,\mathrm{b}_0,E_0},\overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{\theta,\mathrm{b},E}=\overline{X}_{\tilde{\theta},B_\theta,E},$$

where

$$\tilde{\theta}=(1-\theta)\theta_0+\theta\theta_1 \quad \textit{and} \quad B_\theta(u)=\big(\mathrm{b}_0(u)\big)^{1-\theta}\big(a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\big)^{\theta}\mathrm{b}(\rho(u)),\ u>0.$$

b) If $\theta=0$ *and* $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)}<\infty$, *then*

$$(\overline{X}_{\theta_0,\mathrm{b}_0,E_0},\overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{0,\mathrm{b},E}=\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}.$$

c) If $\theta=1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)}<\infty$, *then*

$$(\overline{X}_{\theta_0,\mathrm{b}_0,E_0},\overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{1,\mathrm{b},E}=\overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a,F}\cap\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a,F},$$

where $B_1(u)=\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\mathrm{b}(\rho(u))$, $u>0$.

Proof. Throughout the proof, we use the notation $Y_0=\overline{X}_{\theta_0,\mathrm{b}_0,E_0}$, $Y_1=\overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}$ and $\overline{K}(s,f)=K(s,f;Y_0,Y_1)$, $f\in Y_0+Y_1$, $s>0$.

We start with the proof of a). Let $f\in\overline{Y}_{\theta,\mathrm{b},E}$. Lemma 4.1, Theorem 3.1 a) and the lattice property of $\widetilde{E}$ yield

$$\begin{aligned}\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}}&\sim\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\|_{\widetilde{E}}\\&\gtrsim\big\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)}\big\|_{\widetilde{E}}.\end{aligned}$$

Now using (2.3) and observing that

$$\rho(u)^{-\theta}\mathrm{b}(\rho(u))=u^{\theta_0-\tilde{\theta}}\frac{B_\theta(u)}{\mathrm{b}_0(u)},\quad u>0, \tag{4.5}$$

one deduces that

$$\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}}\gtrsim\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))u^{-\theta_0}\mathrm{b}_0(u)K(u,f)\|_{\widetilde{E}}=\|u^{-\tilde{\theta}}B_\theta(u)K(u,f)\|_{\widetilde{E}}.$$

Thus, the inclusion $\overline{Y}_{\theta,\mathrm{b},E}\hookrightarrow\overline{X}_{\tilde{\theta},B_\theta,E}$ is proved.

Next we proceed with the reverse inclusion. Let $f\in\overline{X}_{\tilde{\theta},B_\theta,E}$. Using again Lemma 4.1, Theorem 3.1 a) and the triangular inequality, we have the estimate

$$\begin{aligned}\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}}&\lesssim\Big\|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)}\Big\|_{\widetilde{E}}\\&+\Big\|\rho(u)^{1-\theta}\mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}}\\&+\Big\|\rho(u)^{1-\theta}\mathrm{b}(\rho(u))\Big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\Big\|_{\widetilde{E}_1(u,\infty)}\Big\|_{\widetilde{E}}.\end{aligned} \tag{4.6}$$

We denote the last three expressions by I_1, I_2 and I_3, respectively, and we have to estimate each one by the norm of the function f in $\overline{X}_{\tilde{\theta},B_\theta,E}$. Let us begin with I_1. Identity (4.5) implies that

$$I_1=\Big\|u^{\theta_0-\tilde{\theta}}\frac{B_\theta(u)}{\mathrm{b}_0(u)}\|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)}\Big\|_{\widetilde{E}},$$

where $\theta_0-\tilde\theta<0$, so applying (4.1) and (4.3) we obtain that $I_1 \lesssim \|f\|_{\overline{X}_{\tilde\theta,B_\theta,E}}$. Indeed,

$$\begin{aligned} I_1 &\lesssim \left\| u^{\theta_0-\tilde\theta}\frac{B_\theta(u)}{\mathrm{b}_0(u)}\int_0^u t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\,\frac{dt}{t}\right\|_{\widetilde{E}} \\ &\lesssim \left\| u^{\theta_0-\tilde\theta}\frac{B_\theta(u)}{\mathrm{b}_0(u)}u^{-\theta_0}\mathrm{b}_0(u)K(u,f)\right\|_{\widetilde{E}} \\ &= \|u^{-\tilde\theta}B_\theta(u)K(u,f)\|_{\widetilde{E}} = \|f\|_{\overline{X}_{\tilde\theta,B_\theta,E}}. \end{aligned}$$

Similarly, using that

$$\rho(u)^{1-\theta}\mathrm{b}(\rho(u)) = u^{\theta_1-\tilde\theta}\frac{B_\theta(u)}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u>0, \tag{4.7}$$

the $\widetilde{L}_1$-bound (4.2) and the Hardy type inequality (4.4) ($\theta_1-\tilde\theta>0$), we have that

$$\begin{aligned} I_2 &= \left\| \rho(u)^{1-\theta}\mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\right\|_{\widetilde{E}} \qquad (4.8)\\ &= \left\| u^{\theta_1-\tilde\theta}\frac{B_\theta(u)}{a(u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\right\|_{\widetilde{E}} \\ &\lesssim \left\| u^{\theta_1-\tilde\theta}\frac{B_\theta(u)}{a(u)}\int_u^\infty t^{-\theta_1}a(t)K(t,f)\,\frac{dt}{t}\right\|_{\widetilde{E}} \\ &\lesssim \left\| u^{\theta_1-\tilde\theta}\frac{B_\theta(u)}{a(u)}u^{-\theta_1}a(u)K(u,f)\right\|_{\widetilde{E}} \\ &= \|u^{-\tilde\theta}B_\theta(u)K(u,f)\|_{\widetilde{E}} = \|f\|_{\overline{X}_{\tilde\theta,B_\theta,E}}. \end{aligned}$$

Finally, we observe that I_3 is bounded by I_2. Indeed, using (4.7) we can identify I_3 in the following way

$$I_3 = \left\| u^{\theta_1-\tilde\theta}\frac{B_\theta(u)}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}\left\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\right\|_{\widetilde{E}_1(u,\infty)}\right\|_{\widetilde{E}}.$$

Since the function $t \rightsquigarrow \|\cdot\|_{\widetilde{F}(t,\infty)}$, $t\in(u,\infty)$, is a non-increasing function and $\theta_1-\tilde\theta>0$, Lemma 4.4 and Lemma 2.3 (iii) give that

$$I_3 \sim \left\| u^{\theta_1-\tilde\theta}\frac{B_\theta(u)}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}\,\mathrm{b}_1(u)\,\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(u,\infty)}\right\|_{\widetilde{E}} \lesssim I_2.$$

Summing up $\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}} \lesssim I_1+I_2+I_3 \lesssim I_1+I_2 \lesssim \|f\|_{\overline{X}_{\tilde\theta,B_\theta,E}}$.

The proof of b) follows the same steps. In fact, let $f\in\overline{Y}_{0,\mathrm{b},E}$. Lemma 4.1, Theorem 3.1 a) and the lattice property of $\widetilde{E}$ yield

$$\|f\|_{\overline{Y}_{0,\mathrm{b},E}} \sim \|\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\|_{\widetilde{E}} \gtrsim \left\|\mathrm{b}(\rho(u))\|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)}\right\|_{\widetilde{E}}.$$

Hence, $f\in\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}$.

Next, we prove the reverse embedding. Let $f \in \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}$. Arguing as in (4.6), we have

$$\begin{aligned}\|f\|_{\overline{Y}_{0,\mathrm{b},E}} &\lesssim \Big\|\mathrm{b}(\rho(u))\|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)}\Big\|_{\widetilde{E}} \\ &\quad+\Big\|\rho(u)\mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}} \\ &\quad+\Big\|\rho(u)\mathrm{b}(\rho(u))\Big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\Big\|_{\widetilde{E}_1(u,\infty)}\Big\|_{\widetilde{E}} \\ &:= I_4+I_5+I_6.\end{aligned}$$

Clearly $I_4=\|f\|_{\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}}$. To estimate I_5 by I_4 one can argue as in (4.9) with $\theta_1-\theta_0>0$ and use (2.3) to obtain

$$\begin{aligned}I_5 &= \Big\|\rho(u)\mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}} \\ &= \Big\|u^{\theta_1-\theta_0}\frac{\mathrm{b}_0(u)}{a(u)}\mathrm{b}(\rho(u))\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}} \\ &\lesssim \Big\|u^{\theta_1-\theta_0}\frac{\mathrm{b}_0(u)}{a(u)}\mathrm{b}(\rho(u))\int_u^\infty t^{-\theta_1}a(t)K(t,f)\ \frac{dt}{t}\Big\|_{\widetilde{E}} \\ &\lesssim \Big\|u^{\theta_1-\theta_0}\frac{\mathrm{b}_0(u)}{a(u)}\mathrm{b}(\rho(u))u^{-\theta_1}a(u)K(u,f)\Big\|_{\widetilde{E}} \\ &= \|u^{-\theta_0}\mathrm{b}_0(u)\mathrm{b}(\rho(u))K(u,f)\|_{\widetilde{E}} \\ &\lesssim \Big\|\mathrm{b}(\rho(u))\|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)}\Big\|_{\widetilde{E}}=I_4.\end{aligned}$$

Now, we estimate I_6 by I_5 and then by I_4. Using the definition of $\rho(u)$ and Lemmas 4.4 and 2.3 (iii) we have that

$$\begin{aligned}I_6 &= \Big\|\rho(u)\mathrm{b}(\rho(u))\Big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\Big\|_{\widetilde{E}_1(u,\infty)}\Big\|_{\widetilde{E}} \\ &= \Big\|u^{\theta_1-\theta_0}\frac{\mathrm{b}_0(u)}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}\mathrm{b}(\rho(u))\Big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\Big\|_{\widetilde{E}_1(u,\infty)}\Big\|_{\widetilde{E}} \\ &\sim \Big\|u^{\theta_1-\theta_0}\frac{\mathrm{b}_0(u)\mathrm{b}_1(u)}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}\mathrm{b}(\rho(u))\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}} \\ &\lesssim \Big\|u^{\theta_1-\theta_0}\frac{\mathrm{b}_0(u)}{a(u)}\mathrm{b}(\rho(u))\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}}=I_5\lesssim I_4.\end{aligned}$$

Then, $\|f\|_{\overline{Y}_{0,\mathrm{b},E}}\lesssim I_4=\|f\|_{\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}}$ and the proof of b) is complete.

Finally, we proceed with the proof of c). Choose $f\in\overline{Y}_{1,\mathrm{b},E}$. Lemma 4.1, Theorem 3.1 a) and the lattice property guarantee that

$$\begin{aligned}\|f\|_{\overline{Y}_{1,\mathrm{b},E}} &\sim \|\rho(u)^{-1}\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\|_{\widetilde{E}} \\ &\gtrsim \Big\|\mathrm{b}(\rho(u))\,\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}}\end{aligned}$$

and

$$\|f\|_{\overline{Y}_{1,\mathrm{b},E}}\gtrsim\Big\|\mathrm{b}(\rho(u))\,\Big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\Big\|_{\widetilde{E}_1(u,\infty)}\Big\|_{\widetilde{E}}$$

and therefore $f \in \overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a,F} \cap \overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a,F}$. Let us prove the reverse embedding. Again, Lemma 4.1, Theorem 3.1 a) and the triangular inequality give that

$$\begin{aligned}\|f\|_{\overline{Y}_{1,\mathrm{b},E}} \lesssim & \left\| \rho(u)^{-1}\mathrm{b}(\rho(u))\|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)} \right\|_{\widetilde{E}} \\ & + \left\| \mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)} \right\|_{\widetilde{E}} \\ & + \left\| \mathrm{b}(\rho(u))\big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\big\|_{\widetilde{E}_1(u,\infty)} \right\|_{\widetilde{E}} := I_7 + I_8 + I_9.\end{aligned}$$

Since $I_8 = \|f\|_{\overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a,F}}$ and $I_9 = \|f\|_{\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a,F}}$, it is enough to estimate I_7. We proceed as before, applying the definition of $\rho(u)$, and using the equations (4.1), (4.3) and (2.5) it follows that

$$\begin{aligned} I_7 &= \left\| \rho(u)^{-1}\mathrm{b}(\rho(u))\|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)} \right\|_{\widetilde{E}} \\ &= \left\| u^{\theta_0-\theta_1}\frac{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}{\mathrm{b}_0(u)}\,\mathrm{b}(\rho(u))\|t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)} \right\|_{\widetilde{E}} \\ &\lesssim \left\| u^{\theta_0-\theta_1}\frac{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}{\mathrm{b}_0(u)}\,\mathrm{b}(\rho(u))\int_0^u t^{-\theta_0}\mathrm{b}_0(t)K(t,f)\,\frac{dt}{t} \right\|_{\widetilde{E}} \\ &\lesssim \left\| u^{-\theta_1}a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\mathrm{b}(\rho(u))K(u,f)\right\|_{\widetilde{E}} \\ &\lesssim \left\| \mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)} \right\|_{\widetilde{E}} = I_8. \end{aligned}$$

Hence

$$\|f\|_{\overline{Y}_{1,\mathrm{b},E}} \lesssim \max\left\{\|f\|_{\overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a,F}},\ \|f\|_{\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a,F}}\right\}$$

and the proof of c) is finished. □

Next we deal with the extreme case $\theta_0 = 0$.

Theorem 4.6. *Let $0 < \theta_1 < 1$ and let E, E_0, E_1 and F be r.i. spaces. Let a, b, b_0, $\mathrm{b}_1 \in SV$ with b_0 and b_1 satisfying $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$, $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$, respectively, and consider the function*

$$\rho(u) = u^{\theta_1}\frac{\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u > 0.$$

a) If $0 < \theta < 1$, then

$$(\overline{X}_{0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{\theta,\mathrm{b},E} = \overline{X}_{\tilde{\theta},B_\theta,E},$$

where

$$\tilde{\theta} = \theta\theta_1 \quad \textit{and} \quad B_\theta(u) = \left(\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\right)^{1-\theta}\left(a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\right)^{\theta}\mathrm{b}(\rho(u)),\ u > 0.$$

b) If $\theta = 0$ and $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, then

$$(\overline{X}_{0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{0,\mathrm{b},E} = \overline{X}_{0,B_0,E} \cap \overline{X}^{\mathcal{L}}_{0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0},$$

where $B_0(u) = \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\mathrm{b}(\rho(u))$, $u > 0$.

c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$*, then*

$$(\overline{X}_{0,\mathrm{b}_0,E_0}, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{1,\mathrm{b},E} = \overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a,F} \cap \overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a,F},$$

where $B_1(u) = \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\mathrm{b}(\rho(u))$*,* $u > 0$*.*

Proof. The proofs of a) and c) follow the same arguments used in the proofs of Theorem 4.5 a) and c). The only differences are the use of Theorem 3.1 b) instead of a) and the use of the inequality $\mathrm{b}_0(u) \lesssim \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}$, $u > 0$.

Next we prove b). As in Theorem 4.5 we use the notation $Y_0 = \overline{X}_{0,\mathrm{b}_0,E_0}$, $Y_1 = \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}$ and $\overline{K}(s,f) = K(s,f;Y_0,Y_1)$, $f \in Y_0 + Y_1$, $s > 0$. Again, Lemma 4.1 establishes that

$$\|f\|_{\overline{Y}_{0,\mathrm{b},E}} \sim \|\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\|_{\widetilde{E}}.$$

Then, to finish the proof it suffices to show that

$$\|\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\|_{\widetilde{E}} \sim \max\left\{\|f\|_{\overline{X}_{0,B_0,E}}, \|f\|_{\overline{X}^{\mathcal{L}}_{0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}}\right\}.$$

Theorem 3.1 b) and (2.5) guarantee that

$$\overline{K}(\rho(u),f) \gtrsim \|\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)}$$

and that

$$\begin{aligned}\overline{K}(\rho(u),f) &\gtrsim \rho(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)} \\ &= u^{\theta_1}\frac{\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)} \\ &\gtrsim u^{\theta_1}\frac{\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{a(u)}u^{-\theta_1}a(u)K(u,f) = \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}K(u,f).\end{aligned}$$

Hence,

$$\|\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\|_{\widetilde{E}} \gtrsim \max\left\{\|f\|_{\overline{X}_{0,B_0,E}}, \|f\|_{\overline{X}^{\mathcal{L}}_{0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}}\right\}.$$

Now, we prove the reverse inequality. Use Theorem3.1 b) and the triangular inequality to obtain that

$$\begin{aligned}\|\mathrm{b}(\rho(u))\,\overline{K}(\rho(u),f)\|_{\widetilde{E}} \lesssim &\left\|\mathrm{b}(\rho(u))\|\mathrm{b}_0(t)K(t,f)\|_{\widetilde{E}_0(0,u)}\right\|_{\widetilde{E}} \\ &+ \left\|\rho(u)\mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\right\|_{\widetilde{E}} \\ &+ \left\|\rho(u)\mathrm{b}(\rho(u))\left\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\right\|_{\widetilde{E}_1(u,\infty)}\right\|_{\widetilde{E}} \\ &:= I_{10} + I_{11} + I_{12}.\end{aligned}$$

The term I_{10} is precisely $\|f\|_{\overline{X}^{\mathcal{L}}_{0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}}$. The other two terms can be estimated by $\|f\|_{\overline{X}_{0,B_0,E}}$ proceeding as we did in (4.9) and with I_3. Indeed,

$$\begin{aligned} I_{11} &= \Big\|\rho(u)\mathrm{b}(\rho(u))\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}} \\ &= \Big\|u^{\theta_1}\frac{B_0(u)}{a(u)}\|t^{-\theta_1}a(t)K(t,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}} \\ &\lesssim \Big\|u^{\theta_1}\frac{B_0(u)}{a(u)}\int_u^\infty t^{-\theta_1}a(t)K(t,f)\ \frac{dt}{t}\Big\|_{\widetilde{E}} \\ &\lesssim \Big\|u^{\theta_1}\frac{B_0(u)}{a(u)}u^{-\theta_1}a(u)K(u,f)\Big\|_{\widetilde{E}} \\ &= \|B_0(u)K(u,f)\|_{\widetilde{E}} = \|f\|_{\overline{X}_{0,B_0,E}} \end{aligned}$$

and

$$\begin{aligned} I_{12} &= \Big\|\rho(u)\mathrm{b}(\rho(u))\big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\big\|_{\widetilde{E}_1(u,\infty)}\Big\|_{\widetilde{E}} \\ &= \Big\|u^{\theta_1}\frac{B_0(u)}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}\big\|\mathrm{b}_1(t)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(t,\infty)}\big\|_{\widetilde{E}_1(u,\infty)}\Big\|_{\widetilde{E}} \\ &\sim \Big\|u^{\theta_1}\frac{B_0(u)}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}\,\mathrm{b}_1(u)\|s^{-\theta_1}a(s)K(s,f)\|_{\widetilde{F}(u,\infty)}\Big\|_{\widetilde{E}} \lesssim I_{11}. \end{aligned}$$

The proof of b) is complete. □

Our last result of this subsection characterizes the reiteration space when the first space in the couple is X_0.

Theorem 4.7. *Let $0 < \theta_1 < 1$, and let E, E_1, F be r.i. spaces. Let a, b, $\mathrm{b}_1 \in SV$ with b_1 satisfying $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$ and consider the function*

$$\rho(u) = u^{\theta_1}\frac{1}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u > 0.$$

The following statements hold:

a) If $0 < \theta < 1$, or $\theta = 0$ and $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, then

$$(X_0, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{\theta,\mathrm{b},E} = \overline{X}_{\tilde{\theta},B_\theta,E},$$

where

$$\tilde{\theta} = \theta\theta_1 \quad \text{and} \quad B_\theta(u) = \big(a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\big)^{\theta}\mathrm{b}(\rho(u)),\ u > 0.$$

b) If $\theta = 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, then

$$(X_0, \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F})_{1,\mathrm{b},E} = \overline{X}^{\mathcal{R}}_{\theta_1,B_1,E,a,F} \cap \overline{X}^{\mathcal{R},\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1,a,F},$$

where $B_1(u) = \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\mathrm{b}(\rho(u))$, $u > 0$.

Proof. Again, as in Theorem 4.5 we use the notation $Y_0 = X_0$, $Y_1 = \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}_1,E_1,a,F}$ and $\overline{K}(s,f) = K(s,f;Y_0,Y_1)$, $f \in Y_0 + Y_1$, $s > 0$. Lemma 4.1 establishes the equivalence

$$\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}} \sim \|\rho(u)^{-\theta}\mathrm{b}(\rho(u))\overline{K}(\rho(u),f)\|_{\widetilde{E}},$$

for $0 \le \theta < 1$. Moreover, (3.2) and (2.5) imply that

$$\begin{aligned}
\|f\|_{\overline{Y}_{\theta,\mathrm{b},E}} &\gtrsim \Big\| \rho(u)^{1-\theta} \mathrm{b}(\rho(u)) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} \|t^{-\theta_1} a(t) K(t,f)\|_{\widetilde{F}(u,\infty)} \Big\|_{\widetilde{E}} \\
&\gtrsim \Big\| u^{\theta_1(1-\theta)} \Big(\frac{1}{a(u)\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}} \Big)^{1-\theta} \mathrm{b}(\rho(u)) \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)} u^{-\theta_1} a(u) K(u,f) \Big\|_{\widetilde{E}} \\
&= \|u^{-\theta\theta_1} B_\theta(u) K(u,f)\|_{\widetilde{E}} = \|f\|_{\tilde{\theta},B_\theta,E}.
\end{aligned}$$

Hence, the inclusion $\overline{Y}_{\theta,\mathrm{b},E} \hookrightarrow \overline{X}_{\tilde{\theta},B_\theta,E}$ is proved. The reverse inclusion can be done similarly to the estimate of I_2 and I_3 in the proof of Theorem 4.5 a).

The case $\theta = 1$ can be proved similarly to Theorem 4.5 c) with $I_7 = 0$. □

4.3. The Space $(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1})_{\theta,\mathrm{b},E}$, $0 < \theta_0 < \theta_1 \le 1$ and $0 \le \theta \le 1$

The results of this subsection can be proved using the same ideas we used to prove those of Sect. 4.2. However, since the proofs are lengthy, we will follow an alternative approach that uses symmetry arguments; besides, some of the proofs will be left to the reader.

Theorem 4.8. *Let $0 < \theta_0 < \theta_1 < 1$ and let E, E_0, E_1 and F be r.i. spaces. Let a, b, b_0, $\mathrm{b}_1 \in SV$ with b_0 satisfying $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$ and consider the function*

$$\rho(u) = u^{\theta_1-\theta_0} \frac{a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{\mathrm{b}_1(u)}, \quad u > 0.$$

Then, the following statements hold:

a) If $0 < \theta < 1$, then

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1})_{\theta,\mathrm{b},E} = \overline{X}_{\tilde{\theta},B_\theta,E},$$

where

$$\tilde{\theta} = (1-\theta)\theta_0 + \theta\theta_1 \quad \text{and} \quad B_\theta(u) = \big(a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\big)^{1-\theta} \big(\mathrm{b}_1(u)\big)^{\theta} \mathrm{b}(\rho(u)),\ u > 0.$$

b) If $\theta = 0$ and $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, then

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1})_{0,\mathrm{b},E} = \overline{X}^{\mathcal{L}}_{\theta_0,B_0,E,a,F} \cap \overline{X}^{\mathcal{L},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a,F},$$

where $B_0(u) = \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)} \mathrm{b}(\rho(u))$, $u > 0$.

c) If $\theta = 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, then

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1})_{1,\mathrm{b},E} = \overline{X}^{\mathcal{R}}_{\theta_1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1}.$$

Proof. We express the interpolation spaces by means of Lemma 2.9

$$\begin{aligned}
(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1})_{\theta,\mathrm{b},E} &= (\overline{X}_{\theta_1,\mathrm{b}_1,E_1}, \overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F})_{1-\theta,\overline{\mathrm{b}},E} \\
&= \big((X_1,X_0)_{1-\theta_1,\overline{\mathrm{b}}_1,E_1}, (X_1,X_0)^{\mathcal{R}}_{1-\theta_0,\overline{\mathrm{b}}_0,E_0,\overline{a},F}\big)_{1-\theta,\overline{\mathrm{b}},E} \qquad (4.9)
\end{aligned}$$

Here the functions $\overline{a}$, $\overline{b}$ and $\overline{b}_i$, $i = 0, 1$, have the usual meaning.

Taking $\theta = 0$ in (4.9) and applying Theorem 4.5 c) we have

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1})_{0,\mathrm{b},E}$$
$$= (X_1, X_0)^{\mathcal{R}}_{1-\theta_0,B_0^{\#},E,\overline{a},F} \cap (X_1, X_0)^{\mathcal{R},\mathcal{R}}_{1-\theta_0,\overline{\mathrm{b}}\circ\rho^{\#},E,\overline{\mathrm{b}}_0,E_0,\overline{a},F},$$

where $\rho^{\#}(u) = u^{\theta_1-\theta_0}\frac{\overline{\mathrm{b}}_1(u)}{\overline{a}(u)\|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,u)}}$ and $B_0^{\#}(u) = \|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,u)}\overline{\mathrm{b}}(\rho^{\#}(u))$, $u > 0$. Since $\|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,\frac{1}{u})} = \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}$ it yields that

$$\overline{B}_0^{\#}(u) = B_0^{\#}\Big(\frac{1}{u}\Big) = \|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,\frac{1}{u})}\overline{\mathrm{b}}\bigg(\Big(\frac{1}{u}\Big)^{\theta_1-\theta_0}\frac{\overline{\mathrm{b}}_1(\frac{1}{u})}{\overline{a}(\frac{1}{u})\|\overline{\mathrm{b}}_0\|_{\widetilde{E}_0(0,\frac{1}{u})}}\bigg)$$
$$= \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\mathrm{b}\Big(u^{\theta_1-\theta_0}\frac{a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{\mathrm{b}_1(u)}\Big) = B_0(u)$$

and $\overline{\overline{\mathrm{b}}\circ\rho^{\#}}(u) = \mathrm{b}(\rho(u))$, $u > 0$, and consequently Lemmas 2.9 and 2.12 show

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{\theta_1,\mathrm{b}_1,E_1})_{0,\mathrm{b},E} = \overline{X}^{\mathcal{L}}_{\theta_0,B_0,E,a,F} \cap \overline{X}^{\mathcal{L},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a,F}.$$

The cases $\theta = 1$ and $0 < \theta < 1$ can be proved similarly. □

Theorem 4.9. *Let $0 < \theta_0 < 1$ and let E, E_0, E_1 and F be r.i. spaces. Let a, b, b_0, $\mathrm{b}_1 \in SV$ with b_0 and b_1 satisfying $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$, $\|\mathrm{b}_1\|_{\widetilde{E}_1(0,1)} < \infty$, respectively, and consider the function*

$$\rho(u) = u^{1-\theta_0}\frac{a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}}{\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}}, \quad u > 0.$$

Then, the following statements hold:

a) If $0 < \theta < 1$, then

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{1,\mathrm{b}_1,E_1})_{\theta,\mathrm{b},E} = \overline{X}_{\tilde{\theta},B_\theta,E},$$

where

$$\tilde{\theta} = (1-\theta)\theta_0 + \theta \quad \textit{and} \quad B_\theta(u) = \big(a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\big)^{1-\theta}\big(\|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\big)^{\theta}\mathrm{b}(\rho(u)), \; u > 0.$$

b) If $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)} < \infty$, then

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{1,\mathrm{b}_1,E_1})_{0,\mathrm{b},E} = \overline{X}^{\mathcal{L}}_{\theta_0,B_0,E,a,F} \cap \overline{X}^{\mathcal{L},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a,F},$$

where $B_0(u) = \|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\mathrm{b}(\rho(u))$, $u > 0$.

c) If $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, then

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F}, \overline{X}_{1,\mathrm{b}_1,E_1})_{1,\mathrm{b},E} = \overline{X}_{1,B_1,E} \cap \overline{X}^{\mathcal{R}}_{1,\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1},$$

where $B_1(u) = \|\mathrm{b}_1\|_{\widetilde{E}_1(0,u)}\mathrm{b}(\rho(u))$, $u > 0$.

Theorem 4.10. *Let $0 < \theta_0 < 1$ and let E, E_0, E_1 and F be r.i. spaces. Let a, b, $\mathrm{b}_0 \in SV$ with b_0 satisfying $\|\mathrm{b}_0\|_{\widetilde{E}_0(1,\infty)} < \infty$ and consider the function*

$$\rho(u) = u^{1-\theta_0}a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}, \quad u > 0.$$

Then, the following statements hold:

a) If $0<\theta<1$, *or* $\theta=1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)}<\infty$, *then*

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F},X_1)_{\theta,\mathrm{b},E}=\overline{X}_{\tilde{\theta},B_\theta,E},$$

where

$$\tilde{\theta}=(1-\theta)\theta_0+\theta \quad \text{and} \quad B_\theta(u)=\big(a(u)\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\big)^{1-\theta}\mathrm{b}(\rho(u)), \quad u>0.$$

b) If $\|\mathrm{b}\|_{\widetilde{E}(1,\infty)}<\infty$, *then*

$$(\overline{X}^{\mathcal{L}}_{\theta_0,\mathrm{b}_0,E_0,a,F},X_1)_{0,\mathrm{b},E}=\overline{X}^{\mathcal{L}}_{\theta_0,B_0,E,a,F}\cap\overline{X}^{\mathcal{L},\mathcal{L}}_{\theta_0,\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0,a,F},$$

where $B_0(u)=\|\mathrm{b}_0\|_{\widetilde{E}_0(u,\infty)}\mathrm{b}(\rho(u))$, $u>0$.

Remark 4.11. We observe that generalized Holmstedt-type formula (3.2) (or (3.6)) holds when the space X_0 (or X_1) is replaced by an intermediate space $\widetilde{X}_0$ of class 0 (or $\widetilde{X}_1$ of class 1, respectively); see [8, Chap. 5] for the definition. Consequently, Theorem 4.7 is also true for any intermediate space $\widetilde{X}_0$ of class 0 and Theorem 4.10 is true for any intermediate space $\widetilde{X}_1$ of class 1.

5. Applications

The applications we consider in this section will involve ordered (quasi)-Banach couples $\overline{X}=(X_0,X_1)$, in the sense that $X_1\hookrightarrow X_0$. First, we briefly review how our conditions adapt to this simpler setting.

5.1. Ordered Couples

Given a real parameter $0\le\theta\le1$, $a,\mathrm{b},c\in SV(0,1)$ and r.i. spaces E,F,G on $(0,1)$, the spaces $\overline{X}_{\theta,\mathrm{b},E}$, $\overline{X}^{\mathcal{L}}_{\theta,\mathrm{b},E,a,F}$ and $\overline{X}^{\mathcal{L},\mathcal{L}}_{\theta,c,E,\mathrm{b},F,a,G}$ are defined just as in Definitions 2.5, 2.7 and 2.11; the only change being that $\widetilde{E}(0,\infty)$ must be replaced by $\widetilde{E}(0,1)$, see [26]. Likewise the spaces $\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}$ and $\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G}$ are defined as

$$\overline{X}^{\mathcal{R}}_{\theta,\mathrm{b},E,a,F}=\Big\{f\in X_0:\big\|\mathrm{b}(t)\|s^{-\theta}a(s)K(s,f)\|_{\widetilde{F}(t,1)}\big\|_{\widetilde{E}(0,1)}<\infty\Big\}$$

and

$$\overline{X}^{\mathcal{R},\mathcal{R}}_{\theta,c,E,\mathrm{b},F,a,G}=\Big\{f\in X_0:\Big\|c(u)\big\|\mathrm{b}(t)\|s^{-\theta}a(s)K(s,f)\|_{\widetilde{G}(t,1)}\big\|_{\widetilde{F}(u,1)}\Big\|_{\widetilde{E}(0,1)}<\infty\Big\}.$$

Of course, all the results in the paper remain true if we work with an ordered couple and use as parameters slowly varying functions on $(0,1)$ and r.i. spaces on $(0,1)$. In these cases, all assumptions concerning the interval $(1,\infty)$ must be omitted.

It is worth to mention that if the couple is ordered, then the scale $\{\overline{X}_{\theta,\mathrm{b},E}\}_{0\le\theta\le1}$ is also ordered.

Lemma 5.1 [26, Lemma 5.2]. *Let* $\overline{X}$ *be an ordered (quasi-) Banach couple,* b_0, $\mathrm{b}_1\in SV(0,1)$ *and* E_0, E_1 *r.i. spaces on* $(0,1)$. *If* $0\le\theta_0<\theta_1\le1$, *then*

$$\overline{X}_{\theta_1,\mathrm{b}_1,E_1}\hookrightarrow\overline{X}_{\theta_0,\mathrm{b}_0,E_0}.$$

5.2. Grand and Small Lebesgue Spaces

Next, we apply our previous results to the grand and small Lebesgue spaces. Following the paper by Fiorenza and Karadzhov [28] we give the following definition:

Definition 5.2. Let (Ω,μ) be a finite measure space with non-atomic measure μ and assume that $\mu(\Omega)=1$. Let $1<p<\infty$ and $\alpha>0$. The space $L^{p),\alpha}(\Omega)$ is the set of all measurable functions f on (Ω,μ) such that

$$\|f\|_{p),\alpha} = \left\| \ell^{-\frac{\alpha}{p}}(t)\|f^*(s)\|_{L_p(t,1)} \right\|_{L_\infty(0,1)} < \infty.$$

The small Lebesgue space $L^{(p,\alpha}(\Omega)$ is the set of all measurable functions f on (Ω,μ) such that

$$\|f\|_{(p,\alpha} = \left\| \ell^{\frac{\alpha}{p'}-1}(t)\|f^*(s)\|_{L_p(0,t)} \right\|_{\widetilde{L}_1(0,1)} < \infty,$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

The classical grand Lebesgue space $L^{p)}(\Omega) := L^{p),1}(\Omega)$ was introduced by Iwaniec and Sbordone in [36] in connection with the integrability properties of the Jacobian under minimal hypothesis. The classical small Lebesgue space $L^{(p}(\Omega) := L^{(p,1}(\Omega)$ was introduced by Fiorenza in [27] as associate to the grand Lebesgue spaces; that is $(L^{(p'})' = L^{p)}$. Since then many authors have studied relevant properties of these spaces, such as interpolation, boundedness of classical operators, generalized versions, etc. For more information about this spaces and their generalizations see the recent paper [29] and the references [2,4,6,13,19,24,30].

We shall also consider ultrasymmetric spaces.

Definition 5.3. Let $1\le p<\infty$, $\mathrm{b}\in SV$ and E an r.i. space. The ultrasymmetric space $L_{p,\mathrm{b},E}(\Omega,\mu)$ is the set of all measurable functions on (Ω,μ) such that

$$\|f\|_{L_{p,\mathrm{b},E}} = \|t^{1/p}\mathrm{b}(t)f^*(t)\|_{\widetilde{E}} < \infty.$$

This class of spaces was introduced and studied by E. Pustylnik [47] and comprises many classical examples as *Lorentz–Karamata* spaces $L_{p,q;\mathrm{b}}$ (see [33,40]), generalized *Lorentz–Zygmund* spaces [43], *Lorentz–Zygmund* spaces $L^{p,q}(\log L)^\alpha$ (see [7,8]) and some Orlicz spaces. In case $E=L_q$ and $b\equiv 1$, we have the classical *Lorentz space* $L_{p,q}$ and the *Lebesgue space* L_p.

For convenience, we will denote the function spaces as $L^{p),\alpha}$, $L^{(p,\alpha}$, L_p, etc..., dropping the dependence with respect to the domain (Ω,μ).

Ultrasymmetric spaces are interpolation spaces for the couple (L_1,L_∞). Indeed, Peetre's well-known formula [8,44]

$$K(t,f;L_1,L_\infty) = \int_0^t f^*(s)\,ds = tf^{**}(t), \quad t>0,$$

and the equivalence $\|t^{1-\theta}\mathrm{b}(t)f^{**}(t)\|_{\widetilde{E}} \sim \|t^{1-\theta}\mathrm{b}(t)f^*(t)\|_{\widetilde{E}}$ for $0<\theta<1$ (see, e.g., [16, Lemma 2.16]), yield the equality

$$L_{p,\mathrm{b},E} = (L_1,L_\infty)_{1-\frac{1}{p},\mathrm{b},E} \tag{5.1}$$

for any r.i. space E, $\mathrm{b}\in SV(0,1)$ and $1<p<\infty$.

Grand and small Lebesgue spaces are limiting interpolation spaces for the couple (L_1, L_p) and (L_p, L_∞), respectively. Moreover they can also be characterized as $\mathcal{R}$ and $\mathcal{L}$-spaces, respectively. In fact, we can observe from Definition 5.2 that

$$L^{p),\alpha} = (L_1, L_p)_{1,\ell^{-\frac{\alpha}{p}}(u),L_\infty}.$$

Then, using the reiteration formula (4.14) from [33] or [22, Th. 6.12], they can be characterized as $\mathcal{R}$ spaces

$$L^{p),\alpha} = (L_1, (L_1, L_\infty)_{1-\frac{1}{p},1,L_p})_{1,\ell^{-\frac{\alpha}{p}}(u),L_\infty} = (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p},\ell^{-\frac{\alpha}{p}}(u),L_\infty,1,L_p}. \tag{5.2}$$

Similarly, using the reiteration formula (3.21) from [33] or [22, Th. 6.11], the small Lebesgue spaces can be seen as $\mathcal{L}$-spaces

$$\begin{aligned} L^{(p,\alpha} &= (L_p, L_\infty)_{0,\ell^{\frac{\alpha}{p'}-1}(u),L_1} \\ &= \big((L_1, L_\infty)_{1-\frac{1}{p},1,L_p}, L_\infty\big)_{0,\ell^{\frac{\alpha}{p'}-1}(u),L_1} \\ &= (L_1, L_\infty)^{\mathcal{L}}_{1-\frac{1}{p},\ell^{\frac{\alpha}{p'}-1}(u),L_1,1,L_p}. \end{aligned} \tag{5.3}$$

Since in Corollary 5.5 from [26] we interpolate the grand Lebesgue spaces with the ultrasymmetric spaces included in them, now Theorem 4.5 allows us to obtain the "dual" situation. In other to do that we need some previous considerations. First of all, if $1 < p_0 < p_1 < \infty$ and $\alpha > 0$ then $(L_{p_0,q;b_0}, L^{p_1),\alpha})$ is an ordered couple. Indeed,

$$L^{p_1),\alpha} \hookrightarrow L^{p_1,\infty}(\log L)^{-\frac{\alpha}{p_1}} \hookrightarrow L_{p_0,q;b_0}.$$

We will also need the following technical lemma.

Lemma 5.4. *[21, Lemma 6.1] If $\sigma + \frac{1}{q} < 0$ with $1 \le q < \infty$ or $q = \infty$ and $\sigma \le 0$, then*

$$\|\ell^\sigma(t)\|_{\widetilde{L}_q(0,u)} \sim \ell^{\sigma+\frac{1}{q}}(u), \quad u \in (0,1). \tag{5.4}$$

If $\sigma + \frac{1}{q} > 0$ with $1 \le q < \infty$, or $q = \infty$ and $\sigma \ge 0$, then

$$\|\ell^\sigma(t)\|_{\widetilde{L}_q(u,1)} \sim \ell^{\sigma+\frac{1}{q}}(u), \quad u \in (0,1/2). \tag{5.5}$$

Notice also that if $\mathrm{b}(t) \sim a(t)$ for all $t \in (0,1/2)$, then the monotonicity properties of the K-functional and the properties of the slowly varying functions imply that

$$\overline{X}_{\theta,\mathrm{b},E} = \overline{X}_{\theta,a,E}.$$

Thus, for any $0 < \theta < 1$ and any r.i. space E,

$$\overline{X}_{\theta,\|\ell^\sigma(t)\|_{\widetilde{L}_q(u,1)},E} = \overline{X}_{\theta,\ell^{\sigma+\frac{1}{q}}(u),E}. \tag{5.6}$$

Corollary 5.5. *Let E, E_0 be r.i. spaces, $\mathrm{b}, \mathrm{b}_0 \in SV(0,1)$, $1 < p_0 < p_1 < \infty$ and $\beta > 0$. Consider the function $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}}\mathrm{b}_0(u)\ell^{\frac{\beta}{p_1}}(u)$, $u \in (0,1)$.*

a) If $0 < \theta < 1$, then

$$\big(L_{p_0,\mathrm{b}_0,E_0}, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E}, \tag{5.7}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $B_\theta(u) = \mathrm{b}_0^{1-\theta}(u)\ell^{\frac{-\beta\theta}{p_1}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

b) If $\theta = 0$*, then*

$$\left(L_{p_0,\mathrm{b}_0,E_0}, L^{p_1),\beta}\right)_{0,\mathrm{b},E} = (L_1, L_\infty)^{\mathcal{L}}_{1-\frac{1}{p_0},\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}. \tag{5.8}$$

c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$*, then*

$$\begin{aligned} &\left(L_{p_0,\mathrm{b}_0,E_0}, L^{p_1),\beta}\right)_{1,\mathrm{b},E} \\ &= (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p_1},B_1,E,1,L_{p_1}} \cap (L_1, L_\infty)^{\mathcal{R},\mathcal{R}}_{1-\frac{1}{p_1},\mathrm{b}\circ\rho,E,\ell^{-\frac{\beta}{p_1}}(u),L_\infty,1,L_{p_1}} \end{aligned} \tag{5.9}$$

where $B_1(u) = \ell^{\frac{-\beta}{p_1}}(u)\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

Proof. Let $0 < \theta < 1$. Using (5.1), (5.2) and Theorem 4.5 a) we obtain that

$$\begin{aligned} &\left(L_{p_0,\mathrm{b}_0,E_0}, L^{p_1),\beta}\right)_{\theta,\mathrm{b},E} \\ &= \left((L_1, L_\infty)_{1-\frac{1}{p_0},\mathrm{b}_0,E_0}, (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p_1},\ell^{-\frac{\beta}{p_1}}(u),L_\infty,1,L_{p_1}}\right)_{\theta,\mathrm{b},E} \\ &= (L_1, L_\infty)_{\widetilde{\theta},B_\theta,E}, \end{aligned}$$

where $\widetilde{\theta} = 1 - \left(\frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right)$ and

$$B_\theta(u) = \mathrm{b}_0^{1-\theta}(u)\|\ell^{-\frac{\beta}{p_1}}(t)\|^{\theta}_{\widetilde{L}_\infty(0,u)}\mathrm{b}\left(u^{\frac{1}{p_0}-\frac{1}{p_1}}\frac{\mathrm{b}_0(u)}{\|\ell^{-\beta/p_1}(t)\|_{\widetilde{L}_\infty(0,u)}}\right), \quad u \in (0,1).$$

We may apply (5.1) to obtain (5.7). The proofs of the cases $\theta = 0, 1$ can be done similarly. □

In particular, the choice in (5.8) of $E = L_1$, $E_0 = L_{p_0}$ and functions $\mathrm{b}_0 \equiv 1$, $\mathrm{b}(u) = \ell^{\frac{\alpha}{p_0'}-1}(u)$, $u \in (0,1)$, gives the following result.

Corollary 5.6. *Let* $1 < p_0 < p_1 < \infty$ *and* $\alpha, \beta > 0$*, then*

$$\left(L_{p_0}, L^{p_1),\beta}\right)_{0,\ell^{\frac{\alpha}{p_0'}-1}(u),L_1} = L^{(p_0,\alpha}. \tag{5.10}$$

Now, we state the interpolation formulae for a couple formed by a grand and a small Lebesgue space. The result recovers Theorem 5.1 from [30], and completes it with the extreme cases $\theta = 0$, 1. Moreover, for $0 < \theta < 1$, this is a special case of [1, Theorem 6.5].

Theorem 5.7. *Let* $1 < p_0 < p_1 < \infty$*,* $1 \le r \le \infty$ *and* $\alpha, \beta > 0$*.*

a) If $0 < \theta < 1$*, then*

$$(L^{(p_0,\alpha}, L^{p_1),\beta})_{\theta,r} = L^{p,r}(\log L)^A,$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $A = \frac{\alpha(1-\theta)}{p_0'} - \frac{\beta\theta}{p_1}$*.*

b) If $\theta = 0$*, then*

$$(L^{(p_0,\alpha}, L^{p_1),\beta})_{0,r} = (L_1, L_\infty)^{\mathcal{L}}_{1-\frac{1}{p_0},\ell^{\frac{\alpha}{p_0'}}(u),L_r,1,L_{p_0}} \cap (L_{p_0}, L^{p_1),\beta})^{\mathcal{L}}_{0,1,L_r,\ell^{\frac{\alpha}{p_0'}-1}(u),L_1}.$$

c) If $\theta = 1$ *and* $\mathrm{b} \in SV(0,1)$ *is such that* $\|\mathrm{b}\|_{\widetilde{L}_r(0,1)} < \infty$*, then*

$$\begin{aligned} &(L^{(p_0,\alpha}, L^{p_1),\beta})_{1,\mathrm{b},L_r} \\ &= (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p_1},B_1,L_r,1,L_{p_1}} \cap (L_1, L_\infty)^{\mathcal{R},\mathcal{R}}_{1-\frac{1}{p_1},\mathrm{b}\circ\rho,L_r,\ell^{-\frac{\beta}{p_1}}(u),L_\infty,1,L_{p_1}}, \end{aligned}$$

where $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}}\ell^{\frac{\alpha}{p_0'}+\frac{\beta}{p_1}}(u)$ *and* $B_1(u) = \ell^{-\frac{\beta}{p_1}}(u)\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

Proof. Let $0<\theta<1$. Applying equality (5.10), Theorem 5.12 from [24] and (5.5) we obtain the identity

$$\big(L^{(p_0,\alpha}, L^{p_1),\beta}\big)_{\theta,r} = \big((L_{p_0}, L^{p_1),\beta})_{0,\ell^{\frac{\alpha}{p_0'}-1}(u),L_1}, L^{p_1),\beta}\big)_{\theta,r} = \big(L_{p_0}, L^{p_1),\beta}\big)_{\theta,\ell^{\frac{\alpha(1-\theta)}{p_0'}}(u),L_r}.$$

Now use (5.7) to establish a). The limiting case $\theta=1$ follows from (5.9).

Finally, assume that $\theta=0$. Then, (5.10) together with Theorem 5.13 from [24] and (5.5) establish that

$$\big(L^{(p_0,\alpha}, L^{p_1),\beta})_{0,r} = \big((L_{p_0}, L^{p_1),\beta})_{0,\ell^{\frac{\alpha}{p_0'}-1}(u),L_1}, L^{p_1),\beta}\big)_{0,r} = \big(L_{p_0}, L^{p_1),\beta}\big)_{0,\ell^{\frac{\alpha}{p_0'}}(u),L_r} \cap \big(L_{p_0}, L^{p_1),\beta}\big)^{\mathcal{L}}_{0,1,L_r,\ell^{\frac{\alpha}{p_0'}-1}(u),L_1}.$$

Now, applying (5.8) we have that

$$\big(L^{(p_0,\alpha}, L^{p_1),\beta})_{0,r} = (L_1,L_\infty)^{\mathcal{L}}_{1-\frac{1}{p_0},\ell^{\frac{\alpha}{p_0'}}(u),L_r,1,L_{p_0}} \cap (L_{p_0}, L^{p_1),\beta})^{\mathcal{L}}_{0,1,L_r,\ell^{\frac{\alpha}{p_0'}-1}(u),L_1}.$$

□

Now, we identify the spaces $\big(L\log L, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E}$ and $\big(L_1, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E}$ for $1<p_1<\infty$, $\beta>0$ and all possible values of $\theta\in[0,1]$. Remember that

$$(L_1,L_\infty)_{0,1,L_1} = L\log L \tag{5.11}$$

and $L^{p_1),\beta} \hookrightarrow L\log L \hookrightarrow L_1$ which makes $(L\log L, L^{p_1),\beta})$ and $(L_1, L^{p_1),\beta})$ ordered couples.

Corollary 5.8. *Let E be an r.i. space, $\mathrm{b}\in SV(0,1)$, $1<p_1<\infty$ and $\beta>0$. Consider the function $\rho(u)=u^{1-\frac{1}{p_1}}\ell^{1+\frac{\beta}{p_1}}(u)$, $u\in(0,1)$.*

a) If $0<\theta<1$, then

$$\big(L\log L, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E}, \tag{5.12}$$

where $\frac{1}{p}=1-\theta+\frac{\theta}{p_1}$ and $B_\theta(u)=\ell^{1-\theta-\frac{\beta\theta}{p_1}}(u)\mathrm{b}(\rho(u))$, $u\in(0,1)$.

b) If $\theta=0$, then

$$\big(L\log L, L^{p_1),\beta}\big)_{0,\mathrm{b},E} = (L_1,L_\infty)_{0,B_0,E} \cap (L_1,L_\infty)^{\mathcal{L}}_{0,\mathrm{b}\circ\rho,E,1,L_1},$$

where $B_0(u)=\ell(u)\mathrm{b}(\rho(u))$, $u\in(0,1)$.

c) If $\theta=1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)}<\infty$, then

$$\big(L\log\ L, L^{p_1),\beta}\big)_{1,\mathrm{b},E} = (L_1,L_\infty)^{\mathcal{R}}_{1-\frac{1}{p_1},B_1,E,1,L_{p_1}} \cap (L_1,L_\infty)^{\mathcal{R},\mathcal{R}}_{1-\frac{1}{p_1},\mathrm{b}\circ\rho,E,\ell^{-\frac{\beta}{p_1}}(u),1,L_{p_1}},$$

where $B_1(u)=\ell^{-\frac{\beta\theta}{p_1}}(u)\mathrm{b}(\rho(u))$, $u\in(0,1)$.

Proof. We prove a). By equalities (5.2), (5.11) and Theorem 4.6, we have that

$$\big(L\log L, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E} = \Big((L_1,L_\infty)_{0,1,L_1}, (L_1,L_\infty)^{\mathcal{R}}_{1-\frac{1}{p_1},\ell^{-\frac{\beta}{p_1}}(u),L_\infty,1,L_{p_1}}\Big)_{\theta,\mathrm{b},E} = (L_1,L_\infty)_{\tilde\theta,B_\theta,E},$$

where $\tilde{\theta} = \theta\big(1 - \frac{1}{p_1}\big)$ and

$$B_\theta(u) = \|1\|^{1-\theta}_{\widetilde{L}_1(u,1)} \|\ell^{-\frac{\beta}{p_1}}(t)\|^{\theta}_{\widetilde{L}_\infty(0,u)} \mathrm{b}\Big(u^{1-\frac{1}{p_1}} \frac{\|1\|_{\widetilde{L}_1(u,1)}}{\|\ell^{-\beta/p_1}(t)\|_{\widetilde{L}_\infty(0,u)}}\Big),\ u \in (0,1).$$

Besides, it follows from equivalences (5.4) and (5.5) that

$$B_\theta(u) \sim \ell^{1-\theta-\frac{\beta\theta}{p_1}}(u)\mathrm{b}(u^{1-\frac{1}{p_1}}\ell^{1+\frac{\beta}{p_1}}(u)), \quad u \in (1,1/2).$$

Hence, (5.6) and (5.1) yield (5.12). The remaining cases can be proved similarly. □

Moreover, Theorem 4.7 enables us to identify the space $\big(L_1, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E}$, $0 \le \theta \le 1$.

Corollary 5.9. *Let E be an r.i. space, $\mathrm{b} \in SV(0,1)$, $1 < p_1 < \infty$ and $\beta > 0$. Consider the function $\rho(u) = u^{1-\frac{1}{p_1}}\ell^{\frac{\beta}{p_1}}(u)$, $u \in (0,1)$.*

a) If $0 \le \theta < 1$, then

$$\big(L_1, L^{p_1),\beta}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E}$$

where $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$ and $B_\theta(u) = \ell^{-\frac{\beta\theta}{p_1}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

b) If $\theta = 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, then

$$\big(L_1, L^{p_1),\beta}\big)_{1,\mathrm{b},E} = (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p_1},B_1,E,1,L_{p_1}} \cap (L_1, L_\infty)^{\mathcal{R},\mathcal{R}}_{1-\frac{1}{p_1},\mathrm{b}\circ\rho,E,\ell^{-\frac{\beta}{p_1}}(u),1,L_{p_1}}$$

where $B_1(u) = \ell^{-\frac{\beta}{p_1}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

Our results also allow to interpolate couples formed by small Lebesgue spaces and ultrasymmetric spaces embedded onto them. Observe that Proposition 5.6 of [12] and Lemma 5.1 establish that if $1 < p_0 < p_1 < \infty$, $\alpha > 0$ and $\beta > 1$, then

$$L_{p_1,\mathrm{b}_1,E_1} \hookrightarrow L^{p_0}(\log L)^{\frac{\beta\alpha}{p_0'-1}} \hookrightarrow L^{(p_0,\alpha}.$$

The proofs of the following results can be carried out using similar arguments to those of the previous corollaries, so we omit the details.

Corollary 5.10. *Let E, E_1 be r.i. spaces, $\mathrm{b} \in SV(0,1)$, $1 < p_0 < p_1 < \infty$ and $\alpha > 0$. Consider the function $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}}\ell^{\frac{\alpha}{p_0'}}(u)\mathrm{b}_1^{-1}(u)$, $u > 0$.*

a) If $0 < \theta < 1$, then

$$\big(L^{(p_0,\alpha}, L_{p_1,\mathrm{b}_1,E_1}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $B_\theta(u) = \ell^{\frac{\alpha(1-\theta)}{p_0'}}(u)\mathrm{b}_1^{\theta}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

b) If $\theta = 0$, then

$$\begin{aligned}&\big(L^{(p_0,\alpha}, L_{p_1,\mathrm{b}_1,E_1}\big)_{0,\mathrm{b},E}\\ &= (L_1, L_\infty)^{\mathcal{L}}_{1-\frac{1}{p_0},B_0,E,1,L_{p_0}} \cap (L_1, L_\infty)^{\mathcal{L},\mathcal{L}}_{1-\frac{1}{p_0},\mathrm{b}\circ\rho,E,\ell^{\frac{\alpha}{p_0'}-1}(u),L_1,1,L_{p_0}},\end{aligned}$$

where $B_0(u) = \ell^{\frac{\alpha}{p_0'}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

c) If $\theta = 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, then

$$\big(L^{(p_0,\alpha}, L_{p_1,\mathrm{b}_1,E_1}\big)_{1,\mathrm{b},E} = (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p_1},\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1}.$$

In particular,

$$\big(L^{(p_0,\alpha}, L_{p_1}\big)_{1,\ell^{\frac{-\beta}{p_1}}(u),L_\infty} = L^{p_1),\beta}.$$

We are also able to interpolate the small Lebesque space $L^{(p_0,\alpha}$ with the Lorentz–Zygmund spaces $L_{\infty,q}(\log L)^\beta$ and $L^\beta_{\exp}$. These spaces consist of all measurable functions f on $(0,1)$ for which the respective norms

$$\|f\|_{\infty,q,\beta} = \Big(\int_0^1 \big(\ell^\beta(t) f^*(t)\big)^q \frac{dt}{t}\Big)^{1/q}, \qquad \text{if } \beta + \frac{1}{q} < 0,\ (1 \le q < \infty)$$

$$\|f\|_{L^\beta_{\exp}} = \sup_{0<t<1} \ell^\beta(t) f^*(t), \qquad \text{if } \beta \le 0,\ (q = \infty)$$

are finite (see [7]). For simplicity, we jointly denote them by $L_{\infty,q,\beta}$, when $1 \le q \le \infty$. They are ultrasymmetric spaces (see [25, Example 4.4]) and they can be identified as interpolation spaces between L_1 and L_∞ in the following way

$$(L_1, L_\infty)_{1,\ell^\beta(u),L_q} = L_{\infty,q,\beta}.$$

Corollary 5.11. *Let E be an r.i. space,* $\mathrm{b} \in SV(0,1)$, $1 < p_0 < \infty$ *and* $1 \le q_1 \le \infty$. *Assume that $\alpha > 0$ and $\beta + \frac{1}{q_1} < 0$, or $\beta \le 0$ if $q_1 = \infty$, and consider the function $\rho(u) = u^{\frac{1}{p_0}} \ell^{\frac{\alpha}{p_0'} - (\beta + \frac{1}{q_1})}(u)$, $u \in (0,1)$.*

a) If $0 < \theta < 1$, then

$$\big(L^{(p_0,\alpha}, L_{\infty,q_1,\beta}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0}$ and $B_\theta(u) = \ell^{(1-\theta)\frac{\alpha}{p_0'} + \theta(\beta + \frac{1}{q_1})}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

b) If $\theta = 0$, then

$$\big(L^{(p_0,\alpha}, L_{\infty,q_1,\beta}\big)_{0,\mathrm{b},E} = \big(L_1, L_\infty\big)^{\mathcal{L}}_{1-\frac{1}{p_0},B_0,E,1,L_{p_0}} \cap \big(L_1, L_\infty\big)^{\mathcal{L},\mathcal{L}}_{1-\frac{1}{p_0},\mathrm{b}\circ\rho,E,\ell^{\frac{\alpha}{p_0'}-1}(u),L_1,1,L_{p_0}},$$

where $B_0(u) = \ell^{\frac{\alpha}{p_0'}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

c) If $\theta = 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, then

$$\big(L^{(p_0,\alpha}, L_{\infty,q_1,\beta}\big)_{1,\mathrm{b},E} = \big(L_1, L_\infty\big)_{1,B_1,E} \cap \big(L_1, L_\infty\big)^{\mathcal{R}}_{1,\mathrm{b}\circ\rho,E,\ell^\beta(u),L_{q_1}}$$

where $B_1(u) = \ell^{\beta + \frac{1}{q_1}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

Finally, we can apply Theorem 4.10 to deduce:

Corollary 5.12. *Let E be an r.i. space,* $\mathrm{b} \in SV(0,1)$, $1 < p_0 < \infty$ *and* $\alpha > 0$. *Consider the function $\rho(u) = u^{\frac{1}{p_0}} \ell^{\frac{\alpha}{p_0'}}(u)$, $u \in (0,1)$.*

a) If $0 < \theta < 1$, or $\theta = 1$ and $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, then

$$\big(L^{(p_0,\alpha}, L_\infty\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0}$ and $B_\theta(u) = \ell^{\frac{\alpha(1-\theta)}{p_0'}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

b) If $\theta = 0$, then

$$\big(L^{(p_0,\alpha}, L_\infty\big)_{0,\mathrm{b},E} = \big(L_1, L_\infty\big)^{\mathcal{L}}_{1-\frac{1}{p_0},B_0,E,1,L_{p_0}} \cap \big(L_1, L_\infty\big)^{\mathcal{L},\mathcal{L}}_{1-\frac{1}{p_0},\mathrm{b}\circ\rho,E,\ell^{\frac{\alpha}{p_0'}-1}(u),L_1,1,L_{p_0}},$$

where $B_0(u) = \ell^{\frac{\alpha}{p_0'}}(u)\mathrm{b}(\rho(u))$, $u \in (0,1)$.

5.3. Generalized Gamma Spaces.

Our results can also be applied to the Generalized Gamma spaces with double weight defined in [30].

Definition 5.13. Let $1 \le p < \infty$, $1 \le q \le \infty$ and w_1, w_2 two weights on $(0,1)$ satisfying the following conditions:

(c1) There exist $K_{12} > 0$ such that $w_2(2t) \le K_{12} w_2(t)$, for all $t \in (0,1/2)$. The space $L^p(0,1;w_2)$ is continuously embedded in $L^1(0,1)$.

(c2) The function $\int_0^t w_2(s)ds$ belongs to $L^{\frac{q}{p}}(0,1;w_1)$.

The *Generalized Gamma* space with double weights $G\Gamma(p,q,w_0,w_1)$ is the set of all measurable functions f on $(0,1)$ such that

$$\|f\|_{G\Gamma} = \left(\int_0^1 w_1(t)\left(\int_0^t w_2(s)(f^*(s))^p\,ds\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} < \infty.$$

These spaces are a generalization of the $G\Gamma(p,q,w_0) := G\Gamma(p,q,w_0,1)$, introduced in [31], while the spaces $G\Gamma(p,\infty,w_0,w_1)$ appeared in [32].

If we assume that $uw_1(u)$ and w_2 are slowly varying functions, we can identify the Generalized Gamma space as an $\mathcal{L}$-space in the following way

$$G\Gamma(p,q,w_1,w_2) = (L_1,L_\infty)^{\mathcal{L}}_{1-\frac{1}{p},(uw_1(u))^{\frac{1}{q}},L_q,(w_2(u))^{\frac{1}{p}},L_p}.$$

Thus, we can apply the results from §4 to interpolate these spaces with ultrasymmetric spaces, with Lorentz-Zygmund spaces $L_{\infty,q}(\log L)^\beta$, $L^\beta_{\exp}$ and also with L_∞. We present here only the first case.

Corollary 5.14. *Let* E, E_1 *be r.i. spaces and* $\mathrm{b}, \mathrm{b}_1, uw_1(u), w_2 \in SV(0,1)$. *Let* $1 < p_0 < p_1 < \infty$, $1 \le q_0 \le \infty$ *and consider the function*

$$\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \frac{(w_2(u))^{\frac{1}{p_0}}\|(tw_1(t))^{\frac{1}{q_0}}\|_{\widetilde{L}_{q_0}(u,1)}}{\mathrm{b}_1(u)}, \quad u \in (0,1).$$

a) If $0 < \theta < 1$, *then*

$$\left(G\Gamma(p_0,q_0,w_1,w_2), L_{p_1,\mathrm{b}_1,E_1}\right)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E}$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and*

$$B_\theta(u) = \left((w_2(u))^{\frac{1}{p_0}}\|(tw_1(t))^{\frac{1}{q_0}}\|_{\widetilde{L}_{q_0}(u,1)}\right)^{1-\theta} \mathrm{b}_1^\theta(u)\mathrm{b}(\rho(u)),\ u \in (0,1).$$

b) If $\theta = 0$ *then*

$$\begin{aligned}\left(G\Gamma(p_0,q_0,w_1,w_2), L_{p_1,\mathrm{b}_1,E_1}\right)_{0,\mathrm{b},E} &= (L_1,L_\infty)^{\mathcal{L}}_{1-\frac{1}{p_0},B_0,E,(w_2(u))^{\frac{1}{p_0}},L_{p_0}} \\ &\cap (L_1,L_\infty)^{\mathcal{L},\mathcal{L}}_{1-\frac{1}{p_0},\mathrm{b}\circ\rho,E,(uw_1(u))^{\frac{1}{q_0}},L_{q_0},(w_2(u))^{\frac{1}{p_0}},L_{p_0}},\end{aligned}$$

where $B_0(u) = \|(tw_1(t))^{\frac{1}{q_0}}\|_{\widetilde{L}_{q_0}(u,1)}\mathrm{b}(\rho(u))$, $u \in (0,1)$.

c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$, *then*

$$\left(G\Gamma(p_0,q_0,w_1,w_2), L_{p_1,\mathrm{b}_1,E_1}\right)_{1,\mathrm{b},E} = (L_1,L_\infty)^{\mathcal{R}}_{1-\frac{1}{p_1},\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1}.$$

The results from [26] allow to obtain the range $p_0 > p_1$.

5.4. *A*- and *B*-Type Spaces.

Finally, we consider the A and B-type spaces studied by Pustylnik in [46].

Definition 5.15. Given $1 < p < \infty$, $\alpha < 1$ and E an r.i. space. The space $A_{p,\alpha,E}$ is the set of all measurable functions f on $(0,1)$ such that

$$\|f\|_{A_{p,\alpha,E}} = \Big\| \ell^{\alpha-1}(t) \int_t^1 s^{\frac{1}{p}} f^{**}(s)\, \frac{ds}{s} \Big\|_{\widetilde{E}} < \infty$$

assumed that the function $(1+u)^{\alpha-1}$ belongs to E (i.e. $\|\ell^{\alpha-1}(t)\|_{\widetilde{E}(0,1)} < \infty$.) The space $B_{p,\mathrm{b},E}$ is the set of all measurable functions f on $(0,1)$ such that

$$\|f\|_{B_{p,\alpha,E}} = \| \sup_{0<s<t} s^{\frac{1}{p}} \ell^{\alpha-1}(s) f^{**}(s) \|_{\widetilde{E}} < \infty.$$

The space of B-type for $\alpha = 0$ first appeared in [16]. General versions of these spaces were studied in [48]. The main advantage of the A and B-type spaces is their optimality in the weak interpolation [46,49].

The A and B-type spaces can be seen as $\mathcal{R}$ and $\mathcal{L}$-spaces, respectively. Indeed,

$$A_{p,\alpha,E} = (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p},\ell^{\alpha-1}(t),E,1,L_1} \quad \text{and} \quad B_{p,\alpha,E} = (L_1, L_\infty)^{\mathcal{L}}_{1-\frac{1}{p},1,E,\ell^{\alpha-1}(t),L_\infty}.$$

Then, we can apply the results from Sect. 4 to obtain the following interpolation formulae.

Corollary 5.16. *Let* E, E_0, E_1 *be r.i. spaces,* $\mathrm{b}, \mathrm{b}_0 \in SV(0,1)$, $1 < p_0 < p_1 < \infty$, $\beta < 1$ *and assume that* $(1+u)^{\beta-1}$ *belongs to* E_1*. Consider the function* $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \mathrm{b}_0(u) \|\ell^{\beta-1}(t)\|^{-1}_{\widetilde{E}_1(0,u)}$, $u \in (0,1)$.

a) If $0 < \theta < 1$*, then*

$$\big(L_{p_0,\mathrm{b}_0,E_0}, A_{p_1,\beta,E_1}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $B_\theta(u) = \mathrm{b}_0^{1-\theta}(u) \|\ell^{\beta-1}(t)\|^{\theta}_{\widetilde{E}_1(0,u)} \mathrm{b}(\rho(u))$, $u \in (0,1)$.

b) If $\theta = 0$*, then*

$$\big(L_{p_0,\mathrm{b}_0,E_0}, A_{p_1,\beta,E_1}\big)_{0,\mathrm{b},E} = \big(L_1, L_\infty\big)^{\mathcal{L}}_{1-\frac{1}{p_0},\mathrm{b}\circ\rho,E,\mathrm{b}_0,E_0}.$$

c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$*, then*

$$\begin{aligned}\big(L_{p_0,\mathrm{b}_0,E_0}, A_{p_1,\beta,E_1}\big)_{1,\mathrm{b},E} \\ = \big(L_1, L_\infty\big)^{\mathcal{R}}_{1-\frac{1}{p_1},B_1,E,1,L_1} \cap \big(L_1, L_\infty\big)^{\mathcal{R},\mathcal{R}}_{1-\frac{1}{p_1},\mathrm{b}\circ\rho,E,\ell^{\beta-1}(u),E_1,1,L_1},\end{aligned}$$

where $B_1(u) = \|\ell^{\beta-1}(t)\|_{\widetilde{E}_1(0,u)} \mathrm{b}(\rho(u))$, $u \in (0,1)$.

Corollary 5.17. *Let* E, E_0, E_1 *be r.i. spaces,* b, $\mathrm{b}_1 \in SV(0,1)$, $1 < p_0 < p_1 < \infty$ *and* $\alpha < 1$*. Consider the function* $\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \ell^{\alpha-1}(u) \varphi_{E_0}(\ell(u)) \mathrm{b}_1^{-1}(u)$, $u \in (0,1)$.

a) If $0 < \theta < 1$*, then*

$$\big(B_{p_0,\alpha,E_0}, L_{p_1,\mathrm{b}_1,E_1}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $B_\theta(u) = \big(\ell^{\alpha-1}(u)\varphi_{E_0}(\ell(u))\big)^{1-\theta} \mathrm{b}_1^{\theta}(u) \mathrm{b}(\rho(u))$, $u \in (0,1)$.

b) If $\theta = 0$*, then*

$$\big(B_{p_0,\alpha,E_0}, L_{p_1,\mathrm{b}_1,E_1}\big)_{0,\mathrm{b},E} = \big(L_1, L_\infty\big)^{\mathcal{L}}_{1-\frac{1}{p_0},B_0,E,\ell^{\alpha-1}(u),L_\infty} \cap \big(L_1, L_\infty\big)^{\mathcal{L},\mathcal{L}}_{1-\frac{1}{p_0},\mathrm{b}\circ\rho,E,1,E_0,\ell^{\alpha-1}(u),L_\infty},$$

where $B_0(u) = \varphi_{E_0}(\ell(u))\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

c) If $\theta = 1$ *and* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$*, then*

$$\big(B_{p_0,\alpha,E_0}, L_{p_1,\mathrm{b}_1,E_1}\big)_{1,\mathrm{b},E} = (L_1, L_\infty)^{\mathcal{R}}_{1-\frac{1}{p_1},\mathrm{b}\circ\rho,E,\mathrm{b}_1,E_1}.$$

In particular, the B-type spaces (or A-type) can be seen as limiting interpolation spaces between the ultrasymmetric spaces and A-type spaces (or B-type, respectively).

Corollary 5.18. *Let* E_0*,* E_1 *be r.i. spaces,* $1 < p_0 < p_1 < \infty$ *and* $\alpha, \beta < 1$*. Then*

$$(L_{p_0,\ell^{\alpha-1}(u),L_\infty}, A_{p_1,\beta,E_1})_{0,1,E_0} = B_{p_0,\alpha,E_0}$$

and

$$(B_{p_0,\alpha,E_0}, L_{p_1,1,L_1})_{1,\ell^{\beta-1}(u),E_1} = A_{p_1,\beta,E_1}.$$

Finally, arguing as in Theorem 5.7 we characterize ultrasymmetric spaces as interpolation spaces between A and B-type spaces.

Corollary 5.19. *Let* E*,* E_0*,* E_1 *be r.i. spaces,* $1 < p_0 < p_1 < \infty$*,* $\alpha, \beta < 1$ *and assume that* $(1+u)^{\beta-1}$ *belongs to* E_1*. Consider the function*

$$\rho(u) = u^{\frac{1}{p_0}-\frac{1}{p_1}} \frac{\ell^{\alpha-1}(u)\varphi_{E_0}(\ell(u))}{\|\ell^{\beta-1}(t)\|_{\widetilde{E}_1(0,u)}}, \qquad u \in (0,1).$$

a) If $0 < \theta < 1$*, then*

$$\big(B_{p_0,\alpha,E_0}, A_{p_1,\beta,E_1}\big)_{\theta,\mathrm{b},E} = L_{p,B_\theta,E},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $B_\theta(u) = \big(\ell^{\alpha-1}(u)\varphi_{E_0}(\ell(u))\big)^{1-\theta}\|\ell^{\beta-1}(t)\|^{\theta}_{\widetilde{E}_1(0,u)}$ $\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

b) If $\theta = 0$*, then*

$$\big(B_{p_0,\alpha,E_0}, A_{p_1,\beta,E_1}\big)_{0,\mathrm{b},E} = \big(L_1, L_\infty\big)^{\mathcal{L}}_{1-\frac{1}{p_0},B_0,E,\ell^{\alpha-1}(u),L_\infty} \cap \big(L_{p_0,\ell^{\alpha-1}(u),L_\infty}, A_{p_1,\beta,E_1}\big)^{\mathcal{L}}_{0,\mathrm{b}(u\varphi_{E_0}(\ell(u))),E,1,E_0},$$

where $B_0(u) = \varphi_{E_0}(\ell(u))\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

c) If $\theta = 1$ *and* $b \in SV(0,1)$ *is such that* $\|\mathrm{b}\|_{\widetilde{E}(0,1)} < \infty$*, then*

$$\big(B_{p_0,\alpha,E_0}, A_{p_1,\beta,E_1}\big)_{1,\mathrm{b},E} = \big(L_1, L_\infty\big)^{\mathcal{R}}_{1-\frac{1}{p_1},B_1,E,1,L_1} \cap \big(L_1, L_\infty\big)^{\mathcal{R},\mathcal{R}}_{1-\frac{1}{p_1},\mathrm{b}\circ\rho,E,\ell^{\beta-1}(u),E_1,1,L_1},$$

where $B_1(u) = \|\ell^{\beta-1}(t)\|_{\widetilde{E}_1(0,u)}\mathrm{b}(\rho(u))$*,* $u \in (0,1)$*.*

Final Remarks. After this paper was completed, we had knowledge of the paper by Leo Doktorski [18]. This paper contains limiting reiteration formulae involving $\mathcal{R}$ and $\mathcal{L}$ spaces. Although it has some intersection with our results, there are substantial differences. One of these is that the interpolation parameters remain within the scale of Lebesgue spaces rather than the richer family of r.i. spaces. However, this restriction allows the author to obtain results in the quasi-Banach case, which we do not.

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