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# The $L^r$-Variational Integral

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**Abstract.** We define the $L^r$-variational integral and we prove that it is equivalent to the $HK_r$-integral defined in 2004 by P. Musial and Y. Sagher in the Studia Mathematica paper *The $L^r$-Henstock–Kurzweil integral*. We prove also the continuity of $L^r$-variation function.

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**Keywords.** $L^r$-Variational Integral, $HK_r$ Integral, Non-absolute integral.

## 1. History and Aim

At the beginning of the 1900s, Denjoy and Perron developed descriptive processes for recovering a function from its derivative that solved known problems of classical Riemann and Lebesgue integrals. Many years later, an equivalent constructive Riemann-type integral process was developed by Henstock and Kurzweil. Both integration processes were generalized quite recently for many different spaces (see [1,11] and [12]) solving the problem of recovering Fourier coefficients in Haar, Walsh and Vilenkin systems (see [9,10,14,15] and [16]). Many properties of these non-absolute integrals were investigated, for example, the Hake property was studied with an abstract differential basis in a topological spaces, in terms of variational measure and in Riesz spaces (see [13,17] and [2]).

To establish pointwise estimates for solutions of elliptic partial differential equations, in 1961 Calderon and Zygmund introduced the $L^r$-derivative (see [3]) and in 1968 L. Gordon described a Perron-type integral, the $P_r$-integral, that recovers a function from its $L^r$-derivative (see [4]). In 2004, Musial and Sagher extended the $P_r$-integral to the $L^r$-Henstock–Kurzweil integral, the $HK_r$-integral, that recovers also a function from its $L^r$-derivative (see [6]). Quite recently the integration by parts formula for the $HK_r$-integral was investigated by Musial and Tulone (see [7]) and the same authors described a norm on the space of $HK_r$-integrable functions and studied the dual and completion of this space (see [8]).

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It is well known that the Henstock–Kurzweil integral is equivalent to the variational integral (see [5]). In this paper, we define the $L^r$-variational integral and we prove that it is equivalent to the $HK_r$-integral.

## 2. Introduction

We will assume that $r \geq 1$ and we will consider the case of the closed interval $[a, b]$.

**Definition 2.1.** A function $f : [a, b] \to \mathbb{R}$ is $L^r$-variational integrable on $[a, b]$ if there exists a function $F \in L^r [a, b]$ with the following property: for each $\varepsilon > 0$ there exist a non-decreasing function $\phi$ defined on $[a, b]$ and a gauge $\delta$, i.e., a positive function, defined on $[a, b]$ such that $\phi(b) - \phi(a) < \varepsilon$ and for any $\delta$-fine tagged interval $(x, [c, d])$, where $[c, d] \subseteq [a, b]$,

$$\left( \frac{1}{d-c} \int_c^d |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r} < \phi(d) - \phi(c). \tag{2.1}$$

We will use the following definition given in [6]

**Definition 2.2.** A function $f : [a, b] \to \mathbb{R}$ is $L^r$-Henstock–Kurzweil integrable on $[a, b]$ if there exists a function $F \in L^r [a, b]$ so that for any $\varepsilon > 0$ there exists a gauge $\delta$ so that for any finite collection of nonoverlapping $\delta$-fine tagged intervals

$$\mathcal{Q} = \{(x_i, [c_i, d_i]), 1 \leq i \leq q\},$$

we have

$$\sum_{i=1}^{q} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

By Theorem 5 in [6], the function $F$ in the Definition 2.2 is unique up to an additive constant, so we can state that for each $x \in (a, b]$

$$F(x) = (HK_r) \int_a^x f.$$

We need the following definition in a later theorem.

**Definition 2.3.** Let $F \in L^r [a, b]$. For $x \in [a, b]$ we say that $F$ is $L^r$-continuous at $x$ if

$$\lim_{h \to 0} \left( \frac{1}{2h} \int_{x-h}^{x+h} |F(y) - F(x)|^r \, dy \right)^{1/r} = 0.$$

If $F$ is $L^r$-continuous for all $x \in E$, we say that $F$ is $L^r$-continuous on $E$.

The Henstock–Kurzweil integral primitive is continuous in the usual sense. In [6] is proved an equivalent result for $L^r$-Henstock–Kurzweil indefinite integral.

**Theorem 2.4.** *The function $F$ in the definition of the $L^r$-Henstock–Kurzweil is $L^r$-continuous on $[a, b]$.*

**Definition 2.5.** Let $\Phi$ be a function defined on the subintervals of $[a, b]$. The function $\Phi$ is superadditive if

$$\Phi([u, v]) + \Phi([v, w]) \leq \Phi([u, w]),$$

whenever $a \leq u < v < w \leq b$. The function $\Phi$ is continuous if for each $c \in (a, b)$,

$$\lim_{x \to c^-} \Phi([x, c]) = 0 = \lim_{x \to c^+} \Phi([c, x])$$

and

$$\lim_{x \to b^-} \Phi([x, b]) = 0 = \lim_{x \to a^+} \Phi([a, x]).$$

*Remark 2.6.* Throughout this paper, if an interval function is said to be continuous, it is to be considered continuous in the sense of Definition 2.5.

**Definition 2.7.** Let $\delta$ be a gauge and let

$$\mathcal{P} = \{(x_i, [c_i, d_i]), 1 \leq i \leq n\}$$

be a $\delta$-fine partition of $[a, b]$. Let

$$W(\mathcal{P}) = \sum_{i=1}^{q} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r}. \quad (2.2)$$

The main tool we need to get the $L^r$-variational integral is the following definition of $L^r$-variation function.

**Definition 2.8.** For each subinterval $[c, d] \subseteq [a, b]$ define

$$\Phi([c, d]) = \Phi(F, \delta, [c, d]) = \sup\{W(\mathcal{P})\}, \quad (2.3)$$

where the supremum is taken over all $\delta$-fine partitions $\mathcal{P}$ of $[c, d]$.

**Theorem 2.9.** *The function $\Phi$ is superadditive.*

*Proof.* Let $u, v$ and $w$ be such that $a \leq u < v < w \leq b$ and let $\varepsilon > 0$. If either $\Phi([u, v]) = \infty$ or $\Phi([v, w]) = \infty$ then surely $\Phi([u, w]) = \infty$ and the assertion holds. Otherwise let $\mathcal{P}_1$ be a partition of $[u, v]$ such that $W(\mathcal{P}_1) > \Phi([u, v]) - \varepsilon$ and let $\mathcal{P}_2$ be a partition of $[v, w]$ such that $W(\mathcal{P}_2) > \Phi([v, w]) - \varepsilon$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, and clearly $W(\mathcal{P}) = W(\mathcal{P}_1) + W(\mathcal{P}_2)$. But $W(\mathcal{P}) \leq \Phi([u, w])$. Therefore,

$$\Phi([u, v]) + \Phi([v, w]) - 2\varepsilon < W(\mathcal{P}_1) + W(\mathcal{P}_2) \leq \Phi([u, w]).$$

□

Now we can prove the following theorem that extends Theorem 11.9 in [5]

## 3. Main Results

**Theorem 3.1.** *A function* $f : [a,b] \to \mathbb{R}$ *is* $L^r$*-Henstock–Kurzweil integrable on* $[a,b]$ *if and only if there exists a function* $F : [a,b] \to \mathbb{R}$ *with the following property: for each* $\varepsilon > 0$ *there exists a superadditive interval function* $\Phi$ *defined on the subintervals of* $[a,b]$ *and a gauge* $\delta$ *defined on* $[a,b]$ *such that* $\Phi([a,b]) < \varepsilon$ *and for any* $\delta$*-fine tagged interval* $(x,[c,d])$*, where* $[c,d] \subseteq [a,b]$ *,*

$$\left(\frac{1}{d-c}\int_c^d |F(y) - F(x) - f(x)(y-x)|^r\, dy\right)^{1/r} < \Phi([c,d]).$$

*Proof.* Suppose there exists a function $F$ with the property stated in the theorem. Let $\varepsilon > 0$ and choose $\Phi$ and $\delta$ according to the hypotheses. If $\mathcal{P} := \{(x_i,[c_i,d_i]), 1 \le i \le n\}$ is a $\delta$-fine tagged partition of $[a,b]$, then

$$\sum_{i=1}^{n}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y-x_i)|^r\, dy\right)^{1/r} \le \sum_{i=1}^{n}\Phi([c_i,d_i]) \le \Phi([a,b]), < \varepsilon$$

and so $f$ is $L^r$-Henstock–Kurzweil integrable on $[a,b]$.

Now suppose that $f$ is $L^r$-Henstock–Kurzweil integrable on $[a,b]$ and let

$$F(x) = (HK_r)\int_a^x f,$$

for each $x \in (a,b]$. Let $\varepsilon > 0$. By hypothesis, there exists a gauge $\delta$ on $[a,b]$ such that

$$\sum_{i=1}^{n}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y-x_i)|^r\, dy\right)^{1/r} < \varepsilon/2,$$

whenever $\mathcal{P}$ is a $\delta$-fine tagged partition of $[a,b]$. Let

$$\mathcal{P} = \{(x_i,[c_i,d_i]), 1 \le i \le n\}$$

and let $W(\mathcal{P})$ be defined as in (2.2) and let $\Phi$ be defined on the subintervals of $[a,b]$ as in (2.3). By Theorem 2.9, $\Phi$ is superadditive. Also,

$$\Phi([a,b]) \le \varepsilon/2 < \varepsilon.$$

Finally, by the definition of $\Phi$, if $(x,[c,d])$ is a $\delta$-fine tagged interval such that $[c,d] \subseteq [a,b]$,

$$\left(\frac{1}{d-c}\int_c^d |F(y) - F(x) - f(x)(y-x)|^r\, dy\right)^{1/r} < \Phi([c,d]).$$

This completes the proof. □

**Theorem 3.2.** *A function* $f : [a,b] \to$ *is* $L^r$*-Henstock–Kurzweil integrable on* $[a,b]$ *if and only if* $f$ *is* $L^r$*-variational integrable on* $[a,b]$*.*

*Proof.* Suppose first that $f$ is $L^r$-variational integrable on $[a,b]$. Let $\varepsilon > 0$ and let $F$, $\delta$ and $\phi$ satisfy the conditions in Definition 2.1. If $\mathcal{P} = \{(x_i,[c_i,d_i]), 1 \le i \le n\}$ is a $\delta$-fine tagged partition of $[a,b]$, then

$$\sum_{i=1}^{n}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r} \le \sum_{i=1}^{n}(\phi(d_i)-\phi(c_i)) = \phi(b)-\phi(a) < \varepsilon$$

and so $f$ is $L^r$-Henstock–Kurzweil integrable on $[a,b]$ and

$$(HK_r)\int_a^b f = F(b)-F(a).$$

Now suppose that $f$ is $L^r$-Henstock–Kurzweil integrable on $[a,b]$ and that for each $x \in (a,b]$,

$$F(x) = (HK_r)\int_a^x f.$$

Let $\varepsilon > 0$. By Theorem 3.1 there exists a superadditive interval function $\Phi$ defined on $[a,b]$ such that $\Phi([a,b]) < \varepsilon$ and

$$\left(\frac{1}{d-c}\int_c^d|F(y)-F(x)-f(x)(y-x)|^r\,dy\right)^{1/r} < \Phi([c,d]),$$

whenever $(x,[c,d])$ is a $\delta$-fine tagged interval such that $[c,d] \subseteq [a,b]$. Define $\phi : [a,b] \to \mathbb{R}$ by $\phi(a) = 0$ and $\phi(x) = \Phi([a,x])$ for all $x \in (a,b]$. If $a \le c < d \le b$, then

$$\phi(d)-\phi(c) = \Phi([a,d])-\Phi([a,c]) \ge \Phi([c,d]) \ge 0$$

and so $\phi$ is non-decreasing. In addition,

$$\phi(b)-\phi(a) = \Phi([a,b]) < \varepsilon.$$

Suppose that $(x,[c,d])$ is a $\delta$-fine tagged interval such that $[c,d] \subseteq [a,b]$. Then,

$$\left(\frac{1}{d-c}\int_c^d|F(y)-F(x)-f(x)(y-x)|^r\,dy\right)^{1/r} \le \Phi([c,d]) \le \phi(d)-\phi(c).$$

Hence, the function $f$ is $L^r$-variational integrable on $[a,b]$. This completes the proof. □

**Corollary 3.3.** *If $f$ is $L^r$-variational integrable on $[a,b]$, then the function $F$ which satisfies the conditions of Definition 2.1 is unique up to an additive constant.*

We now prove the continuity of the interval function $\Phi$.

**Proposition 3.4.** *Let $f$ be $L^r$-variational integrable on $[a,b]$ and let $F$ be a function that satisfies (2.1). Let $\delta$ be a gauge, $\Phi = \Phi(\delta, F)$ be as in (2.3), and assume that $\Phi([a,b])$ is finite. Then, $\Phi$ is continuous.*

*Proof.* We will prove that $\lim_{x\to c^-} \Phi([x,c]) = 0$ for each $c \in (a,b]$; the proof for right-handed limits is similar. Suppose by way of contradiction that $\lim_{x\to c^-} \Phi([x,c])$ either fails to exist or exists and is not equal to zero. Since $\Phi$ is nonnegative, there exists $\eta > 0$ such that $\limsup_{x\to c^-} \Phi([x,c]) > \eta$. Let us see that for every $\xi \in [a,c)$, $\Phi([\xi,c]) > \eta$. Fix $\xi$, there exists $\xi < \zeta < c$ such that $\Phi([\zeta,c]) > \eta$. Since $\Phi$ is superadditive, we have that

$$\Phi([\xi,c]) \geq \Phi([\xi,\zeta]) + \Phi([\zeta,c]) \geq \Phi([\zeta,c]) > \eta.$$

Consequently, for each $\xi \in [a,c)$, there exists $\mathcal{P}_\xi$, a $\delta$-fine tagged partition of $[\xi,c]$ such that $W(\mathcal{P}_\xi) > \eta$.

We now prove that we can make the following three assumptions about $\mathcal{P}_x$ :

1. $\mathcal{P}_x$ contains at least two tagged intervals,
2. $c$ is a tag of $\mathcal{P}_x$, and
3. the interval containing $c$ is arbitrarily small.

Fix $x$ and $\varepsilon > 0$. Choose $y \in (\max(x, c-\varepsilon), c)$. By Cousin's Lemma there exists $\mathcal{Q}$, a $\delta$-fine tagged partition of $[x,y]$. Define $\mathcal{P}_x = \mathcal{Q} \cup \mathcal{P}_y$. We then have

$$W(\mathcal{P}_x) = W(\mathcal{Q}) + W(\mathcal{P}_y) \geq W(\mathcal{P}_y) > \eta.$$

If $c$ is the tag of its interval, then $\mathcal{P}_x$ has the desired properties.

Now suppose that $c$ is not the tag of its interval. Let $s$ and $t$ be such that $(t,[s,c])$ is the tagged interval which contains $c$. It is possible that $s = t$ but we assume that $t < c$.

It suffices to show that

$$\lim_{u\to c^-} W(\{(t,[s,u]),(c,[u,c])\}) = W(\{(t,[s,c])\}).$$

Note that

$$\begin{aligned} &W(\{(t,[s,u]),(c,[u,c])\}) \\ &\quad = \left(\frac{1}{u-s}\int_s^u |F(y) - F(t) - f(t)(y-t)|^r\, dy\right)^{1/r} \\ &\qquad + \left(\frac{1}{c-u}\int_u^c |F(y) - F(c) - f(c)(y-c)|^r\, dy\right)^{1/r}. \end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)-f(c)(y-c)|^r\,dy\right)^{1/r}\\
&=\left(\frac{1}{c-u}\right)^{1/r}\left(\int_u^c |F(y)-F(c)-f(c)(y-c)|^r\,dy\right)^{1/r}\\
&\le\left(\frac{1}{c-u}\right)^{1/r}\left(\int_u^c |F(y)-F(c)|^r\,dy\right)^{1/r}\\
&\quad+\left(\frac{1}{c-u}\right)^{1/r}\left(\int_u^c |f(c)(y-c)|^r\,dy\right)^{1/r}\\
&\le\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)|^r\,dy\right)^{1/r}\\
&\quad+|f(c)|\left(\frac{1}{c-u}\right)^{1/r}\left(\int_u^c |(c-u)|^r\,dy\right)^{1/r}\\
&=\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)|^r\,dy\right)^{1/r}+|f(c)|(c-u).
\end{aligned}$$

By Theorem 2.4 the function $F$ is $L^r$-continuous at each point of $[a,b]$, and so we have that

$$\begin{aligned}
&\lim_{u\to c^-}\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)-f(c)(y-c)|^r\,dy\right)^{1/r}\\
&\le\lim_{u\to c^-}\left[\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)|^r\,dy\right)^{1/r}+|f(c)|(c-u)\right]=0.
\end{aligned}\tag{3.1}$$

We also have that

$$\begin{aligned}
&\lim_{u\to c^-}\left(\frac{1}{u-s}\int_s^u |F(y)-F(t)-f(t)(y-t)|^r\,dy\right)^{1/r}\\
&=\left(\frac{1}{c-s}\int_s^c |F(y)-F(t)-f(t)(y-t)|^r\,dy\right)^{1/r}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{u\to c^-} W(\{(t,[s,u]),(c,[u,c])\})\\
&=\lim_{u\to c^-} W(\{(t,[s,u])\})=W(\{(t,[s,c])\}).
\end{aligned}$$

We now prove the proposition. Set $x_1=a$ and write

$$\begin{aligned}
\mathcal{P}_{x_1}&=\mathcal{Q}_1\cup(c,[x_2,c])\\
\mathcal{P}_{x_2}&=\mathcal{Q}_2\cup(c,[x_3,c])\\
&\;\;\vdots\\
\mathcal{P}_{x_k}&=\mathcal{Q}_k\cup(c,[x_{k+1},c]).
\end{aligned}$$

By the result proved above, we may assume that for each $k$, $c-x_k<1/k$ and, therefore, that $x_k\to c$.

For each $n$, the collection

$$\mathcal{P}'_n = \bigcup_{k=1}^{n} \mathcal{Q}_k$$

is a $\delta$-fine tagged partition of $[a, x_{n+1}]$. Hence,

$$W\left(\mathcal{P}'_n\right) = \sum_{k=1}^{n} W\left(\mathcal{Q}_k\right) \leq \Phi\left([a, x_{n+1}]\right) \leq \Phi\left([a, b]\right) < \infty.$$

This shows that the series

$$\sum_{k=1}^{\infty} W\left(\mathcal{Q}_k\right)$$

converges and hence

$$\lim_{k \to \infty} W\left(\mathcal{Q}_k\right) = 0.$$

We then have for each $k$,

$$\begin{aligned} \eta &< W\left(\mathcal{P}_{x_k}\right) \\ &= W\left(\mathcal{Q}_k\right) + \left(\frac{1}{c - x_{k+1}} \int_{x_{k+1}}^{c} \left|F\left(y\right) - F\left(c\right) - f\left(c\right)\left(y - c\right)\right|^r dy\right)^{1/r}. \end{aligned}$$

By (3.1), the term on the right goes to zero; therefore, the entire right side of the equality goes to zero. This contradiction completes the proof. □

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