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On Weak*-Extensible Subspaces of Banach Spaces

G. Martínez-Cervantes and J. Rodríguez

Abstract. Let X be a Banach space and $Y \subseteq X$ be a closed subspace. We prove that if the quotient X/Y is weakly Lindelöf determined or weak Asplund, then for every w^*-convergent sequence $(y_n^*)_{n\in\mathbb{N}}$ in Y^* there exist a subsequence $(y_{n_k}^*)_{k\in\mathbb{N}}$ and a w^*-convergent sequence $(x_k^*)_{k\in\mathbb{N}}$ in X^* such that $x_k^*|_Y = y_{n_k}^*$ for all $k \in \mathbb{N}$. As an application, we obtain that Y is Grothendieck whenever X is Grothendieck and X/Y is reflexive, which answers a question raised by González and Kania.

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1. Introduction

Throughout this paper, X is a Banach space. We denote by w^* the weak* topology on its (topological) dual X^*. The space X is said to be *Grothendieck* if every w^*-convergent sequence in X^* is weakly convergent. This property has been widely studied over the years, we refer the reader to the recent survey [7] for complete information on it. By a "subspace" of a Banach space we mean a closed linear subspace. If $Y \subseteq X$ is a subspace, then (i) the quotient X/Y is Grothendieck whenever X is Grothendieck, and (ii) X is Grothendieck whenever Y and X/Y are Grothendieck (see, e.g., [3, 2.4.e]). In general, the property of being Grothendieck is not inherited by subspaces (for instance, c_0 is not Grothendieck while ℓ_∞ is). However, this is the case for complemented subspaces or, more generally, subspaces satisfying the following property:

Definition 1.1. A subspace $Y \subseteq X$ is said to be w^*-*extensible* in X if for every w^*-convergent sequence $(y_n^*)_{n\in\mathbb{N}}$ in Y^* there exist a subsequence $(y_{n_k}^*)_{k\in\mathbb{N}}$ and a w^*-convergent sequence $(x_k^*)_{k\in\mathbb{N}}$ in X^* such that $x_k^*|_Y = y_{n_k}^*$ for all $k \in \mathbb{N}$.

Indeed, it is easy to show that a Banach space is Grothendieck if (and only if) every w^*-convergent sequence in its dual admits a weakly convergent subsequence. Thus, a subspace $Y \subseteq X$ is Grothendieck whenever X is Grothendieck and Y is w^*-extensible in X. The concept of w^*-extensible subspace was studied in [4,11,15] (there the definition was given by replacing

"w^*-convergent" by "w^*-null"; both definitions are easily seen to be equivalent). Independently, it was also considered in [12, Sect. 3] in connection with set-valued integration. Note that every subspace is w^*-extensible in X whenever B_{X^*} (the closed unit ball of X^*, that we just call the "dual ball" of X) is w^*-sequentially compact (cf. [3, 2.4.f]). However, this observation does not provide new results on the stability of the Grothendieck property under subspaces, because the only Grothendieck spaces having w^*-sequentially compact dual ball are the reflexive ones.

In this note, we focus on finding sufficient conditions for the w^*-extensibility of a subspace $Y \subseteq X$ which depend only on the quotient X/Y. Our motivation stems from the question (raised in [7, Problem 23]) of whether Y is Grothendieck whenever X is Grothendieck and X/Y is reflexive. It is known that the separable injectivity of c_0 (Sobczyk's theorem) implies that if X/Y is separable, then Y is w^*-extensible in X in a stronger sense, namely, the condition of Definition 1.1 holds without passing to subsequences (see, e.g., [1, Theorem 2.3 and Proposition 2.5]). Our main result is the following (see below for unexplained notation):

Theorem 1.2. *Let $Y \subseteq X$ be a subspace. Suppose that X/Y satisfies one of the following conditions:*

(i) Every non-empty w^*-closed subset of $B_{(X/Y)^*}$ has a G_δ-point (in the relative w^*-topology).

(ii) $\mathrm{dens}(X/Y) < \mathfrak{s}$.

Then Y is w^*-extensible in X.

Given a compact Hausdorff topological space K, a point $t \in K$ is called a *G_δ-point* (in K) if there is a sequence of open subsets of K whose intersection is $\{t\}$. Corson compacta have G_δ-points (see, e.g., [6, Theorem 14.41]), and the same holds for any non-empty w^*-closed subset in the dual ball of a weak Asplund space (see, e.g., the proof of [5, Theorem 2.1.2]). Thus, we get the following corollary covering the case when X/Y is reflexive.

Corollary 1.3. *Let $Y \subseteq X$ be a subspace. If X/Y is weakly Lindelöf determined or weak Asplund, then Y is w^*-extensible in X.*

As an application of the above we get an affirmative answer to [7, Problem 23]:

Corollary 1.4. *Let $Y \subseteq X$ be a subspace. If X is Grothendieck and X/Y is reflexive, then Y is Grothendieck.*

As to condition (ii) in Theorem 1.2, recall that the density character of a Banach space Z, denoted by $\mathrm{dens}(Z)$, is the smallest cardinality of a dense subset of Z. For our purposes, we just mention that the *splitting number* $\mathfrak{s}$ is the minimum of all cardinals κ for which there is a compact Hausdorff topological space of weight κ that is not sequentially compact. In general, $\omega_1 \leq \mathfrak{s} \leq \mathfrak{c}$. So, under CH, cardinality strictly less than $\mathfrak{s}$ just means countable. However, in other models there are uncountable sets of cardinality strictly less than $\mathfrak{s}$. We refer the reader to [14] for detailed information on $\mathfrak{s}$ and other cardinal characteristics of the continuum.

Any of the conditions in Theorem 1.2 implies that $B_{(X/Y)^*}$ is w^*-sequentially compact (see [5, Lemma 2.1.1] and note that the weight of $(B_{(X/Y)^*}, w^*)$ coincides with $\mathrm{dens}(X/Y)$), which is certainly not enough to guarantee that Y is w^*-extensible in X (see Remark 2.5). The ideas that we use in this paper are similar to those used by Hagler and Sullivan [9] to study sufficient conditions for the dual ball of a Banach space to be w^*-sequentially compact. By the way, as another consequence of Theorem 1.2 we obtain a generalization of [9, Theorem 1] (see Corollary 2.6).

The proof of Theorem 1.2 and some further remarks are included in the next section. We follow standard Banach space terminology as it can be found in [5] and [6].

2. Proof of Theorem 1.2 and Further Remarks

By a "compact space" we mean a compact Hausdorff topological space. The *weight* of a compact space K, denoted by $\mathrm{weight}(K)$, is the smallest cardinality of a base of K. The following notion was introduced in [10]:

Definition 2.1. Let L be a compact space and $K \subseteq L$ be a closed set. We say that L is a *countably discrete extension* of K if $L \setminus K$ consists of countably many isolated points.

Countably discrete extensions turn out to be a useful tool to study twisted sums of c_0 and $C(K)$-spaces, see [2] and [10]. As it can be seen in the proof of Theorem 1.2, countably discrete extensions appear in a natural way when dealing with sequential properties and twisted sums. Lemma 2.2 below isolates two properties of compact spaces which are stable under countably discrete extensions, both of them implying sequential compactness. Nevertheless, sequential compactness itself is not stable under countably discrete extensions (see Remark 2.5).

Lemma 2.2. *Let L be a compact space which is a countably discrete extension of a closed set $K \subseteq L$. Then*

(i) if every non-empty closed subset of K has a G_δ-point (in its relative topology), then the same property holds for L;
(ii) $\mathrm{weight}(L) = \mathrm{weight}(K)$ *whenever K is infinite.*

Therefore, if either every non-empty closed subset of K has a G_δ-point (in its relative topology) or $\mathrm{weight}(K) < \mathfrak{s}$, *then L is sequentially compact.*

Proof. (i) Let $M \subseteq L$ be a non-empty closed set. If $M \subseteq K$, then M has a G_δ-point (in the relative topology) by the hypothesis. On the other hand, if $M \cap (L \setminus K) \neq \emptyset$, then M contains a point which is isolated in L and so a G_δ-point (in M).

(ii) Let $R : C(L) \to C(K)$ be the bounded linear operator defined by $R(f) = f|_K$ for all $f \in C(K)$. Then R is surjective and $C(K)$ is isomorphic to $C(L)/\ker R$. The fact that L is a countably discrete extension of K implies that $\ker R$ is finite-dimensional or isometrically isomorphic to c_0. In any case, $\ker R$ is separable and so $\mathrm{dens}(C(K)) = \mathrm{dens}(C(L))$. The conclusion follows from the equality $\mathrm{dens}(C(S)) = \mathrm{weight}(S)$, which holds for any infinite compact space S (see, e.g., [13, Proposition 7.6.5] or [6, Exercise 14.36]).

The last statement of the lemma follows from [5, Lemma 2.1.1] and [14, Theorem 6.1], respectively. □

We will use below the well-known fact that $\mathrm{dens}(Z) = \mathrm{weight}(B_{Z^*}, w^*)$ for any Banach space Z.

Proof of Theorem 1.2. Let $(y_n^*)_{n\in\mathbb{N}}$ be a w^*-convergent sequence in Y^*. Without loss of generality, we can assume that $(y_n^*)_{n\in\mathbb{N}}$ is w^*-null and contained in B_{Y^*}. Clearly, there is nothing to prove if $y_n^* = 0$ for infinitely many $n \in \mathbb{N}$. So, we can assume further that $y_n^* \neq 0$ for all $n \in \mathbb{N}$. By the Hahn–Banach theorem, for each $n \in \mathbb{N}$ there is $z_n^* \in X^*$ with $z_n^*|_Y = y_n^*$ and $\|z_n^*\| = \|y_n^*\| \leq 1$.

Let $q : X \to X/Y$ be the quotient operator. It is well-known that its adjoint $q^* : (X/Y)^* \to X^*$ is an isometric isomorphism from $(X/Y)^*$ onto $Y^\perp$. In addition, q^* is w^*-to-w^*-continuous; hence, $K := B_{X^*} \cap Y^\perp = q^*(B_{(X/Y)^*})$ is w^*-compact. Observe that for every $x^* \in X^* \setminus Y^\perp$ there exist $x \in Y$, $\alpha > 0$ and $m \in \mathbb{N}$ such that $x^*(x) > \alpha > z_n^*(x)$ for every $n \geq m$. Hence $L := K \cup \{z_n^* : n \in \mathbb{N}\} \subseteq B_{X^*}$ is w^*-closed (so that L is w^*-compact) and each z_n^* is w^*-isolated in L (bear in mind that $z_n^*|_Y = y_n^* \neq 0$). Then (L, w^*) is a countably discrete extension of (K, w^*), with (K, w^*) and $(B_{(X/Y)^*}, w^*)$ being homeomorphic. Bearing in mind that $\mathrm{dens}(X/Y)$ coincides with the weight of $(B_{(X/Y)^*}, w^*)$, from Lemma 2.2 it follows that L is sequentially compact and, therefore, $(z_n^*)_{n\in\mathbb{N}}$ admits a w^*-convergent subsequence. The proof is finished. □

If X is Grothendieck and $Y \subseteq X$ is a subspace such that X/Y is separable, then Y is Grothendieck (see [8, Proposition 3.1]). This fact can also be seen as a consequence of Corollary 1.4, because every separable Grothendieck space is reflexive.

As an immediate application of Theorem 1.2 we also obtain an affirmative answer to [7, Problem 22] (note that $\mathfrak{p} \leq \mathfrak{s}$ and that the strict inequality $\mathfrak{p} < \mathfrak{s}$ is consistent with ZFC, see, e.g., [14, Theorems 3.1 and 5.4]):

Corollary 2.3. *Let $Y \subseteq X$ be a subspace such that* $\mathrm{dens}(X/Y) < \mathfrak{s}$. *If X is Grothendieck, then Y is Grothendieck.*

Remark 2.4. In fact, the previous corollary is a particular case of Corollary 1.4. Indeed, on the one hand, the assumption that $\mathrm{dens}(X/Y) < \mathfrak{s}$ implies that $B_{(X/Y)^*}$ is w^*-sequentially compact. On the other hand, the Grothendieck property is preserved by quotients and any Grothendieck space with w^*-sequentially compact dual ball is reflexive (cf. [1, Proposition 6.18]).

Remark 2.5. In general, the w^*-sequential compactness of $B_{(X/Y)^*}$ is not enough to guarantee that a subspace $Y \subseteq X$ is w^*-extensible in X. Indeed, it is easy to check that *if both B_{Y^*} and $B_{(X/Y)^*}$ are w^*-sequentially compact and Y is w^*-extensible in X, then B_{X^*} is w^*-sequentially compact as well* (see the proof of [4, Proposition 6]). On the other hand, there exists a Banach space X such that B_{X^*} is not w^*-sequentially compact although $B_{(X/Y)^*}$

is w^*-sequentially compact for some separable subspace $Y \subseteq X$ (see [9], cf. [3, Sect. 4.8]).

Hagler and Sullivan proved in [9, Theorem 1] that B_{X^*} is w^*-sequentially compact whenever there is a subspace $Y \subseteq X$ such that B_{Y^*} is w^*-sequentially compact and X/Y has an equivalent Gâteaux smooth norm. Since every Banach space admitting an equivalent Gâteaux smooth norm is weak Asplund (see, e.g., [5, Corollary 4.2.5]), the following corollary generalizes that result:

Corollary 2.6. *Let $Y \subseteq X$ be a subspace such that B_{Y^*} is w^*-sequentially compact. If X/Y satisfies any of the conditions in Theorem 1.2, then B_{X^*} is w^*-sequentially compact.*

Note added in proof The condition that every weak* compact subset has a G_δ point (with respect to the weak* topology), is implied by the space being a Gateaux differentiability space (see, e.g., the proof of [5, Theorem 2.1.2]). Thus, Corollary 1.3 also applies when X/Y is a Gateaux differentiability space. This fact was kindly pointed out to us by Warren Moors.

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G. Martínez-Cervantes
Dpto. de Matemáticas, Facultad de Ciencias
Universidad de Alicante
03080 Alicante
Spain
e-mail: gonzalo.martinez@ua.es

J. Rodríguez
Dpto. de Ingeniería y Tecnología de Computadores, Facultad de Informática
Universidad de Murcia
30100 Espinardo
Murcia
Spain
e-mail: joserr@um.es

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