Periodic Solutions to Second-Order Nonlinear Differential Equations in Banach Spaces

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Abstract. In this article, we deal with the existence of solutions for the following second-order differential equation:

$$\begin{cases} u''(t) = f(t, u(t)) + h(t) \\ u(a) - u(b) = u'(a) - u'(b) = 0, \end{cases}$$

where $\mathbb{B}$ is a reflexive real Banach space, $f : [a, b] \times \mathbb{B} \rightarrow \mathbb{B}$ is a sequentially weak-strong continuous mapping, and $h : [a, b] \rightarrow \mathbb{B}$ is a continuous function on $\mathbb{B}$. The proofs are obtained using a recent generalization of the well-known Bolzano–Poincaré–Miranda theorem to infinite-dimensional Banach spaces. In the last section, we present three examples of application of the general result.

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1. Introduction

The main purpose of this article is to study the existence of solutions of the second-order dynamical equation

$$\begin{cases} u''(t) = f(t, u(t)) + h(t) \\ u(a) - u(b) = u'(a) - u'(b) = 0, \end{cases} \quad (1.1)$$

where $\mathbb{B}$ is a reflexive real Banach space, $f : [a, b] \times \mathbb{B} \rightarrow \mathbb{B}$ is a continuous mapping, and $h : [a, b] \rightarrow \mathbb{B}$ is a continuous function on $\mathbb{B}$. During the previous decades, some classical tools have been used in the study of periodic solutions of Eq. (1.1) in finite-dimensional Banach spaces, including the method of upper and lower solutions [6, 10, 13], and degree theory and fixed point theory [8, 12]. In this sense, in [20], the author gave a generalization of Miranda–Poincaré theorem and using this generalization proved theorems about the

existence for systems of k equations $x'' = f(t, x, x')$, where $f : [0, 1] \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ is a vector function, subject to various boundary conditions.

Recently, the existence of periodic solutions of second-order differential equations on a general Banach space has been treated in several papers; for instance, see [14, 17, 19]. In [14], operator-valued Fourier multiplier theorems are used to study this problem. In [17], the authors developed an infinite-dimensional version of Poincaré–Miranda theorem and they showed their relation with viability theory for differential inclusions and how to apply them, in particular, in the context of constrained PDEs. On the other hand, in [19], variational methods are used to study the existence of two solutions of superlinear second-order Hamiltonian systems that have damped and impulsive effects.

In this paper, we are able to give sufficient conditions to obtain existence of periodic solutions for Problem (1.1). Namely, we obtain that when $\mathbb{B}$ is a reflexive Banach space and $f : [a, b] \times \mathbb{B} \rightarrow \mathbb{B}$ is a sequentially weak–strong continuous mapping satisfying conditions (3.2) and (3.4) (see them below), then, using a generalization of Bolzano–Poincaré–Miranda theorem [1] and Leray–Schauder Alternative theorem (for instance, see [4]), Problem (1.1) admits a classical solution.

2. Notations and Preliminaries

In this section, we shall recall some definitions and results that are needed later on.

Throughout this article, $\mathbb{B}$ will denote a real reflexive Banach space with norm $\|\cdot\|$ and we shall denote its dual by $\mathbb{B}^*$. The closed ball of center x_0 and radius r is denoted by $B_r[x_0]$ and the sphere of center x_0 and radius r by $\partial B_r[x_0]$. The symbol “ $\rightharpoonup$ ” is used to denote the convergence of a sequence in $\mathbb{B}$ with respect to the weak topology $\sigma(\mathbb{B}, \mathbb{B}^*)$, as well as the symbol “ $\rightarrow$ ” is used to denote the convergence of a sequence in $\mathbb{B}$ with respect to the norm topology.

Recall that the *normalized duality mapping* $J : \mathbb{B} \rightarrow 2^{\mathbb{B}^*}$ is defined by

$$J(x) := \{j \in \mathbb{B}^* : \langle x, j \rangle = \|x\|^2, \|j\| = \|x\|\}, \quad x \in \mathbb{B}.$$

Let $\langle y, x \rangle_+ := \max\{\langle y, j \rangle : j \in J(x)\}$ and let $\langle y, x \rangle_- := \min\{\langle y, j \rangle : j \in J(x)\}$. For more details, see [5].

Denote by $\mathcal{C}([a, b], \mathbb{B})$ the space of $\mathbb{B}$ -valued continuous functions on $[a, b]$ with the norm $\|u\|_\infty = \sup\{\|u(t)\| : t \in [a, b]\}$, and denote by $L^1(a, b, \mathbb{B})$ the space of $\mathbb{B}$ -valued Bochner integrable functions on $[a, b]$ with the norm $\|u\|_1 = \int_a^b \|u(t)\| dt$. It is well known (for example, see [18]) that, if $u \in \mathcal{C}([a, b], \mathbb{B})$, then $u \in L^1(a, b, \mathbb{B})$ and, for each $t \in [a, b]$, we have that $\frac{d}{dt}(\int_a^t u(s) ds) = u(t)$.

Let K be a nonempty subset of $\mathbb{B}$. A mapping $F : K \rightarrow \mathbb{B}$ is said to be *compact* if $F(C)$ is relatively compact whenever C is a bounded subset of K . If, moreover, F is continuous in K , then F is called *completely continuous*. Clearly, every continuous mapping with closed domain in a finite-dimensional Banach space is completely continuous. On the other hand, F is said to be

sequentially weak-strong continuous mapping if given any sequence (x_n) in K , such that $x_n \rightharpoonup x \in K$ the strong convergence $F(x_n) \rightarrow F(x)$ holds. It is clear that if $\mathbb{B}$ is a reflexive Banach space, K is a closed convex subset of $\mathbb{B}$, and $F : K \rightarrow \mathbb{B}$ is a sequentially weak-strong continuous mapping, then F is completely continuous.

Theorem 2.1 (Schauder's Fixed Point Theorem). *Let K be a nonempty, bounded, closed, and convex subset of a Banach space $\mathbb{B}$. If $F : K \rightarrow K$ is completely continuous, then F has a fixed point.*

The hypothesis that is used to produce a more convenient form of Schauder's fixed point theorem is called *Leray-Schauder boundary condition*: Given a mapping $F : \mathbb{B} \rightarrow \mathbb{B}$, the mapping F satisfies this condition if there exists $r > 0$, such that $\|x\| = r$ implies $F(x) \neq \lambda x$ for all $\lambda > 1$. The result that replaces the requirement in Schauder's fixed point theorem is the following one (see, e.g., [4]):

Theorem 2.2 (Leray-Schauder Alternative). *Let $\mathbb{B}$ be a Banach space. If $F : \mathbb{B} \rightarrow \mathbb{B}$ is a completely continuous mapping and F satisfies the Leray-Schauder condition, then F has a fixed point.*

We state below Ascoli-Arzelà's theorem (see [7, p.34]) exactly in the form we need later as follows:

Theorem 2.3 (Ascoli-Arzelà). *Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space and let $K \subseteq \mathcal{C}([a, b], \mathbb{B})$ be a family of functions, such that*

- (i) $\|u(t)\| \leq M$, for all $t \in [a, b]$, $u \in K$.
- (ii) K is equi-uniformly continuous, i.e., for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$, such that $\|u(t) - u(s)\| \leq \epsilon$, if $|t - s| \leq \delta(\epsilon)$, for all $u \in K$.
- (iii) For each $t \in [a, b]$, the set $\{u(t) : u \in K\}$ is compact in $\mathbb{B}$.

Then, K is compact in $\mathcal{C}([a, b], \mathbb{B})$.

Recently, in [1, Theorem 3.1], an extension of the well-known Bolzano-Poincaré-Miranda theorem to infinite-dimensional Banach spaces is obtained. A particular version of this result, which we will use, is the following one (see Remark 3.1 and Corollary 3.1 in [1]).

Proposition 2.1. *Given a reflexive Banach space $\mathbb{B}$, a point $z \in \mathbb{B}$, and a number $R > 0$, let $F : B_R[z] \rightarrow \mathbb{B}$ be a weak-strong continuous mapping. Assume that there exists a functional $[\cdot, \cdot]_c : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{R}$ satisfying*

- (C₁) $[x, x]_c > 0$, for all $x \in \mathbb{B}$ with $x \neq 0$;
- (C₂) $[\lambda x, x]_c = \lambda [x, x]_c$, for all $x \in \mathbb{B}$ and $\lambda \in \mathbb{R}$.

If $[F(x), x - z]_c$ has constant sign for all $x \in \partial B_R[z]$, then F has at least a zero in $B_R[z]$.

We finish this section recalling several basic definitions and relevant results related to Sobolev spaces.

Let Ω be a measurable subset of $\mathbb{R}^N$ which for simplicity will be assumed to be bounded. The Sobolev space $W^{m,p}(\Omega)$ (see [3, 18]) is the Banach space

of all functions in $L^p(\Omega)$ whose weak derivatives up to order m also belong to $L^p(\Omega)$. The norm in $W^{m,p}(\Omega)$ is given by

$$\|u\|_{m,p} = \|u\|_p + \sum_{1 \leq |\alpha| \leq m} \|D^\alpha u\|_p,$$

or by the equivalent norm

$$\|u\|_{m,p} = \left(\|u\|_p^p + \sum_{1 \leq |\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}},$$

where $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ is a multi-index of non-negative integers with order $|\alpha| = \sum_{i=1}^N \alpha_i$ and $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}} u$. $W_0^{m,p}(\Omega)$ is the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$. Here, $\mathcal{C}_c^\infty(\Omega)$ denotes the space of infinitely differentiable functions $\phi: \Omega \rightarrow \mathbb{R}$ with compact support in Ω .

The next inequality is known as “Poincaré’s inequality” and it is very useful in the study of Dirichlet elliptic equations (for instance, see [18, Section 13.2]) as follows:

Proposition 2.2. *There exists a positive constant $c = c(\Omega, p)$, such that*

$$\|w\|_p \leq c \|\nabla w\|_p, \quad (2.1)$$

for all $w \in W_0^{1,p}(\Omega)$.

As a consequence of the above inequality, we get to $\|\nabla u\|_2$ is an equivalent norm on $W_0^{1,2}(\Omega)$, namely

$$\frac{1}{\sqrt{c^2 + 1}} \|w\|_{1,2} \leq \|\nabla w\|_2 \leq \|w\|_{1,2}, \quad \text{for all } w \in W_0^{1,2}(\Omega), \quad (2.2)$$

where we are considering that $\|w\|_{1,2} = \sqrt{\|w\|_2^2 + \|\nabla w\|_2^2}$, and $\|\nabla w\|_2 = \sqrt{\sum_{i=1}^n \left\| \frac{\partial w}{\partial x_i} \right\|_2^2}$.

3. Main Results

In 1971, Knobloch [16] showed that if $f: [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function which is locally Lipschitzian in u , the condition: there exists $R > 0$, such that $\langle f(t, u), u \rangle \geq 0$ for all $t \in [0, T]$ and $u \in \mathbb{R}^n$ satisfying $\|u\| = R$ is a sufficient condition for the existence of a solution for Problem (1.1). In our case, we have to work with conditions (3.2) and (3.4) (see below).

For our subsequent study, we introduce, for each element of the Banach space $\mathcal{C}([a, b], \mathbb{B})$, an auxiliary mapping and we will prove the existence and uniqueness of zeros of this mapping. Let $v \in \mathcal{C}([a, b], \mathbb{B})$. We define $F_v: \mathbb{B} \rightarrow \mathbb{B}$ as

$$F_v(x) = \int_a^b f(\tau, \int_a^\tau v(s) \, ds + x) \, d\tau. \quad (3.1)$$

Lemma 3.1. *Let $v \in \mathcal{C}([a, b], \mathbb{B})$ and let $f : [a, b] \times \mathbb{B} \rightarrow \mathbb{B}$ be a sequentially weak-strong continuous function satisfying that*

$$\begin{aligned} &\text{for each } M > 0, \text{ there exists } \delta_M > 0, \text{ such that } \langle f(t, y + x), x \rangle_- \geq 0, \\ &\text{for all } t \in [a, b], y \in B_M[0] \text{ and } \|x\| \geq \delta_M. \end{aligned} \quad (3.2)$$

Then, the mapping $F_v : \mathbb{B} \rightarrow \mathbb{B}$ given by (3.1) has at least a zero in $\mathbb{B}$.

Proof. Let $v \in \mathcal{C}([a, b], \mathbb{B})$. First, we are going to see that F_v is sequentially weak-strong continuous. Let $\{x_n\}$ be a bounded sequence in $\mathbb{B}$, i.e., there exists $r > 0$, such that $x_n \in B_r[0]$ for all $n \in \mathbb{N}$. Since $\mathbb{B}$ is reflexive, there exists a weakly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, say $x_{n_k} \rightharpoonup x \in B_r[0]$. We consider the sequence of functions given by $g_{n_k}(t) := f(t, \int_a^t v(s) ds + x_{n_k})$, for $t \in [a, b]$. By hypothesis, f is sequentially weak-strong continuous; then, for each $t \in [a, b]$, $g_{n_k}(t)$ converges strongly to $g(t) := f(t, \int_a^t v(s) ds + x)$. Furthermore, for each n_k , we get to $\|g_{n_k}(t)\| \leq K$ for all $t \in [a, b]$, where $K \geq 0$ is the constant of boundedness of $f([a, b] \times B_{r+(b-a)\|v\|_\infty}[0])$. Thus, for each $k \in \mathbb{N}$

$$\begin{aligned} \|F_v(x_{n_k}) - F_v(x)\| &\leq \int_a^b \left\| f(t, \int_a^t v(s) ds + x_{n_k}) - f(t, \int_a^t v(s) ds + x) \right\| dt \\ &= \int_a^b \|g_{n_k}(t) - g(t)\| dt, \end{aligned}$$

which implies, by the dominated convergence theorem, the strong convergence of $\{F_v(x_{n_k})\}$ to $F_v(x)$.

On the other hand, if we consider the functional $[\cdot, \cdot]_c : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{R}$ given by

$$[x, y]_c = \langle x, y \rangle_+ := \sup_{j \in J(y)} \langle x, j \rangle = \max_{j \in J(y)} \langle x, j \rangle,$$

then $[\cdot, \cdot]_c$ satisfies (C_1) – (C_2) . Moreover, taking $M := (b - a)\|v\|_\infty$, we know that, for all $\tau \in [a, b]$, $\int_a^\tau v(s) ds \in B_M[0]$. By (3.2), there exists $\delta_M > 0$, such that $\langle f(t, \int_a^t v(s) ds + x), x \rangle_- \geq 0$, for all $t \in [a, b]$ and $\|x\| \geq \delta_M$. Therefore, for all $x \in \partial B_{\delta_M}[0]$, we have that

$$[F_v(x), x]_c = \langle F_v(x), x \rangle_+ \geq \int_a^b \langle f(\tau, \int_a^\tau v(s) ds + x), x \rangle_- d\tau \geq 0.$$

Applying Proposition 2.1, we deduce that F_v has at least a zero in $B_{\delta_M}[0]$. $\square$

Remark 3.1. Bearing in mind Proposition 2.1 and the proof of Lemma 3.1, in Lemma 3.1, we can, clearly, replace Condition (3.2) by

$$\begin{aligned} &\text{for each } M > 0 \text{ there exists } \delta_M > 0 \text{ such that } \langle f(t, y + x), x \rangle_+ \leq 0 \\ &\text{for all } t \in [a, b], y \in B_M[0] \text{ and } \|x\| \geq \delta_M. \end{aligned} \quad (3.3)$$

Lemma 3.2. *Let $v \in \mathcal{C}([a, b], \mathbb{B})$ and let $f : [a, b] \times \mathbb{B} \rightarrow \mathbb{B}$ be a strictly accretive operator with respect to the second variable, that is*

$$\langle f(t, x) - f(t, y), x - y \rangle_- > 0 \text{ for all } t \in [a, b], x, y \in \mathbb{B}, \text{ with } x \neq y. \quad (3.4)$$

Then, if there is a zero of the operator $F_v : \mathbb{B} \rightarrow \mathbb{B}$, given by (3.1), it is unique.

Proof. Note that, by (3.4), F_v is strictly accretive, that is, $\langle F_v(x) - F_v(y), j \rangle > 0$ for all $x, y \in \mathbb{B}$, with $x \neq y$ and for all $j \in J(x - y)$. Indeed, for each $x, y \in \mathbb{B}$ and $j \in J(x - y)$, we have that

$$\begin{aligned} \langle F_v(x) - F_v(y), j \rangle &= \left\langle \int_a^b (f(\tau, \int_a^\tau v(s) ds + x) - f(\tau, \int_a^\tau v(s) ds + y)) d\tau, j \right\rangle \\ &= \int_a^b \langle f(\tau, \int_a^\tau v(s) ds + x) - f(\tau, \int_a^\tau v(s) ds + y), j \rangle d\tau. \end{aligned}$$

Since $j \in J(x - y) = J(x + \int_a^\tau v(s) ds - (y + \int_a^\tau v(s) ds))$, from (3.4), we obtain that $\langle F_v(x) - F_v(y), j \rangle > 0$ for all $x, y \in \mathbb{B}$, with $x \neq y$ and for all $j \in J(x - y)$.

Now, suppose that $z_1, z_2 \in \mathbb{B}$ are two zeros of F_v with $z_1 \neq z_2$. Since F_v is strictly monotone, $\langle F_v(z_1) - F_v(z_2), j \rangle > 0$ for all $j \in J(z_1 - z_2)$ which contradicts that $F_v(z_i) = 0$, $i = 1, 2$. Therefore, $z_1 = z_2$, that is, if F_v has a zero, it is unique. $\square$

Remark 3.2. Checking the proof of Lemma 3.2, we can replace the hypothesis of strict accretiveness with respect to the second variable (3.4) by being a strictly dissipative operator, that is

$$\begin{aligned} f : [a, b] \times \mathbb{B} \rightarrow \mathbb{B} \text{ satisfies } \langle f(t, x) - f(t, y), x - y \rangle_+ &< 0 \\ \text{for all } t \in [a, b], x, y \in \mathbb{B}, \text{ with } x \neq y. \end{aligned} \quad (3.5)$$

At this point, we can prove the main result of this work related to the existence of a periodic solution of Problem (1.1).

Theorem 3.1. *Let $f : [a, b] \times \mathbb{B} \rightarrow \mathbb{B}$ be a sequentially weak-strong continuous function satisfying (3.2) and (3.4) (or (3.3) and (3.5), respectively). Let $h : [a, b] \rightarrow \mathbb{B}$ be a continuous function with $\int_a^b h(t) dt = 0$. Then, Problem (1.1) has at least one solution $u \in C^2([a, b], \mathbb{B})$ whenever one of the following assumptions holds:*

- (A₁). [A condition of boundedness] f is bounded.
- (A₂). [A growth condition] There exists a non-decreasing function $\Gamma : [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\lim_{s \rightarrow \infty} \frac{\Gamma(s + \delta_s)}{s} < \frac{1}{2(b - a)^2},$$

where δ_s is the constant given in (3.2), such that

$$\|f(t, x)\| \leq \Gamma(\|x\|), \quad \text{for all } (t, x) \in [a, b] \times \mathbb{B}.$$

Proof. For the sake of clarity, the proof will be divided into five steps.

Step 1. Notice that if we consider $v(t) = u'(t)$, then Problem (1.1) is equivalent to solve the following problem:

$$\begin{cases} v'(t) = f(t, \int_a^t v(s) ds + D) + h(t) \\ v(a) = v(b), \int_a^b v(t) dt = 0, \end{cases} \quad (3.6)$$

where D is any element of $\mathbb{B}$ depending of v .

Step 2. By Lemmas 3.1 and 3.2, for each $v \in \mathcal{C}([a, b], \mathbb{B})$, there exists a unique zero D_v for F_v in $B_R[0]$, where $R := \delta_M$ and $M := (b - a) \|v\|_\infty$.

Integrating Eq. (3.6), we deduce that the previous problem is equivalent to the following integral equation:

$$v(t) = \int_a^t f(\tau, \int_a^\tau v(s) ds + D_v) d\tau + \int_a^t h(\tau) d\tau + v(a),$$

where we take

$$v(a) := -\frac{1}{b-a} \int_a^b \left(\int_a^t (f(\tau, \int_a^\tau v(s) ds + D_v) + h(\tau)) d\tau \right) dt,$$

and D_v is the unique zero of F_v , because in this case, it is clear that $\int_a^b v(t) dt = 0$. Thus, we consider the operator $T : \mathcal{C}([a, b], \mathbb{B}) \rightarrow \mathcal{C}([a, b], \mathbb{B})$, given by

$$T(v)(t) := \int_a^t f(\tau, \int_a^\tau v(s) ds + D_v) d\tau + \int_a^t h(\tau) d\tau + v(a),$$

and, therefore, to find a fixed point for the operator T is equivalent to find a solution of Eq. (3.6).

Step 3. Let us see that T is continuous. Let $\{v_n\}$ be a sequence in $\mathcal{C}([a, b], \mathbb{B})$ that converges strongly to $v \in \mathcal{C}([a, b], \mathbb{B})$. Note that $\{D_{v_n}\}$ is a bounded sequence, and since $\mathbb{B}$ is a reflexive Banach space, we may assume that $D_{v_n} \rightharpoonup D$. We can argue as in proof of Lemma 3.1 where now we can define $g_n(t) := f(t, \int_a^t v_n(s) ds + D_{v_n})$ and $g(t) = f(t, \int_a^t v(s) ds + D)$. The uniqueness of the zeros for F_v allows us to conclude that $D_v = D$.

On the other hand, for each $t \in [a, b]$

$$\begin{aligned} \|T(v_n)(t) - T(v)(t)\| &\leq \int_a^t \|g_n(\tau) - g(\tau)\| d\tau + \|v_n(a) - v(a)\| \\ &\leq \int_a^b \|g_n(\tau) - g(\tau)\| d\tau + \|v_n(a) - v(a)\| \\ &\leq 2 \int_a^b \|g_n(\tau) - g(\tau)\| d\tau, \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\|T(v_n) - T(v)\|_\infty \leq 2 \int_a^b \|g_n(\tau) - g(\tau)\| d\tau$, for all $n \in \mathbb{N}$.

Since f is sequentially weak-strong continuous, $\|g_n(\tau) - g(\tau)\| \rightarrow 0$ as $n \rightarrow \infty$ for all $\tau \in [a, b]$ and, furthermore, $\|g_n(t)\| \leq K$ for all $t \in [a, b]$ where $K \geq 0$ is the constant of boundedness of $f([a, b] \times B_{2(b-a)+\rho})$ and $\rho > 0$ satisfying $\|v_n\|_\infty \leq \rho$ for all $n \in \mathbb{N}$, because $\{v_n\}$ is a convergent sequence. Then, by the dominated convergence theorem, we obtain $T(v_n) \rightarrow T(v)$ as $n \rightarrow \infty$, which means that T is continuous.

Step 4. We next prove that T is compact. Let C be a bounded subset of $\mathcal{C}([a, b], \mathbb{B})$ and let us see that $\overline{T(C)}$ is compact. To prove this, we shall make use of the Ascoli–Arzelà Theorem.

In the first place, let $0 < \varrho = \varrho(C) := \sup\{\|v\|_\infty : v \in C\} < \infty$, because C is bounded. Then, for any $v \in C$, we get to $\|\int_a^\tau v(s) ds + D_v\| \leq$

$(b-a)\varrho + \delta_{(b-a)\varrho}$ for all $\tau \in [a, b]$, where $\delta_{(b-a)\varrho}$ is the constant given in (3.2). Since f is sequentially weak-strong continuous, there exists $K > 0$, such that $\|f(t, x)\| \leq K$ for all $t \in [a, b]$ and $x \in B_R[0]$, where $R := (b-a)\varrho + \delta_{(b-a)\varrho}$. Then, for any $t \in [a, b]$ and $v \in C$, we have that

$$\begin{aligned} \|T(v)(t)\| &\leq 2 \int_a^b (\|f(\tau, \int_a^\tau v(s) ds + D_v)\| + \|h(\tau)\|) d\tau \\ &\leq 2(b-a)(K + \|h\|_\infty), \end{aligned}$$

that is, (i) holds. On the other hand, for any $\varepsilon > 0$, if we take $\delta(\varepsilon) = \varepsilon/(K + \|h\|_\infty) > 0$, then, for all $v \in C$ and $t, s \in [a, b]$ with $t > s$ and $|t - s| \leq \delta(\varepsilon)$, we obtain that

$$\begin{aligned} \|T(v)(t) - T(v)(s)\| &\leq \int_s^t (\|f(\tau, \int_a^\tau v(s) ds + D_v)\| + \|h(\tau)\|) d\tau \\ &\leq (t-s)(K + \|h\|_\infty) < \varepsilon, \end{aligned}$$

which means that $T(C)$ is equi-continuous. Finally, let us see that, for each $t \in [a, b]$, the set $\{T(v)(t) : v \in C\}$ is compact in $\mathbb{B}$. Fix $v \in C$. For each $t \in [a, b]$, we have that

$$\int_a^t f(\tau, \int_a^\tau v(s) ds + D_v) d\tau \in (t-a) \overline{\text{conv}}(f([0, t] \times B_R[0])).$$

Since $\mathbb{B}$ is reflexive, $[0, t] \times B_R[0]$ is weak compact and, since f is sequentially weak-strong continuous, $f([0, t] \times B_R[0])$ is compact. By Mazur's Compactness Theorem, we obtain that $(t-a) \overline{\text{conv}}(f([0, t] \times B_R[0]))$ is compact, for any $t \in [a, b]$.

Therefore, from Ascoli-Arzelà Theorem, we deduce that T is a compact operator.

Step 5. Let us prove that T has a fixed point in $\mathcal{C}([a, b], \mathbb{B})$. It is easy to see that, for any $v \in \mathcal{C}([a, b], \mathbb{B})$

$$\|T(v)(t)\| \leq 2 \int_a^b \|f(\tau, \int_a^\tau v(s) ds + D_v)\| d\tau + 2(b-a) \|h\|_\infty \quad \forall t \in [a, b]. \quad (3.7)$$

- (a) Suppose that f is bounded. Then, there exists $\kappa \geq 0$, such that $\|f(t, x)\| \leq \kappa$ for all $(t, x) \in [a, b] \times \mathbb{B}$. Bearing in mind (3.7), one can prove that $T(\mathbb{B}) \subseteq B_S[0]$, where $S := 2(b-a)(\kappa + \|h\|_\infty)$. Thus, using Schauder's Fixed Point Theorem, we can ensure that $T|_{B_S[0]} : B_S[0] \rightarrow B_S[0]$ has a fixed point.
- (b) Suppose that (A_2) holds. Let us see that T satisfies the Leray-Schauder boundary condition on some closed ball. From (3.7) and (A_2) , for any $v \in \mathcal{C}([a, b], \mathbb{B})$

$$\begin{aligned} \|T(v)\|_\infty &\leq 2(b-a) \left(\Gamma((b-a) \|v\|_\infty + \|D_v\|) + \|h\|_\infty \right) \\ &\leq 2(b-a) \left(\Gamma((b-a) \|v\|_\infty + \delta_{(b-a)\varrho} \|v\|_\infty) + \|h\|_\infty \right). \end{aligned}$$

From the hypothesis of Γ , we deduce that there exists $S > 0$ such that, for all $\|v\|_\infty \geq S$

$$\begin{aligned} \frac{\|T(v)\|_\infty}{\|v\|_\infty} &\leq 2(b-a)^2 \frac{\Gamma((b-a)\|v\|_\infty + \delta_{(b-a)\|v\|_\infty})}{(b-a)\|v\|_\infty} \\ &\quad + 2(b-a) \frac{\|h\|_\infty}{\|v\|_\infty} \\ &< 1. \end{aligned}$$

Then, the Leray–Schauder boundary condition on the ball $B_S[0]$ holds. Therefore, by Leray–Schauder alternative Theorem, there exists at least a fixed point of T .

□

Remark 3.3. The method of proof is relying on fixed point theorems of Leray–Schauder type, which does not yield uniqueness. It is an open question to ensure that the solution is unique or the possibility of multiple solutions under the assumptions of our main theorem. However, in the absence of condition (3.2), the existence of any solution cannot be guaranteed (see example 4.2 below). On the other hand, if conditions (A_1) and (A_2) fail, we cannot claim the uniqueness of solutions: for example, the following second-order differential equation has multiple solutions:

$$\begin{cases} u''(t) = -u(t) \\ u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0. \end{cases}$$

Finally, we point out that the hypothesis $\int_a^b h(t) dt = 0$ cannot be removed in our main result (see Example 4.2 below).

4. Examples

In this section, we present three examples of application of Theorem 3.1. Two of them for a system of n ($n \in \mathbb{N}$) second-order differential equations and the last one for a partial differential equation.

Example 4.1. If $h_1, \dots, h_n : [a, b] \rightarrow \mathbb{R}$ are continuous functions with $\int_a^b h_i(t) dt = 0$, then the problem

$$\begin{cases} u_1''(t) = \frac{u_1(t)}{1 + |u_1(t)| + \dots + |u_n(t)|} + h_1(t) \\ \vdots \\ u_n''(t) = \frac{u_n(t)}{1 + |u_1(t)| + \dots + |u_n(t)|} + h_n(t) \\ u_1(a) = u_1(b), \dots, u_n(a) = u_n(b), \\ u_1'(a) = u_1'(b), \dots, u_n'(a) = u_n'(b), \end{cases} \quad (4.1)$$

has at least a solution in $\mathcal{C}^2([a, b], \mathbb{R}^n)$. Indeed, just to note that (4.1) can be rewritten as Problem (1.1) where $h := (h_1, \dots, h_n)$, $u = (u_1, \dots, u_n)$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by $f(x) = \frac{x}{1+\|x\|_1}$. Here, $\|\cdot\|_1$ denotes the 1-norm of $\mathbb{R}^n$.

It is clear that f is bounded and weak-strong continuous. On the other hand, for each $M > 0$, taking $\delta_M = M$, we have that $\langle f(y+x), x \rangle_- \geq 0$ for all $y \in B_M[0]$ and $\|x\|_1 \geq \delta_M$. Indeed, fix $y \in B_M[0]$ and $\|x\|_1 \geq \delta_M$, for any $j \in J(x)$

$$\begin{aligned} \langle f(y+x), j \rangle &= \frac{1}{1+\|y+x\|_1} \langle y+x, j \rangle = \frac{1}{1+\|y+x\|_1} (\|x\|_1^2 + \langle y, x \rangle) \\ &\geq \frac{1}{1+\|y+x\|_1} (\|x\|_1^2 - \|x\|_1 \|y\|_1) \\ &= \frac{1}{1+\|y+x\|_1} (\|x\|_1 - \|y\|_1) \|x\|_1 \geq 0, \end{aligned}$$

since $\|y\|_1 \leq M$ and $\|x\|_1 \geq \delta_M = M$.

Furthermore, f is strictly accretive, because, for all $x, y \in \mathbb{R}^n$ with $x \neq y$, let j be an element of $J(x-y)$; then

$$\begin{aligned} \langle f(x) - f(y), j \rangle &= \frac{1}{(1+\|x\|_1)(1+\|y\|_1)} \langle x-y+x\|y\|_1 - y\|x\|_1, j \rangle \\ &= \frac{1}{(1+\|x\|_1)(1+\|y\|_1)} \left(\|x-y\|_1^2 \right. \\ &\quad \left. + \langle (x-y)\|y\|_1 + y(\|y\|_1 - \|x\|_1), j \rangle \right) \\ &\geq \frac{\|x-y\|_1^2}{(1+\|x\|_1)(1+\|y\|_1)} > 0. \end{aligned}$$

By Theorem 3.1 (A_1), we deduce that Problem (4.1) has at least one solution $u \in \mathcal{C}^2([a, b], \mathbb{R}^n)$.

Remark 4.1. By carefully checking the above example, we find that other kinds of systems of equations can be solved. Just consider another norm $\|\cdot\|$ in $\mathbb{R}^n$. For example, we can ensure the existence of at least a periodic solution of a system of nonlinear second-order differential equations of the following type:

$$u''(t) = \frac{u(t)}{1+\|u(t)\|} + h(t).$$

The following example shows that in Theorem 3.1, neither the hypothesis $\int_a^b h(t) dt = 0$ nor condition (3.2) can be omitted.

Example 4.2. The following problem does not have solution:

$$\begin{cases} u''(t) = \frac{u(t)}{1+|u(t)|} + 2 \\ u(0) = u(1), \\ u'(0) = u'(1). \end{cases} \quad (4.2)$$

Indeed, suppose that u_0 is a solution of (4.2), then if we call $v(t) = u'_0(t)$, we have

$$v'(t) = \frac{\int_0^t v(s)ds + u_0(0)}{1 + \left| \int_0^t v(s)ds + u_0(0) \right|} + 2,$$

Then,

$$0 = u'_0(1) - u'_0(0) = v(1) - v(0) = \int_0^1 \frac{\int_0^t v(s)ds + u_0(0)}{1 + \left| \int_0^t v(s)ds + u_0(0) \right|} dt + 2,$$

This means that

$$\int_0^1 \frac{\int_0^t v(s)ds + u_0(0)}{1 + \left| \int_0^t v(s)ds + u_0(0) \right|} dt = -2,$$

which is a contradiction.

On the other hand, it is not difficult to see that the function $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(t, x) = \frac{x}{1+|x|}$ is under the conditions of Remark 4.1; therefore, hypothesis $\int_a^b h(t)dt = 0$ cannot be omitted.

If we call $f(t, x) = \frac{x}{1+|x|} + 2$ and $h \equiv 0$, we may say that in the main result, condition (3.2) cannot be removed.

We next give an example in the case f is unbounded and, moreover, the inequality required for Γ in assumption (A_2) holds for any closed interval $[a, b]$.

Example 4.3. Let $h_1, \dots, h_n : [a, b] \rightarrow \mathbb{R}$ be continuous functions with $\int_a^b h_i(t) dt = 0$ and let $0 < \alpha < 1$. Then, the problem

$$\begin{cases} u''_1(t) = \frac{u_1(t) \sqrt{(u_1^2(t) + \dots + u_n^2(t))^\alpha}}{1 + \sqrt{u_1^2(t) + \dots + u_n^2(t)}} + h_1(t) \\ \vdots \\ u''_n(t) = \frac{u_n(t) \sqrt{(u_1^2(t) + \dots + u_n^2(t))^\alpha}}{1 + \sqrt{u_1^2(t) + \dots + u_n^2(t)}} + h_n(t) \\ u_1(a) = u_1(b), \dots, u_n(a) = u_n(b), \\ u'_1(a) = u'_1(b), \dots, u'_n(a) = u'_n(b) \end{cases} \quad (4.3)$$

has at least a solution in $\mathcal{C}^2([a, b], \mathbb{R}^n)$. This problem can be rewritten as Problem (1.1), where $h := (h_1, \dots, h_n)$, $u = (u_1, \dots, u_n)$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by $f(x) = \frac{x \|x\|_2^\alpha}{1 + \|x\|_2}$. Here, $\|\cdot\|_2$ denotes the Euclidean norm of $\mathbb{R}^n$.

It is immediate that f weak-strong continuous, because $\mathbb{R}^n$ is finite dimensional and f is continuous. We claim that f satisfies (3.2). Indeed, for each $M > 0$, taking $\delta_M = M > 0$, we deduce that

$$\langle f(y+x), x \rangle = \frac{\|y+x\|_2^\alpha}{1 + \|y+x\|_2} (\|x\|_2^2 + \langle y, x \rangle)$$

$$\geq \frac{\|y + x\|_2^\alpha \|x\|_2}{1 + \|y + x\|_2} (\|x\|_2 - \|y\|_2) \geq 0,$$

for all $x, y \in \mathbb{R}^n$, with $\|y\|_2 \leq M$ and $\|x\|_2 \geq \delta_M = M$.

We now prove that f is strictly accretive. Let $x, y \in \mathbb{R}^n$. Note that

$$\begin{aligned} \langle f(x) - f(y), x - y \rangle &= \frac{1}{(1 + \|x\|_2)(1 + \|y\|_2)} \langle \|x\|_2^\alpha (1 + \|y\|_2)x \\ &\quad - \|y\|_2^\alpha (1 + \|x\|_2)y, x - y \rangle \\ &= \frac{1}{(1 + \|x\|_2)(1 + \|y\|_2)} \left(\langle \|x\|_2^\alpha x - \|y\|_2^\alpha y, x - y \rangle \right. \\ &\quad \left. + \langle \|x\|_2^\alpha \|y\|_2 x - \|y\|_2^\alpha \|x\|_2 y, x - y \rangle \right). \end{aligned}$$

In [11, Proof of Lemma 1], the authors prove that (in general, for $\alpha \geq 0$)

$$\langle \|x\|_2^\alpha x - \|y\|_2^\alpha y, x - y \rangle \geq \frac{1}{2} \|x - y\|_2^2 (\|x\|_2^\alpha + \|y\|_2^\alpha).$$

On the other hand, if $\|y\|_2 > \|x\|_2$, from Cauchy–Bunyakovsky–Schwarz inequality

$$\begin{aligned} &\langle \|x\|_2^\alpha \|y\|_2 x - \|y\|_2^\alpha \|x\|_2 y, x - y \rangle \\ &= \langle \|x\|_2^\alpha \|y\|_2 (x - y) + (\|x\|_2^\alpha \|y\|_2 - \|y\|_2^\alpha \|x\|_2) y, x - y \rangle \\ &= \|x\|_2^\alpha \|y\|_2 \|x - y\|_2^2 + (\|x\|_2^\alpha \|y\|_2 - \|y\|_2^\alpha \|x\|_2) \langle y, x - y \rangle \\ &= \|x\|_2^\alpha \|y\|_2 \|x - y\|_2^2 + \|x\|_2^\alpha \|y\|_2^\alpha (\|y\|_2^{1-\alpha} - \|x\|_2^{1-\alpha}) \langle y, x - y \rangle \\ &\geq \|x\|_2^\alpha \|y\|_2 \|x - y\|_2^2 - \|x\|_2^\alpha \|y\|_2^\alpha (\|y\|_2^{1-\alpha} - \|x\|_2^{1-\alpha}) \|y\|_2 \|x - y\|_2 \\ &= \|x\|_2^\alpha \|y\|_2 \|x - y\|_2 \left(\|x - y\|_2 - \|y\|_2^\alpha (\|y\|_2^{1-\alpha} - \|x\|_2^{1-\alpha}) \right) \\ &= \|x\|_2^\alpha \|y\|_2 \|x - y\|_2 \left(\|x - y\|_2 - \|y\|_2 + \|y\|_2^\alpha \|x\|_2^{1-\alpha} \right) \\ &\geq \|x\|_2^\alpha \|y\|_2 \|x - y\|_2 \left(\|x - y\|_2 - \|y\|_2 + \|x\|_2 \right) \geq 0. \end{aligned}$$

Therefore, $\langle f(x) - f(y), x - y \rangle > 0$ for all $x, y \in \mathbb{R}^n$ with $\|y\|_2 > \|x\|_2$. This one, together with the symmetry of scalar product, implies that the above inequality holds for all $x, y \in \mathbb{R}^n$, $x \neq y$, that is, f is strictly accretive.

Finally, let us see that (A_2) holds. Indeed, just take the function $\Gamma : [0, \infty) \rightarrow [0, \infty)$ defined by $\Gamma(s) = s^{\alpha+1}/(1+s)$. In this case, we have that $\|f(t, x)\|_2 \leq \Gamma(\|x\|_2)$ for all $(t, x) \in [a, b] \times \mathbb{R}^n$ and moreover

$$\lim_{s \rightarrow \infty} \frac{\Gamma(s + \delta_s)}{s} = \lim_{s \rightarrow \infty} \frac{\Gamma(2s)}{s} = \lim_{s \rightarrow \infty} \frac{2^\alpha}{s^{1-\alpha}} = 0 < \frac{1}{2(b-a)^2},$$

since $\delta_s = s$ is the constant given in (3.2) in this case, and $0 < \alpha < 1$.

Example 4.4. Let Ω be an open convex and bounded subset of $\mathbb{R}^n$ and consider $\rho : [a, b] \times \Omega \rightarrow \mathbb{R}$ a function in $\mathcal{C}([a, b], L^2(\Omega))$.

$$\begin{cases} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial^2 \phi}{\partial t^2}(t, x) - \rho(t, x) \right) = \phi(t, x) \text{ in } (a, b) \times \Omega, \\ \phi(a, x) - \phi(b, x) = \frac{\partial \phi}{\partial t}(a, x) - \frac{\partial \phi}{\partial t}(b, x) = 0 \text{ in } \Omega, \\ \frac{\partial^2 \phi}{\partial t^2}(t, x) - \rho(t, x) = 0 \text{ on } (a, b) \times \partial\Omega. \end{cases} \quad (4.4)$$

Definition 4.1. A function ϕ is said to be a strong solution of Problem (4.4) if the following assumptions are satisfied:

- (i) $\phi \in \mathcal{C}^2([a, b], L^2(\Omega))$,
- (ii) $\frac{\partial^2 \phi}{\partial t^2}(t, \cdot) \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ for all $t \in [a, b]$,
- (iii) ϕ satisfies Eq. (4.4) in the sense of distributions over Ω .

To do this study using the above results, consider the operator $\Delta : L^2(\Omega) \rightarrow L^2(\Omega)$ with domain $D(\Delta) := W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ and, given $y \in D(\Delta)$, define $\Delta y = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} y(x)$, where the partial derivatives are taken in the sense of distributions over Ω . Therefore, if we call $u(t) := \phi(t, \cdot) \in L^2(\Omega)$ and $h(t) := \rho(t, \cdot) \in L^2(\Omega)$. We interpret and rewrite Problem (4.4) as follows:

$$\begin{cases} \Delta(u''(t) - h(t)) = u(t) \\ u(a) - u(b) = u'(a) - u'(b) = 0. \end{cases} \quad (4.5)$$

Our idea is to show that there exists at least $u \in \mathcal{C}^2([a, b]; L^2(\Omega))$ which is a solution of (4.5). To get it, we are going to assume that $h \in \mathcal{C}([a, b], L^2(\Omega))$ and, moreover, $\int_a^b h(t) dt = 0$.

It is well known (for instance, see [2]) that Δ is an m -dissipative operator in $L^2(\Omega)$. Moreover, using Green's formula, we obtain that for $u \in D(\Delta)$

$$\langle \Delta u, u \rangle = - \int_{\Omega} \langle \nabla u, \nabla u \rangle = - \|\nabla u\|_2^2;$$

this implies, by Poincaré's inequality (see Proposition 2.2), that

$$\|\Delta u\|_2 \|u\|_2 \geq \|\nabla u\|_2^2 \geq \frac{1}{c} \|\nabla u\|_2 \|u\|_2$$

and, bearing in mind (2.2), we obtain that

$$\frac{1}{c\sqrt{c^2+1}} \|u\|_{1,2} \leq \|\Delta u\|_2 \quad \text{for all } u \in D(\Delta). \quad (4.6)$$

The inequality given in (4.6) says that Δ is an m - ψ -expansive dissipative operator (it is enough to take $\psi(t) = t/(c\sqrt{c^2+1})$). Then, Remark 4.12 in [9] guarantees that there exists $\Delta^{-1} : L^2(\Omega) \rightarrow D(\Delta)$ and it is a strictly m -dissipative operator in $L^2(\Omega)$. This fact allows us to rewrite Problem (4.5) as follows:

$$\begin{cases} u''(t) = \Delta^{-1}(u(t)) + h(t) \\ u(a) - u(b) = u'(a) - u'(b) = 0. \end{cases} \quad (4.7)$$

Finally, to show that Problem (4.7) admits at least a solution, we only have to check that $\Delta^{-1} : L^2(\Omega) \rightarrow D(\Delta)$ satisfies the conditions of Theorem 3.1.

First, let us see that $\Delta^{-1} : L^2(\Omega) \rightarrow D(\Delta)$ is sequentially weak-strong continuous. Indeed, from inequality (4.6), it is clear that

$$\|\Delta^{-1}u\|_2 \leq \|\Delta^{-1}u\|_{1,2} \leq c\sqrt{c^2+1}\|u\|_2, \quad (4.8)$$

which means that $\Delta^{-1} : L^2(\Omega) \rightarrow D(\Delta)$ is continuous, because it is linear, then this operator is also weak-weak-continuous. On the other hand, if M is a bounded subset of $L^2(\Omega)$, from (4.8), we have that $\Delta^{-1}(M)$ is a bounded subset of $W^{1,2}(\Omega)$; since the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact (Rellich–Kondrachov’s theorem), we have that $\Delta^{-1}(M)$ relatively compact in $L^2(\Omega)$; since Δ^{-1} is continuous, we may conclude that Δ is a weak-strong continuous map.

On the other hand, let us see that $\Delta^{-1} : L^2(\Omega) \rightarrow D(\Delta)$ satisfies Condition (3.3). Indeed, let $M > 0$ and consider $g \in B_M[0] \subseteq L^2(\Omega)$; we have to check that $\langle \Delta^{-1}(u+g), u \rangle = \langle \Delta^{-1}u, u \rangle + \langle \Delta^{-1}g, u \rangle$.

If we call $z := \Delta^{-1}(u) \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, we may consider that $\Delta z = u$; therefore, by using Green’s formula, Cauchy–Bunyakovsky–Schwarz inequality, and (4.8), we derive that

$$\begin{aligned} \langle \Delta^{-1}(u+g), u \rangle &\leq -\|\nabla z\|_2^2 + \|\Delta^{-1}g\|_2 \|u\|_2 \\ &\leq -\|\nabla z\|_2^2 + c\sqrt{c^2+1}\|g\|_2 \|u\|_2. \end{aligned}$$

This means that for every $g \in B_M[0]$

$$\langle \Delta^{-1}(u+g), u \rangle \leq -\|\nabla z\|_2^2 + c\sqrt{c^2+1}M\|u\|_2.$$

Now, using (2.2), we get

$$\langle \Delta^{-1}(u+g), u \rangle \leq -\frac{1}{c^2+1}\|z\|_{1,2}^2 + c\sqrt{c^2+1}M\|u\|_2.$$

Finally, since it is clear that $\|\Delta z\|_2 \leq \|\Delta z\|_{1,2} \leq \|z\|_{1,2}$, we conclude that

$$\begin{aligned} \langle \Delta^{-1}(u+g), u \rangle &\leq -\frac{1}{c^2+1}\|u\|_2^2 + c\sqrt{c^2+1}M\|u\|_2 \\ &= \|u\|_2(c\sqrt{c^2+1}M - \frac{1}{c^2+1}\|u\|_2) \leq 0, \end{aligned}$$

whenever $Mc(c^2+1)^{3/2} \leq \|u\|_2$; hence, we may take $\delta_M := Mc(c^2+1)^{3/2}$.

Second, we have already seen that $\Delta^{-1} : L^2(\Omega) \rightarrow D(\Delta)$ is a strictly dissipative operator, and thus, it satisfies Condition (3.5).

Finally, if we define $\Gamma : [0, \infty) \rightarrow [0, \infty)$ by $\Gamma(s) := c\sqrt{c^2+1}s$. From (4.8), we have that $\|\Delta^{-1}u\|_2 \leq \Gamma(\|u\|_2)$ for all $u \in L^2(\Omega)$. Thus

$$\lim_{s \rightarrow \infty} \frac{\Gamma(s + \delta_s)}{s} = c\sqrt{c^2+1} + c^2(c^2+1)^2.$$

Therefore, as a consequence of Theorem 3.1, under the assumption

$$c\sqrt{c^2+1} + c^2(c^2+1)^2 < \frac{1}{2(b-a)^2}, \quad (4.9)$$

where $\mathbf{c} = \mathbf{c}(\Omega)$ is the constant in the Poincaré's inequality (2.1), we have existence of periodic solutions of Problem (4.7), and hence the following:

Proposition 4.1. *Let Ω be an open convex and bounded subset of $\mathbb{R}^n$ and $a, b \in \mathbb{R}$ satisfying (4.9). Let $\rho \in \mathcal{C}([a, b], L^2(\Omega))$ with $\int_a^b \rho(t, \cdot) dt = 0$. Then, Problem (4.4) has at least one strong solution.*

Remark 4.2. The optimal constant $\mathbf{c}$ in the Poincaré inequality (Proposition 2.2) is sometimes known as the Poincaré constant for the domain Ω . Determining the Poincaré constant is, in general, a very hard task that depends upon the value of p and the geometry of the domain Ω . Nevertheless, it is well known, see [15, Theorem 3], that if $\Omega := \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), 0 < y < 1 - x\}$, then the Poincaré constant $\mathbf{c} = \mathbf{c}(\Omega, 2) = \frac{1}{\pi}$. Therefore, from Proposition 4.1, we may derive that Problem (4.4) on this domain Ω has at least one strong solution whenever $0 < b - a \leq 1$.

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