The Canonical Module of GT-Varieties and the Normal Bundle of RL-Varieties

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Abstract. In this paper, we study the geometry of GT -varieties X_d with group a finite cyclic group $\Gamma \subset \mathrm{GL}(n+1, \mathbb{K})$ of order d . We prove that the homogeneous ideal $I(X_d)$ of X_d is generated by binomials of degree at most 3 and we provide examples reaching this bound. We give a combinatorial description of the canonical module of the homogeneous coordinate ring of X_d and we show that it is generated by monomial invariants of Γ of degree d and $2d$. This allows us to characterize the Castelnuovo–Mumford regularity of the homogeneous coordinate ring of X_d . Finally, we compute the cohomology table of the normal bundle of the so-called RL -varieties. They are projections of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ which naturally arise from level GT -varieties.

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1. Introduction

Through this paper, $\mathbb{K}$ denotes an algebraically closed field of characteristic zero, $R = \mathbb{K}[x_0, \dots, x_n]$ and $\mathrm{GL}(n+1, \mathbb{K})$ denotes the group of invertible matrices of size $(n+1) \times (n+1)$ with coefficients in $\mathbb{K}$.

In [20], Mezzetti et al. related the existence of homogeneous artinian ideals $I \subset R$ generated by homogeneous forms $F_1, \dots, F_r$ of degree d failing the weak Lefschetz property in degree $d-1$ to the existence of rational projective varieties of $\mathbb{P}^{\binom{n+d}{n}-r-1}$ satisfying a Laplace equation. They called I a *Togliatti system*. Since then, Togliatti systems have been extensively studied as one can see in [1, 6, 9, 18, 19, 21, 22].

Any Togliatti system I induces a morphism $\varphi_I : \mathbb{P}^n \rightarrow \mathbb{P}^{r-1}$ defined as $(F_1, \dots, F_r)$, its image $\varphi_I(\mathbb{P}^n)$ is called the *variety parameterized by I* . In [19], the authors introduced a new family of Togliatti systems parameterizing varieties with a special geometric property. They called *GT-system* with cyclic group $\mathbb{Z}/d\mathbb{Z}$ any Togliatti system parameterizing a Galois covering with group $\mathbb{Z}/d\mathbb{Z}$. *GT-systems* and the varieties parameterized by them have been subsequently studied in [6, 7, 9]; in the latter reference, the authors applied invariant theory methods to tackle them.

To be more precise, fix integers $2 \leq n < d$ and e a d th primitive root of 1 $\in \mathbb{K}$. We denote by $M_{d;\alpha_0, \dots, \alpha_n}$ the diagonal matrix $\mathrm{diag}(e^{\alpha_0}, \dots, e^{\alpha_n})$, where $0 \leq \alpha_0 \leq \dots \leq \alpha_n < d$ are integers, such that $\mathrm{GCD}(\alpha_0, \dots, \alpha_n, d) = 1$ and $\alpha_i < \alpha_j$ for some $i \neq j$. We set $\Gamma := \langle M_{d;\alpha_0, \dots, \alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ a finite cyclic group of order d and $R^\Gamma = \{p \in R \mid g(p) = p, \forall g \in \Gamma\}$ the ring of invariants of Γ . The cyclic extension $\bar{\Gamma} \subset \mathrm{GL}(n+1, \mathbb{K})$ of Γ is the finite abelian group of order d^2 generated by $M_{d;\alpha_0, \dots, \alpha_n}$ and $M_{d;1, \dots, 1} = \mathrm{diag}(e, \dots, e)$. The ring of invariants of $\bar{\Gamma}$ is $R^{\bar{\Gamma}} = \{p \in R^\Gamma \mid \deg(p) = td, 0 \leq t\}$, often called the d th Veronese subalgebra of R^Γ . In [6], it is proved that $R^{\bar{\Gamma}} = \mathbb{K}[m_1, \dots, m_{\mu_d}]$, where $m_1, \dots, m_{\mu_d}$ are all the monomial invariants of Γ of degree d , and it is shown that the ideal $I_d = (m_1, \dots, m_{\mu_d})$ is a *GT-system*. They called *GT-variety* with group Γ the variety $X_d := \varphi_{I_d}(\mathbb{P}^n)$ parameterized by I_d and they started to investigate its geometry. In [13], Gröbner posed the problem of determining whether a monomial projection of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ parameterized by the set $\mathcal{M}_{n,d} \subset R$ of all monomials of degree d is an arithmetically Cohen Macaulay (shortly aCM) variety. Motivated by this long-standing problem, the authors of [6] proved that any *GT-variety* X_d is an aCM variety, showing that the homogeneous coordinate ring of X_d is isomorphic to $R^{\bar{\Gamma}}$. They also tackled the problem of finding a minimal free resolution of the homogeneous ideal $I(X_d)$ of X_d , which they determined for all *GT-surfaces*. Recently, in [7], the notions of *GT-system* and *GT-variety* have been generalized to any finite group acting on R , non-necessarily cyclic or even abelian.

In this paper, we address three topics regarding *GT-varieties*. The first two questions concern explicitly the geometry of any *GT-variety* X_d , in contrast with the last one, which deals with the cohomology of the normal bundle of some smooth rational varieties naturally arising from *GT-varieties*.

First, we find a set of generators of the homogeneous ideal $I(X_d)$ of X_d . We prove that $I(X_d)$ is generated by homogeneous binomials of degree at most 3 (Theorem 3.11). We exhibit examples of homogeneous ideals of GT -varieties reaching this bound, which also show that it depends on the group Γ . Second, we determine the algebraic structure of the canonical module ω_{X_d} of the homogeneous coordinate ring of X_d . We identify ω_{X_d} with the ideal $\text{relint}(I_d) = (x_0^{a_0} \dots x_n^{a_n} \in R^{\overline{\Gamma}} \mid 0 \neq a_0 \dots a_n)$ of $R^{\overline{\Gamma}}$ generated by the relative interior of the semigroup ring $\mathbb{K}[m_1, \dots, m_{\mu_d}]$ (Proposition 4.1). We prove that $\text{relint}(I_d)$ is generated by monomials of degree d and $2d$ (Theorem 4.2). This connection allows us to compute the Castelnuovo–Mumford regularity of $R^{\overline{\Gamma}}$.

Finally, we introduce a new family of smooth rational monomial projections of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ naturally associated with level GT -varieties. A GT -variety X_d is called *level* if the ideal $\text{relint}(I_d)$ is generated only by monomials of degree d and, hence, $R^{\overline{\Gamma}}$ is a level ring. An *RL-variety* $\mathcal{X}_d$ associated with a level GT -variety X_d is a monomial projection of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ parameterized by the set of monomials $\mathcal{M}_{n,d} \setminus \{m = x_0^{a_0} \dots x_n^{a_n} \in \text{relint}(I_d) \mid \deg(m) = d\}$. The name *RL-variety* is conceived to emphasize the relation with the relative interior and the levelness. We give examples of *RL*-varieties in any dimension. Inspired by the recent work of Alzati and Re [3], we contribute to the classical open problem of computing the cohomology of the normal bundle of smooth rational varieties (Theorem 5.9). Most results and examples of this topic focus on smooth rational curves and surfaces; see, for instance, [2, 10, 25]. We determine the cohomology table of the normal bundle of any *RL*-variety, shading new light on higher dimensions.

Let us see how this work is organized. In Sect. 2, we gather the basic definitions and results needed in the body of this paper. Section 3 is entirely devoted to find a set of homogeneous binomial generators of the homogeneous ideal $I(X_d)$ of any GT -variety X_d . We establish that the homogeneous coordinate ring of X_d is isomorphic to $R^{\overline{\Gamma}}$ and that $I(X_d)$ is a homogeneous prime binomial ideal. Our main result (Theorem 3.11) proves that $I(X_d)$ is generated by binomials of degree at most 3. We give families of examples of GT -varieties whose homogeneous ideals are minimally generated by binomials of degree 2 and 3. In Sect. 4, we investigate the algebraic structure of the canonical module ω_{X_d} of the homogeneous coordinate ring of X_d . We identify ω_{X_d} with the ideal $\text{relint}(I_d)$ of $R^{\overline{\Gamma}}$, which gives us a combinatorial description of ω_{X_d} . In Theorem 4.2, we show that $\text{relint}(I_d)$ is generated by monomials of degree d and $2d$. We further study GT -varieties X_d where $R^{\overline{\Gamma}}$ is a level ring, i.e., the canonical module $\text{relint}(I_d)$ of $R^{\overline{\Gamma}}$ is generated in only one degree. In particular, we study varieties whose homogeneous coordinate ring $R^{\overline{\Gamma}}$ is level and $\text{relint}(I_d)$ is minimally generated in degree d . Afterwards, in Theorem 4.10, we characterize the Castelnuovo–Mumford regularity of $R^{\overline{\Gamma}}$.

Finally in Sect. 5, we introduce the notions of a level GT -variety and its associated *RL*-variety and we give examples of any dimension. Using the

new methods of [3], we compute the cohomology table of the normal bundle of any RL -variety (see Theorem 5.9).

2. Preliminaries

In this section, we introduce the main objects and results we use in the body of this paper. First, we define semigroups and normal semigroups, we relate them to invariant theory of finite groups, and we see a geometrical interpretation of these objects. For more details, the reader can look at [4, 14, 27]. Finally, we define the weak Lefschetz property, we recall the notions of GT -systems and GT -varieties, and we collect some basic results on this topic.

2.1. Semigroup Rings and Rings of Invariants

By a *semigroup*, we mean a finitely generated additive subsemigroup $H = \langle h_1, \dots, h_t \rangle \subset \mathbb{Z}_{\geq 0}^{n+1}$. $L(H)$ is the additive subgroup of $\mathbb{Z}^{n+1}$ generated by H . We denote by $\mathbb{K}[H] \subseteq R$ the semigroup ring associated with H , i.e., the graded $\mathbb{K}$ -algebra generated by the monomials $X^{h_j} = x_0^{a_0^j} \dots x_n^{a_n^j} \in R$ associated with the points $h_j = (a_0^j, \dots, a_n^j) \in H$, $j = 1, \dots, t$.

Definition 2.1. A semigroup $H \subset \mathbb{Z}^{n+1}$ is called *normal* if it satisfies the following condition: if $zh \in H$ for some $h \in L(H)$ and $0 \neq z \in \mathbb{Z}_{\geq 0}$, then $h \in H$.

A large family of normal semigroups comes from invariant theory, precisely those associated with finite abelian groups acting on R . To be more precise, let $\Lambda \simeq \mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_r\mathbb{Z}$ and choose d_i -th primitive roots e_i of $1 \in \mathbb{K}$, $i = 1, \dots, r$. Therefore, Λ can be linearly represented in $\mathrm{GL}(n+1, \mathbb{K})$ by means of r diagonal matrices $\mathrm{diag}(e_i^{u_{0,i}}, \dots, e_i^{u_{n,i}})$, where $u_{j,i} \in \mathbb{Z}_{\geq 0}$, $0 \leq j \leq n$, $1 \leq i \leq r$. Let $R^\Lambda := \{p \in R \mid \lambda(p) = p \text{ for all } \lambda \in \Lambda\}$ be the ring of invariants of Λ acting on R . Since Λ acts diagonally, each monomial $x_0^{a_0} \dots x_n^{a_n} \in R$ is mapped into a multiple of itself by every $\lambda \in \Lambda$, and a polynomial $p \in R^\Lambda$ if and only if all its monomials are invariants of Λ . Thus, by the Noether's degree bound (see [27, Theorem 2.1.4]), R^Λ has a finite basis consisting of monomials of degree at most the order of Λ . By a basis of R^Λ , we mean a set of elements $\{\theta_1, \dots, \theta_l\} \subset R^\Lambda$ which minimally generates R^Λ as a $\mathbb{K}$ -algebra, i.e., $R^\Lambda = \mathbb{K}[\theta_1, \dots, \theta_l]$. Let $X^{h_1}, \dots, X^{h_t}$ be a monomial basis of R^Λ and $H = \langle h_1, \dots, h_t \rangle$. Then, $R^\Lambda = \mathbb{K}[H]$. Furthermore, a monomial $x_0^{a_0} \dots x_n^{a_n} \in R^\Lambda$ if and only if $(a_0, \dots, a_n)$ satisfies the system of congruences

$$a_0 u_{0,i} + \dots + a_n u_{n,i} \equiv 0 \pmod{d_i}, \quad i = 1, \dots, r. \quad (1)$$

Now, if $w \in L(H)$ is such that $zw \in H$ for some $z \in \mathbb{Z}_{\geq 0}$, then $w \in H$, so H is normal. By [14, Theorem 1] or [15, Proposition 13], $\mathbb{K}[H]$ is Cohen–Macaulay.

More generally, let $G \subset \mathrm{GL}(n+1, \mathbb{K})$ be a finite group. Geometrically, the ring R^G of invariants can be regarded as the coordinate ring of the quotient of $\mathbb{A}^{n+1}$ by G . To be more precise, set $\{f_1, \dots, f_t\}$ a basis of R^G , often called a set of *fundamental invariants* of G , and let $\mathbb{K}[w_1, \dots, w_t]$

be the polynomial ring in the new variables $w_1, \dots, w_t$. Then, the quotient of $\mathbb{A}^{n+1}$ by G is given by the morphism $\pi : \mathbb{A}^{n+1} \rightarrow \pi(\mathbb{A}^{n+1}) \subset \mathbb{A}^t$, such that $\pi(a_0, \dots, a_n) = (f_1(a_0, \dots, a_n), \dots, f_t(a_0, \dots, a_n))$. Even further, π is a Galois covering of $\pi(\mathbb{A}^{n+1})$ with group G . For further details on quotients varieties, we refer the reader to [24]. The ideal $I(\pi(\mathbb{A}^{n+1}))$ of the quotient variety is called the *ideal of syzygies* among the invariants $f_1, \dots, f_t$; it is the kernel of the homomorphism defined by $w_i \rightarrow f_i$, $i = 1, \dots, t$. We denote it by $\text{syz}(f_1, \dots, f_t)$. We summarize all these facts in the following proposition.

Proposition 2.2. *Let $G \subset \text{GL}(n+1, \mathbb{K})$ be a finite linear group, $\{f_1, \dots, f_t\}$ be a set of fundamental invariants, and let $\pi : \mathbb{A}^{n+1} \rightarrow \mathbb{A}^t$ be the induced morphism. Then*

- (i) $\pi(\mathbb{A}^{n+1})$ is the quotient of $\mathbb{A}^{n+1}$ by G with affine coordinate ring R^G .
- (ii) $R^G \cong \mathbb{K}[w_1, \dots, w_t] / \text{syz}(f_1, \dots, f_t)$.
- (iii) π is a Galois covering of $\pi(\mathbb{A}^{n+1})$ with group G .

Proof. See [26, Section 6]. □

The cardinality of a general orbit $G(a)$, $a \in \mathbb{A}^{n+1}$, is called the degree of the covering. Moreover, if we can find a homogeneous set of fundamental invariants $\{f_1, \dots, f_t\}$ of G , such that $\pi : \mathbb{P}^n \rightarrow \mathbb{P}^{t-1}$ is a morphism, then the projective version of Proposition 2.2 is true.

2.2. GT-Systems and GT-Varieties

Let $I \subset R$ be a homogeneous artinian ideal. We say that I has the *weak Lefschetz property* (WLP) if there is a linear form $L \in R_1$, such that, for all integers j , the multiplication map

$$\times L : (R/I)_{j-1} \rightarrow (R/I)_j$$

has maximal rank. In [20], Mezzetti, Miró-Roig, and Ottaviani proved that the failure of the WLP is related to the existence of varieties satisfying at least one Laplace equation of order greater than 2. More precisely, they proved:

Theorem 2.3. *Let $I \subset R$ be an artinian ideal generated by r forms $F_1, \dots, F_r$ of degree d and let I^{-1} be its Macaulay inverse system. If $r \leq \binom{n+d-1}{n-1}$, then the following conditions are equivalent:*

- (i) I fails the WLP in degree $d-1$;
- (ii) $F_1, \dots, F_r$ become $\mathbb{K}$ -linearly dependent on a general hyperplane H of $\mathbb{P}^n$;
- (iii) the n -dimensional variety $Y = \overline{\varphi(\mathbb{P}^n)}$, where $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+d-1}{n}-r-1}$ is the rational morphism associated with $(I^{-1})_d$, satisfies at least one Laplace equation of order $d-1$.

Proof. See [20, Theorem 3.2]. □

Motivated by the above results, Mezzetti, Miró-Roig, and Ottaviani introduced the following definitions (see [18, 20]):

Definition 2.4. Let $I \subset R$ be an artinian ideal generated by $r \leq \binom{n+d-1}{n-1}$ forms of degree d . We say that

- (i) I is a *Togliatti system* if it fails the WLP in degree $d - 1$.
- (ii) I is a *monomial Togliatti system* if, in addition, I can be generated by monomials.

In particular, a Togliatti system is called *smooth* if the variety Y in Theorem 2.3(iii) is smooth. The name is in honour of Togliatti who proved that, for $n = 2$, the only smooth Togliatti system of cubics is

$$I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset \mathbb{K}[x_0, x_1, x_2]$$

([28, 29]). The systematic study of Togliatti systems was initiated in [20] and it has been continuing in [1, 18, 19, 21, 22]. Precisely, in [18], it was introduced the notion of *GT-system* with group a finite cyclic group. Recently, in [8], this notion has been generalized as follows.

Definition 2.5. A *GT-system* with a finite group G is an artinian ideal $I_d \subset R$ generated by r forms $F_1, \dots, F_r$ of degree d , such that

- (i) I_d is a Togliatti system.
- (ii) The morphism $\varphi_{I_d}: \mathbb{P}^n \rightarrow \mathbb{P}^{r-1}$ defined by $(F_1, \dots, F_r)$ is a Galois covering with group G .

If conditions (i) and (ii) holds, we say that $\varphi_{I_d}(\mathbb{P}^n)$ is a *GT-variety with group G* .

GT-systems with group a finite cyclic group have been extensively studied in [6, 7, 19], while in [8], the authors investigate *GT-systems* with the dihedral group acting on $\mathbb{K}[x_0, x_1, x_2]$. In the last two references, invariant theory techniques have been applied to tackle both objects. Fix $G \subset \mathrm{GL}(n+1, \mathbb{K})$ a finite group of order d . Assume that the ring R^G has a basis $\mathcal{B}$ formed by homogeneous invariants of G of degree d and set I_d the ideal generated by $\mathcal{B}$. Keeping this notation, we have the following.

Proposition 2.6. *If $|\mathcal{B}| \leq \binom{d+n-1}{n-1}$, then I_d is a GT-system with group G .*

Proof. Since I_d contains a homogeneous system of parameters of R^G , it is an artinian ideal. By Proposition 2.2, the associated morphism φ_{I_d} is a Galois covering with group G . By Theorem 2.3, it is enough to prove that I_d fails the WLP in degree $d - 1$, i.e., for any linear form $L \in R_1$, the multiplication map $\times L: (R/I_d)_{d-1} \rightarrow (R/I_d)_d$ is not injective. Let $L \in R_1$ and consider $F := \prod_{I_d \neq g \in G} g(L) \in R_{d-1}$. We have that $L \cdot F = \prod_{g \in G} g(L) \in R^G$ and, hence, $F \in \ker(\times L)$. $\square$

Let us see an illustrative example.

Example 2.7. (i) Fix $n = 2$, $d = 5$, and e a 5th primitive root of 1. The finite cyclic group $\mathbb{Z}/5\mathbb{Z}$ can be linearly represented by $\Gamma = \langle \mathrm{diag}(1, e, e^3) \rangle \subset \mathrm{GL}(3, \mathbb{K})$. All the monomial invariants of Γ of degree 5 are: $x_0^5, x_0^2x_1^2x_2, x_0x_1x_2^3, x_1^5, x_2^5$ (see [6, Example 2.15]). In total, we have $r = 5$ monomials. The inequality $r \leq \binom{n+d-1}{n-1}$ is satisfied and the ideal $I_5 \subset R$ generated by them fails the WLP in degree 4. The morphism $\varphi_{I_5}: \mathbb{P}^2 \rightarrow \mathbb{P}^4$ is a Galois covering of degree 5 with group Γ (see [6, Corollary 3.4] and [19, Theorem 3.4]). Actually, $\varphi_{I_5}(\mathbb{P}^2)$ is the quotient surface of $\mathbb{P}^2$ by Γ .

- (ii) Fix $n = 3, d = 4$ and e a 4th primitive root of 1. The diagonal matrix $\text{diag}(1, e, e^2, e^3)$ generates a finite cyclic subgroup $\Gamma \subset \text{GL}(4, \mathbb{K})$ of order 4. There are exactly $r = 10$ monomial invariants of Γ of degree 4: $x_0^4, x_0^2 x_1 x_3, x_0^2 x_2^2, x_0 x_1^2 x_2, x_0 x_2 x_3^2, x_1^4, x_1^2 x_3^2, x_1 x_2^2 x_3, x_2^4, x_3^4$ (see [9, Example 3.2]). The inequality $r \leq \binom{n+d-1}{n-1}$ is satisfied and the ideal I_4 generated by them fails the WLP in degree 3. The associated morphism $\varphi_{I_4} : \mathbb{P}^3 \rightarrow \mathbb{P}^9$ is a Galois covering with group Γ (see [9, Proposition 3.3] and [6, Corollary 3.4]). Similarly to (i), $\varphi_{I_4}(\mathbb{P}^3)$ is the quotient threefold of $\mathbb{P}^3$ by Γ .

In [6, 8], and previously in [9], the authors focus on the geometry of GT -varieties with group a finite cyclic group or a dihedral group. In [6, Theorem 3.2], it is proved that all GT -varieties with group a finite cyclic group are aCM varieties and in [8, Proposition 4.3], it is shown its analogous for GT -surfaces with a dihedral group. Furthermore, in [6, Theorem 4.14] and [8, Theorem 4.6], it is determined a minimal free resolution of GT -surfaces with group a finite cyclic group and a dihedral group, respectively. Reference [9] is devoted to find a minimal set of homogeneous binomial generators of the homogeneous ideal of certain GT -threefolds with group a finite cyclic group. Our goal is to extend the results obtained so far for GT -surfaces and GT -threefolds to arbitrary n -dimensional GT -varieties with group a finite cyclic group.

3. On the Homogeneous Ideal of GT-Varieties

In this section, we look for a system of generators of the homogeneous ideal of GT -varieties with group a finite cyclic group. Using combinatorial techniques, we prove that all these ideals are generated by homogeneous binomials of degree 2 and 3. We begin introducing some notations.

Notation 3.1. Fix integers $2 \leq n < d$ and e a d th primitive root of $1 \in \mathbb{K}$. We denote by $M_{d;\alpha_0, \dots, \alpha_n}$ the diagonal matrix $\text{diag}(e^{\alpha_0}, \dots, e^{\alpha_n})$, where $0 \leq \alpha_0 \leq \dots \leq \alpha_n < d$ are integers, such that $\text{GCD}(d, \alpha_0, \dots, \alpha_n) = 1$ and $\alpha_i < \alpha_j$ for some $i \neq j$.

Along this section, we fix a finite cyclic group $\Gamma := \langle M_{d;\alpha_0, \dots, \alpha_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ of order d and we consider the ring R^Γ of invariants of Γ endowed with the natural grading $R^\Gamma = \bigoplus_{t \geq 1} R_t^\Gamma$, $R_t^\Gamma := R_t \cap R^\Gamma$. The cyclic extension $\bar{\Gamma} \subset \text{GL}(n+1, \mathbb{K})$ of Γ is the finite abelian group of order d^2 generated by $M_{d;\alpha_0, \dots, \alpha_n}$ and $M_{d;1, \dots, 1}$. We consider the ring $R^{\bar{\Gamma}}$ of invariants of $\bar{\Gamma}$ with the grading $R^{\bar{\Gamma}} = \bigoplus_{t \geq 1} R_t^{\bar{\Gamma}}$, $R_t^{\bar{\Gamma}} := R_{td}^\Gamma = R_{td} \cap R^\Gamma$, often called the d th Veronese subalgebra of R^Γ . We denote by $\mathcal{M}_d := \{m_1, \dots, m_{\mu_d}\} \subset R$ the set of all monomial invariants of Γ of degree d , ordered lexicographically. We denote $I_d \subset R$ the monomial artinian ideal generated by $\mathcal{M}_d$. By $\varphi_{I_d} : \mathbb{P}^n \rightarrow \mathbb{P}^{\mu_d-1}$, we denote the associated morphism and we set $X_d := \varphi_{I_d}(\mathbb{P}^n) \subset \mathbb{P}^{\mu_d-1}$ its image. In [6], it is established the following.

Theorem 3.2. (i) $\mathcal{M}_d$ is a basis of $R^{\bar{\Gamma}}$.

- (ii) R^Γ is the coordinate ring of X_d . Hence, X_d is an aCM monomial projection of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ from the inverse system I_d^{-1} .
- (iii) If $\mu_d \leq \binom{d+n-1}{n-1}$, then I_d is a GT-system with group Γ . In this case, we call X_d a GT-variety with group Γ .

Proof. See [6, Theorem 3.1-3.3 and Corollary 3.4]. $\square$

Remark 3.3. Through the rest of this paper, if the condition $\mu_d \leq \binom{d+n-1}{n-1}$ is satisfied, we refer to I_d as a GT-system and to X_d as a GT-variety.

Example 3.4. (i) Fix integers $2 \leq n$ and fix $\Gamma = \langle M_{n+1;0,1,2,\dots,n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ (see [6, Example 3.6(iii)]). In [7, Theorem 4.8], it is proved that the condition $\mu_{n+1} \leq \binom{2n}{n-1}$ is satisfied; thus, I_d is a monomial Togliatti system. By Theorem 3.2(ii), I_d is a GT-system.

(ii) Fix integers $2 = n < d$ and fix $\Gamma = \langle M_{d;0,\alpha_1,\alpha_2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ (see [19] and [6, Example 3.6(i)]). In [19, Theorem 3.4], it is proved that the condition $\mu_d \leq \binom{n-1+d}{n-1}$ is satisfied; thus, I_d is a Togliatti system. By Theorem 3.2(iii), I_d is a GT-system.

Let $w_1, \dots, w_{\mu_d}$ be new variables and set $S := \mathbb{K}[w_1, \dots, w_{\mu_d}]$. We denote by $I(X_d) \subset S$ the homogeneous ideal of X_d . By Proposition 2.2, $I(X_d)$ is the kernel of the morphism $\rho: S \rightarrow R$ given by $\rho(w_i) = m_i$. It holds that $I(X_d)$ is the homogeneous binomial prime ideal generated by

$$\begin{aligned} \mathcal{W}_d &= \{w_{i_1} \dots w_{i_k} - w_{j_1} \dots w_{j_k} \in S \text{ such that } m_{i_1} \dots m_{i_k} \\ &= m_{j_1} \dots m_{j_k}, k \geq 2\}. \end{aligned}$$

For any $k \geq 2$, we denote by $\mathcal{W}_{d,k}$ the set of all binomials of $\mathcal{W}_d$ of degree exactly k . Our goal is to prove that $I(X_d)$ is generated by binomials of degree 2 and 3, i.e., the ideal $I(X_d) = \langle \mathcal{W}_{d,2}, \mathcal{W}_{d,3} \rangle$. Through families of examples in 3.8, we observe that this bound is sharp. We start with some definitions.

Definition 3.5. Fixed $k \geq 2$, we define a *suitable k -binomial* to be a non-zero binomial $w^\alpha = w^{\alpha+} - w^{\alpha-} := \prod_{l=1}^k w_{i_l} - \prod_{l=1}^k w_{j_l} \in \mathcal{W}_{d,k}$, i.e., $\prod_{l=1}^k m_{i_l} = \prod_{l=1}^k m_{j_l}$.

Definition 3.6. Given a suitable k -binomial $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathcal{W}_{d,k}$, we denote by $\mathrm{supp}_+(w^\alpha)$ (respectively $\mathrm{supp}_-(w^\alpha)$) the support of the monomial $w^{\alpha+}$ (respectively, support of $w^{\alpha-}$). We say that w^α is *non-trivial* if $\mathrm{supp}_+(w^\alpha) \cap \mathrm{supp}_-(w^\alpha) = \emptyset$. Otherwise, we say that w^α is *trivial*.

Definition 3.7. Let $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathcal{W}_{d,k}$ be a non-trivial suitable k -binomial. By an $I(X_d)_k$ -sequence from $w^{\alpha+}$ to $w^{\alpha-}$, we mean a finite sequence $(w^1, \dots, w^t)$ of monomials of S satisfying the following two conditions:

- (i) $w^1 = w^{\alpha+}$, $w^t = w^{\alpha-}$ and
- (ii) For all $1 \leq j < t$, $w^j - w^{j+1}$ is a trivial suitable k -binomial.

Example 3.8. Let $\Gamma = \langle M_{6;0,1,2,3} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ be a finite group of order 6. There are $\mu_6 = 16$ monomial invariants of Γ ; we have

$$\begin{aligned} \mathcal{M}_6 = \{ & x_0^6, x_0^4 x_2^2, x_0^3 x_1 x_2 x_3, x_0^3 x_2^3, x_0^2 x_1^3 x_3, x_0^2 x_1^2 x_2^2, x_0^2 x_3^4, \\ & x_0 x_1^4 x_2, x_0 x_1 x_2 x_3^3, x_0 x_2^3 x_3^2, x_1^6, x_1^3 x_3^3, \\ & x_1^2 x_2^2 x_3^2, x_1 x_2^4 x_3, x_2^6, x_3^6 \}, \end{aligned}$$

By Theorem 3.2(iii), the ideal I_6 generated by them is a GT -system and its associated variety X_6 is a GT -variety. The homogeneous binomials $w_1 w_{15} - w_4^2$ and $w_3 w_{12} w_{15} - w_6 w_9 w_{14}$ are non-trivial suitable binomials of degree 2 and 3, respectively. Indeed, $\rho(w_1 w_{15}) = \rho(w_1) \rho(w_{15}) = (x_0^6)(x_2^6) = (x_0^3 x_2^3)^2 = \rho(w_4)^2$, and on the other hand, $\rho(w_3) \rho(w_{12}) \rho(w_{15}) = (x_0^3 x_1 x_2 x_3)(x_1^3 x_3^3)(x_2^6) = (x_0^2 x_1^2 x_2^2)(x_0 x_1 x_2 x_3^3)(x_1 x_2^4 x_3) = \rho(w_6) \rho(w_9) (\rho(w_{14}))$. Finally, $\{w_3 w_{12} w_{15}, w_5 w_9 w_{15}, w_6 w_9 w_{14}\}$ is an $I(X_6)_3$ -sequence from $w_3 w_{12} w_{15}$ to $w_6 w_9 w_{14}$.

The following result characterizes the inclusion of ideals $\langle \mathcal{W}_{d,k} \rangle \subset \langle \mathcal{W}_{d,k-1} \rangle$, $k \geq 3$. It is key to prove our main results, which we state later.

Proposition 3.9. *Fix $k \geq 3$ and let $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathcal{W}_{d,k}$ be a suitable k -binomial. Then, $w^\alpha \in \langle \mathcal{W}_{d,k-1} \rangle$ if and only if there exists an $I(X_d)_k$ -sequence from $w^{\alpha+}$ to $w^{\alpha-}$.*

Proof. We apply the same arguments as in [9, Proposition 5.4]. □

We need the following lemma:

Lemma 3.10. *Any sequence of $2d - 1$ integers in $\{0, \dots, d - 1\}$ contains some subsequence of d integers, the sum of which is a multiple of d .*

Proof. See [11, Theorem]. □

Theorem 3.11. *Let $\Gamma = \langle M_{d;\alpha_0, \dots, \alpha_n} \rangle \subset \mathrm{GL}(n + 1, \mathbb{K})$ be a cyclic group of order d and $X_d \subset \mathbb{P}^{\mu_d - 1}$ the variety parameterized by the ideal $I_d = (m_1, \dots, m_{\mu_d})$ generated by all monomial invariants of Γ of degree d . Then, the homogeneous ideal $I(X_d)$ of X_d is generated by quadrics and cubics. Precisely, $I(X_d) = \langle \mathcal{W}_{d,2}, \mathcal{W}_{d,3} \rangle$.*

Proof. First, we prove that for all $k \geq 4$, any non-trivial suitable k -binomial admits an $I(X_d)_k$ -sequence. By Proposition 3.9, this implies $\langle \mathcal{W}_{d,k} \rangle \subset \langle \mathcal{W}_{d,k-1} \rangle$. Fix $k \geq 4$ and let $w^\alpha = w^{\alpha+} - w^{\alpha-} = w_{i_1} \dots w_{i_k} - w_{j_1} \dots w_{j_k}$ be a non-trivial suitable k -binomial. For each w_{i_l} (respectively, w_{j_l}), let $m_{i_l} = x_0^{a_0^l} \dots x_n^{a_n^l}$ be its associated monomial (respectively, $m_{j_l} = x_0^{b_0^l} \dots x_n^{b_n^l}$), $l = 1, \dots, k$. We have that

$$\sum_{l=1}^k a_s^l = \sum_{l=1}^k b_s^l, \quad 0 \leq s \leq n. \quad (2)$$

We consider the monomials m_{i_1}, m_{j_1} , and for each $0 \leq s \leq n$, we define

$$c_s = \begin{cases} 0 & \text{if } a_s^1 > b_s^1 \\ b_s^1 - a_s^1 & \text{otherwise.} \end{cases}$$

This gives rise to a non-zero monomial $m = x_0^{c_0} \dots x_n^{c_n} \in R$ of degree strictly smaller than d . Clearly, m divides $m_{i_2} \dots m_{i_k}$ (see (2)). Thus, we

consider $m' = (m_{i_2} \dots m_{i_k})/m$, which is a monomial of degree at least $(k-2)d \geq 2d$. We write $m' = x_0^{f_0} \dots x_n^{f_n}$. We define the sequence of integers $L = (\alpha_0, f_0, \alpha_0, \dots, \alpha_n, f_n, \alpha_n)$. Since L has length at least $(k-2)d \geq 2d$, by Lemma 3.10, we can find a subsequence $L' = (\alpha_0, g_0, \alpha_0, \dots, \alpha_n, g_n, \alpha_n) \subset L$ of d elements whose sum is a multiple of d , which implies that the monomial $x_0^{g_0} \dots x_n^{g_n} \in R^\Gamma$. By Theorem 3.2(i), we can decompose

$$m_{i_2} \dots m_{i_k} = m_{i_2}^1 \dots m_{i_k}^1,$$

where all $m_{i_l}^1 \in R^\Gamma$, $2 \leq l \leq k$, are monomials of degree d and in particular

$$m_{i_k}^1 = x_0^{g_0} \dots x_n^{g_n}.$$

Notice that we have $m_{i_1} \dots m_{i_k} = m_{i_1} m_{i_2}^1 \dots m_{i_k}^1$. We define $w^2 \in S$ to be the monomial $\rho^{-1}(m_{i_1}) \rho^{-1}(m_{i_2}^1) \dots \rho^{-1}(m_{i_k}^1)$. By construction, $w^{\alpha_+} - w^2$ is a trivial suitable k -binomial. Observe that $m = x_0^{c_0} \dots x_n^{c_n}$ divides $m_{i_2}^1 \dots m_{i_{k-1}}^1$, thus m_{j_1} divides $m_{i_1} m_{i_2}^1 \dots m_{i_{k-1}}^1$. Applying the same argument as before, we factorize

$$m_{i_1} m_{i_2}^1 \dots m_{i_{k-1}}^1 = m_{i_1}^2 m_{i_2}^2 \dots m_{i_{k-1}}^2,$$

where $m_{i_1}^2 = m_{j_1}$ and all $m_{i_l}^2 \in R^\Gamma$, $2 \leq l \leq k-1$, are monomials of degree d . We set $w^3 = \rho^{-1}(m_{i_1}^2) \dots \rho^{-1}(m_{i_{k-1}}^2) \cdot \rho^{-1}(m_{i_k}^1)$. Since $m_{i_1} m_{i_2}^1 \dots m_{i_{k-1}}^1 m_{i_k}^1 = m_{i_1}^2 m_{i_2}^2 \dots m_{i_{k-1}}^2 m_{i_k}^1$, $w^2 - w^3$ is a suitable trivial k -binomial. Furthermore, since $m_{i_1}^2 = m_{j_1}$, also $w^3 - w^{\alpha_-}$ is a trivial suitable k -binomial. Therefore, $(w_{i_1} \dots w_{i_n}, w^2, w^3, w_{j_1} \dots w_{j_n})$ is an $I(X_d)_k$ -sequence, from which it follows that $\langle \mathcal{W}_{d,k} \rangle \subset \langle \mathcal{W}_{d,k-1} \rangle$. The argument we have developed only requires that $(k-2)d \geq 2d$, which it is satisfied for all $k \geq 4$. Thus, we have proved that for all $k \geq 4$

$$\langle \mathcal{W}_{d,k} \rangle \subset \langle \mathcal{W}_{d,k-1} \rangle \subset \dots \subset \langle \mathcal{W}_{d,3} \rangle,$$

which completes the proof. $\square$

Despite $\mathcal{W}_{d,2}$ always belongs to a minimal set of homogeneous binomial generators of $I(X_d)$, it is not the case for $\mathcal{W}_{d,3}$. Even further, this fact depends on the action of the group $\Gamma = \langle M_{d;\alpha_0, \dots, \alpha_n} \rangle$, as we illustrate in the following examples.

- Example 3.12.* (i) Fix integers $2 = n < d$ and fix $\Gamma = \langle M_{d;0,\alpha_1,\alpha_2} \rangle \subset \text{GL}(3, \mathbb{K})$, a finite cyclic group of order d with $\alpha_1 < \alpha_2$. By [6, Corollary 4.16], the homogeneous ideal $I(X_d)$ of the GT -surface X_d is minimally generated by binomials of degree 2 and 3 if $\text{GCD}(\alpha_1, d) = \text{GCD}(\alpha_2, d) = \text{GCD}(\alpha_2 - 1, d) = 1$ and $I(X_d)$ is minimally generated by binomials of degree 2 otherwise.
- (ii) Fix integers $3 = n < d$ and fix $\Gamma = \langle M_{d;0,1,2,3} \rangle \subset \text{GL}(4, \mathbb{K})$. In [9, Corollary 5.7], it is determined a minimal set of homogeneous binomials generators of the homogeneous ideal $I(X_d)$. Precisely, it is proved that $I(X_d)$ is minimally generated by binomials of degree 2 if d is even and $I(X_d)$ is minimally generated by binomials of degree 2 and 3 if d is odd.

- (iii) Fix integers $2 \leq n < d$ and fix $\Gamma = \langle M_{d;0,\alpha_1,\dots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$. If there exist integers $0 < \alpha_i < \alpha_j$, such that $\mathrm{GCD}(\alpha_i, d) = \mathrm{GCD}(\alpha_j, d) = \mathrm{GCD}(\alpha_j - 1, d) = 1$, then $I(X_d)$ is minimally generated by binomials of degree 2 and 3. Indeed, let V_d be the GT-surface associated with the action of the cyclic group $\Lambda := \langle M_{d;0,\alpha_i,\alpha_j} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ of order d acting on $\mathbb{K}[x_0, x_i, x_j]$. Under a suitable identification of variables, we have that the homogeneous ideal $I(V_d) = I(X_d) \cap \mathbb{K}[\rho^{-1}(x_0), \rho^{-1}(x_i), \rho^{-1}(x_j)]$. Therefore, if a minimal set of homogeneous binomial generators of $I(V_d)$ contains a binomial of degree 3, the same holds for $I(X_d)$.

4. The Canonical Module of GT-Varieties

The algebraic structure of the canonical module ω_X of the homogeneous coordinate ring of an aCM projective variety X plays a central role in its geometry (see [4, 5, 16]). For example, it can lead us to derive information on the Hilbert function and series, as well as on the Castelnuovo–Mumford regularity, of the homogeneous coordinate ring of X . Let $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \mathrm{GL}(n+1)$ be a cyclic group of order d (see Notation 3.1). We denote by $I_d = (m_1, \dots, m_{\mu_d}) \subset R$ the ideal generated by all monomial invariants of Γ of degree d , $\varphi_{I_d} : \mathbb{P}^n \rightarrow \mathbb{P}^{\mu_d-1}$ the morphism induced by I_d and $X_d = \varphi_{I_d}(\mathbb{P}^n) \subset \mathbb{P}^{\mu_d-1}$ its image. X_d is an aCM projective variety and its homogeneous coordinate ring $S/I(X_d)$ is isomorphic to $R^{\bar{\Gamma}}$ (see Theorem 3.2). In this section, we deal with the canonical module ω_{X_d} of $S/I(X_d)$. We identify ω_{X_d} with an ideal of $R^{\bar{\Gamma}}$, and we prove that it is generated by monomials of degree d and $2d$. We focus on varieties X_d the homogeneous coordinate rings $S/I(X_d)$ of which are level ring, i.e., their canonical modules ω_{X_d} are generated in only one degree. Afterwards, we characterize the Castelnuovo–Mumford regularity of $R^{\bar{\Gamma}}$.

Let $m = x_0^{a_0} \dots x_n^{a_n} \in R$ be a monomial; we say that $l_m := (a_0, \dots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ is the lattice point associated with m . Conversely, given a lattice point $l = (a_0, \dots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$, we say that $m_l := x_0^{a_0} \dots x_n^{a_n}$ is the monomial associated with l . We denote $H_d \subset \mathbb{Z}_{\geq 0}^{n+1}$ the semigroup generated by the lattice points $l_{m_1}, \dots, l_{m_{\mu_d}}$ associated to $\mathcal{M}_d := \{m_1, \dots, m_{\mu_d}\}$. By Theorem 3.2(i), we have that $R^{\bar{\Gamma}} = \mathbb{K}[H_d]$ and the semigroup H_d coincides with the set of all solutions $(a_0, \dots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ of the systems

$$(*)_{t,r} = \begin{cases} y_0 + y_1 + \dots + y_n = td \\ \alpha_0 y_0 + \alpha_1 y_1 + \dots + \alpha_n y_n = rd \end{cases} \quad t \geq 1, \quad r = 0, \dots, \alpha_n t.$$

Therefore, H_d is a normal semigroup (see Sect. 2) with $H_d^+ := \{(a_0, \dots, a_n) \in H_d \mid a_i > 0, i = 0, \dots, n\} \neq \emptyset$; for instance, $(d, \dots, d) \in H_d^+$. It holds that H_d^+ coincides with the relative interior of H_d . We set $\mathrm{relint}(I_d) := (x_0^{a_0} \dots x_n^{a_n} \in R^{\bar{\Gamma}} \mid 0 \neq a_0 \dots a_n)$ the ideal of $R^{\bar{\Gamma}}$ generated by all monomials associated with H_d^+ . We have the following.

Proposition 4.1. *$\mathrm{relint}(I_d)$ is the canonical module of $R^{\bar{\Gamma}}$.*

Proof. See [4, Theorem 6.3.5(b)] $\square$

For a complete exposition of the canonical module of normal semigroup rings, we refer the reader to [4].

We denote by $\mathcal{C}_{d,k} \subset \text{relint}(I_d)$ the set of all monomials of degree kd . We have:

Theorem 4.2. *For any cyclic group $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ of order d , $\text{relint}(I_d) = \langle \mathcal{C}_{d,1}, \mathcal{C}_{d,2} \rangle$.*

Proof. It is enough to show that for any monomial $m \in \mathcal{C}_{d,k}$, $k \geq 3$, there exists a monomial $m' \in \mathcal{C}_{d,k-1}$ which divides m . This proves that for $k \geq 3$, $\langle \mathcal{C}_{d,k} \rangle \subset \langle \mathcal{C}_{d,k-1} \rangle \subset \dots \subset \langle \mathcal{C}_{d,2} \rangle$.

We fix an integer $k \geq 3$, a monomial $m = x_0^{a_0} \dots x_n^{a_n} \in \mathcal{C}_{d,k}$ and we set $m_1 = m/(x_0 \dots x_n) = x_0^{a_0-1} \dots x_n^{a_n-1}$. Since $d \geq n+1$ and $k \geq 3$, m' is a monomial of degree $kd - (n+1) \geq 2d$. We define the sequence of integers $L = (\alpha_0, \overset{a_0-1}{\cdot}, \alpha_0, \dots, \alpha_n, \overset{a_n-1}{\cdot}, \alpha_n)$, by [11, Theorem] there exists a subsequence $L' = (\alpha_0, \overset{b_0}{\cdot}, \alpha_0, \dots, \alpha_n, \overset{b_n}{\cdot}, \alpha_n) \subset L$ of d integers the sum of which is a multiple of d . Therefore, L' gives rise a monomial $m_2 := x_0^{b_0} \dots x_n^{b_n} \in R^{\overline{\Gamma}}$ of degree d which divides m_1 . Hence, we can factorize $m = m_2 m'$ and by construction $m' \in \mathcal{C}_{d,k-1}$ is the required monomial. $\square$

Example 4.3. Let $\Gamma = \langle M_{6;0,1,2,3} \rangle \subset \text{GL}(4, \mathbb{K})$ be a cyclic group of order 6. We have that $\mathcal{C}_{6,1} = \{x_0^3 x_1 x_2 x_3, x_0 x_1 x_2 x_3^3\}$ (see Example 3.8) and

$$\begin{aligned} \mathcal{C}_{6,2} = \{ & x_0^3 x_1 x_2 x_3, x_0^7 x_1 x_2 x_3^3, x_0^6 x_1^2 x_2^2 x_3^2, x_0^6 x_1 x_2^4 x_3, x_0^5 x_1^4 x_2 x_3^2, x_0^5 x_1^3 x_2^3 x_3, x_0^5 x_1 x_2 x_3^5, \\ & x_0^4 x_1^5 x_2^2 x_3, x_0^4 x_1^2 x_2^2 x_3^4, x_0^4 x_1 x_2^4 x_3^3, x_0^3 x_1^7 x_2 x_3, x_0^3 x_1^4 x_2 x_3^4, x_0^3 x_1^2 x_2^3 x_3^3, x_0^3 x_1^2 x_2^5 x_3^2, \\ & x_0^3 x_1 x_2^7 x_3, x_0^3 x_1 x_2 x_3^7, x_0^2 x_1^5 x_2^2 x_3^3, x_0^2 x_1^4 x_2^4 x_3^2, x_0^2 x_1^3 x_2^6 x_3, x_0^2 x_1^2 x_2^2 x_3^6, x_0^2 x_1 x_2^4 x_3^5, \\ & x_0 x_1^7 x_2 x_3^3, x_0 x_1^6 x_2^3 x_3^2, x_0 x_1^5 x_2^5 x_3, x_0 x_1^4 x_2 x_3^6, x_0 x_1^3 x_2^3 x_3^5, x_0 x_1^2 x_2^5 x_3^4, x_0 x_1 x_2^7 x_3^3, \\ & x_0 x_1 x_2 x_3^9 \}. \end{aligned}$$

Only the following four monomials $x_0^2 x_1^4 x_2^4 x_3^2, x_0^2 x_1^3 x_2^6 x_3, x_0 x_1^6 x_2^3 x_3^2, x_0 x_1^5 x_2^5 x_3 \in \mathcal{C}_{6,2}$ do not belong to the ideal $\langle \mathcal{C}_{6,1} \rangle \subset R^{\overline{\Gamma}}$. From this observation and Theorem 4.2, we obtain that the canonical module of $R^{\overline{\Gamma}}$ is the ideal

$$\text{relint}(I_6) = (x_0^3 x_1 x_2 x_3, x_0^2 x_1^4 x_2^4 x_3^2, x_0^2 x_1^3 x_2^6 x_3, x_0 x_1^6 x_2^3 x_3^2, x_0 x_1^5 x_2^5 x_3, x_0 x_1 x_2 x_3^3).$$

In [9], a minimal free resolution of $R^{\overline{\Gamma}}$ is computed. Set $S := \mathbb{K}[w_1, \dots, w_{16}]$ (see Example 3.8); precisely, we have

$$\begin{aligned} 0 \rightarrow S(-14)^4 \oplus S(-15)^2 &\rightarrow S(-13)^{108} \oplus S(-14)^7 \rightarrow S(-12)^{803} \\ &\rightarrow S(-11)^{2850} \rightarrow S(-10)^{6237} \rightarrow \\ S(-9)^{9064} \rightarrow S(-7)^6 \oplus S(-8)^{8811} &\rightarrow S(-6)^{258} \oplus S(-7)^{5352} \\ &\rightarrow S(-5)^{844} \oplus S(-6)^{1638} \rightarrow \\ S(-4)^{796} \oplus S(-5)^{184} \rightarrow S(-3)^{322} \oplus S(-4)^{13} \\ &\rightarrow S(-2)^{57} \rightarrow S \rightarrow S/I(X_6) \rightarrow 0, \end{aligned}$$

verifying as well Theorem 4.2.

We recall that $R^{\overline{\Gamma}}$ is a *level* ring if its canonical module $\text{relint}(I_d)$ is generated in only one degree and $R^{\overline{\Gamma}}$ is a *Gorenstein* ring if it is a level ring

and $\text{relint}(I_d)$ is principal. As a consequence of Theorem 4.2, we have that $R^{\bar{\Gamma}}$ is a level ring if and only if $\text{relint}(I_d) = \langle \mathcal{C}_{d,1} \rangle$ or $\mathcal{C}_{d,1} = \emptyset$. In [6, Corollary 4.13(ii)], it is shown that the homogeneous coordinate ring of any GT -surface is level. However, the same assertion is not true for GT -varieties of higher dimensions; see, for instance, [9]. We investigate further this property, which will play an important role in the last section of this work. Let us first present an interesting family of examples of Gorenstein GT -varieties.

Proposition 4.4. *Fix an even integer $2 \leq n$ and $\Gamma = \langle M_{n+1;0,1,2,\dots,n} \rangle \subset \text{GL}(n+1, \mathbb{K})$, a cyclic group of order $n+1$. Then, $R^{\bar{\Gamma}}$ is a Gorenstein ring.*

Proof. Notice that $m = x_0 \dots x_n \in R^{\bar{\Gamma}}$. Indeed m is of degree $n+1$ and it is satisfied that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$, which is a multiple of $n+1$, since n is even. It is straightforward to see that m divides any monomial of $\text{relint}(I_{n+1})$. Hence, $\text{relint}(I_{n+1}) = (x_0 \dots x_n)$ and the proof is complete. $\square$

Proposition 4.5. *Fix integers $2 \leq n < d$, $1 \leq k$ and assume that n is even. Let $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ and $\Gamma_k = \langle M_{kd;\alpha_0,\dots,\alpha_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ be finite cyclic groups of order d and kd , respectively. If $R^{\bar{\Gamma}}$ is a Gorenstein ring, then $R^{\bar{\Gamma}_k}$ is a level ring.*

Proof. We denote $I_d \subset R$ (respectively, $I_{kd} \subset R$) the ideal generated by all monomials of degree d (respectively, kd) which are invariants of Γ (respectively, Γ_k). We write $\text{relint}(I_d) = \langle m \rangle$. We want to prove that any monomial $m' \in \mathcal{C}_{d,2kd}$ is divisible by a monomial $\bar{m} \in \mathcal{C}_{d,kd}$. We fix $m' = x_0^{a_0} \dots x_n^{a_n} \in \mathcal{C}_{d,2kd}$. Notice that m' is also an invariant of Γ , so $m' \in \mathcal{C}_{d,k}$ and by hypothesis m divides m' . We define $m_1 = \frac{m'}{m}$; since $m_1 \in R^{\bar{\Gamma}}$ is a monomial of degree $(2k-1)d$, by Theorem 3.2(i), there are monomials $m_2, \dots, m_{2k} \in R^{\bar{\Gamma}}$ of degree d , such that $m_1 = m_2 \dots m_{2k}$, and hence $m' = mm_2 \dots m_{2k}$. For each monomial m_i , $1 \leq i \leq 2k$, there is a unique integer $r_i \geq 0$, such that the lattice point l_{m_i} is a solution of the system $(*)_{1,r_i}$ induced by $\bar{\Gamma}$. By Lemma 3.10, there is a subsequence $(r_{i_1}, \dots, r_{i_k}) \subset (r_1, \dots, r_{2k})$ of k integers the sum of which is a multiple of k . Therefore, we obtain that $m_{i_1} \dots m_{i_k} \in R^{\bar{\Gamma}_k}$ and $\bar{m} = m' / (m_{i_1} \dots m_{i_k}) \in \text{relint}(I_{kd})$ is the required monomial. $\square$

Corollary 4.6. *Fix integers $2 \leq n$, $1 \leq k$ with n even and fix $\Gamma_k = \langle M_{k(n+1);0,1,2,\dots,n} \rangle \subset \text{GL}(n+1, \mathbb{K})$, a finite cyclic group of order $k(n+1)$. Then, $R^{\bar{\Gamma}_k}$ is a level ring.*

Proof. It follows directly from Propositions 4.4 and 4.5. $\square$

The rest of this section concerns the Castelnuovo–Mumford regularity $\text{reg}(R^{\bar{\Gamma}})$ of $R^{\bar{\Gamma}}$ with $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ a cyclic group of order d as in Notation 3.1. We characterize $\text{reg}(R^{\bar{\Gamma}})$ in terms of $\text{relint}(I_d)$. First, we need some preparation.

Definition 4.7. A set $\{y_0, \dots, y_n\} \subset R^{\bar{\Gamma}}$ of n homogeneous elements is said to be a *homogeneous system of parameters*, shortly h.s.o.p, if $R^{\bar{\Gamma}}$ is a finitely generated $\mathbb{K}[y_0, \dots, y_n]$ -module.

For sake of completeness, we prove the following.

Proposition 4.8. $\{x_0^d, \dots, x_n^d\}$ is an h.s.o.p of $R^{\bar{\Gamma}}$.

Proof. We consider the graded quotient algebra $A := R^{\bar{\Gamma}}/\langle x_0^d, \dots, x_n^d \rangle = \bigoplus_{t \geq 1} A_t$, $A_t = R_t^{\bar{\Gamma}}/\langle x_0^d, \dots, x_n^d \rangle_{td} = R_{td}^{\Gamma}/\langle x_0^d, \dots, x_n^d \rangle_{td}$. Therefore, for $t \geq n+1$, we have that $A_t = \langle 0 \rangle$ and for $1 \leq t \leq n$, a $\mathbb{K}$ -basis of A_t is formed by the set of all monomials $m = x_0^{a_0} \dots x_n^{a_n} \in R^{\Gamma}$ of degree td , such that $a_0 < d, \dots, a_n < d$. We write $\theta_1, \dots, \theta_D$ the set of all such monomials and $\theta_0 = 1$. Then, it is clear that $R^{\bar{\Gamma}} = \langle \theta_0, \theta_1, \dots, \theta_D \rangle$ as a $\mathbb{K}[x_0^d, \dots, x_n^d]$ -module. $\square$

We call $\{\theta_0, \theta_1, \dots, \theta_D\}$ a set of *secondary invariants* of $\bar{\Gamma}$. For $1 \leq i \leq D$, we denote $\delta_i = \deg(\theta_i)$. We set $e_0 = 1$ and we define e_j , $j = 1, \dots, n$, the multiplicities of the sequence of degrees $(\delta_1, \delta_1, \dots, \delta_D)$. Notice that $e_1 = \mu_d - (n+1)$. Moreover, we have the following.

Proposition 4.9. (i) The number of secondary invariants of $R^{\bar{\Gamma}}$ is $D+1 = d^{n-1}$.

(ii) The Hilbert series $\text{HS}(R^{\bar{\Gamma}}, z)$ of the ring $R^{\bar{\Gamma}}$ is

$$\text{HS}(R^{\bar{\Gamma}}, z) = \frac{1 + e_1 z + \dots + e_n z^n}{(1-z)^{n+1}}. \quad (3)$$

In particular, the degree of X_d is d^{n-1} .

Proof. See [27, Proposition 2.3.6]. $\square$

Theorem 4.10. With the above notation

$$n \leq \text{reg}(R^{\bar{\Gamma}}) \leq n+1.$$

The equality $\text{reg}(R^{\bar{\Gamma}}) = n+1$ holds if and only if $\mathcal{C}_{d,1} \neq \emptyset$.

Proof. The right inequality follows immediately from (3). We set $m = x_0^d \dots x_n^d$ and $m' = x_0^{d-1} \dots x_n^{d-1}$. Lemma 3.10 assures the existence of a monomial of degree $(n-1)d$ in $R^{\bar{\Gamma}}$ dividing m' , and hence, it assures the existence of secondary invariants of degrees smaller or equal to $(n-1)d$, so $e_{n-1} > 0$ which gives us the left inequality. Now, $\text{reg}(R^{\bar{\Gamma}}) = n+1$ if and only if $e_n > 0$. If $e_n > 0$, then there exists a secondary invariant θ of degree nd and we obtain $m/\theta \in \mathcal{C}_{d,1}$. Conversely, let $p = x_0^{a_0} \dots x_n^{a_n} \in \mathcal{C}_{1,d}$. Notice that necessarily $a_i < d$, $i = 0, \dots, n$; thus, m/p is a secondary invariant of degree nd . $\square$

To end this section, we present some examples illustrating the last results. They also bring to light how the Hilbert series and regularity of $R^{\bar{\Gamma}}$ can be deduced by just looking at the set of invariants of $\bar{\Gamma}$ of degree smaller or equal to nd , and vice versa.

Example 4.11. (i) Fix integers $2 = n < d$ and fix $\Gamma = \langle M_{d;0,\alpha_1,\alpha_2} \rangle \subset \text{GL}(3, \mathbb{K})$ with $0 < \alpha_1 < \alpha_2$. Let $\lambda, 0 < \mu \leq \frac{d}{\text{GCD}(a,d)}$ be the uniquely determined integers satisfying $\alpha_2 = \lambda \frac{\alpha_1}{\text{GCD}(\alpha_1,d)} + \mu \frac{d}{\text{GCD}(\alpha_1,d)}$. In [6, Proposition 4.12], it is proved that

$$\text{HS}(R^{\bar{\Gamma}}, z) = \frac{\frac{d-\theta(\alpha_1,\alpha_2,d)+2}{2}z^2 + \frac{d+\theta(\alpha_1,\alpha_2,d)-4}{2}z + 1}{(1-z)^3},$$

where $\theta(\alpha_1, \alpha_2, d) = \text{GCD}(\alpha_1, d) + \text{GCD}(\lambda, d') + \text{GCD}(\lambda - 1, d')$. By Proposition 4.9, there are exactly $\frac{d+\theta(\alpha_1, \alpha_2, d)-4}{2}$ secondary invariants of degree d and $\frac{d-\theta(\alpha_1, \alpha_2, d)+2}{2}$ secondary invariants of degree $2d$. From this, it can be easily deduced that $\text{reg}(R^{\bar{\Gamma}}) = 3$.

- (ii) Fix $n = 3$, $d = 4$ and fix $\Gamma = \langle M_{4,0,1,2,3} \rangle \subset \text{GL}(4, \mathbb{K})$. We denote I_4 its associated GT -system. The ideal I_4 is generated by the following 10 monomials:

$$x_0^4, x_0^2 x_2^2, x_0^2 x_1 x_3, x_0 x_1^2 x_2, x_0 x_1 x_3^2, x_1^4, x_1^2 x_3^2, x_1 x_2^2 x_3, x_2^4, x_3^4.$$

Therefore, $\text{relint}(I_4) = \langle \mathcal{C}_{4,2} \rangle$ with $\mathcal{C}_{4,2} = \{x_0^4 x_1 x_2^2 x_3, x_0^3 x_1^3 x_2 x_3, x_0^3 x_1 x_2 x_3^2, x_0 x_1^3 x_2^2 x_3, x_0^2 x_1^2 x_2^2 x_3^2, x_0^2 x_1 x_2^4 x_3, x_0 x_1^4 x_2 x_3^2, x_0 x_1^2 x_2^4 x_3^2, x_0 x_1 x_2^3 x_3^3\}$. There are 6 and 9 secondary invariants of degree d and $2d$, respectively, and by Theorem 4.10, $\text{reg}(R^{\bar{\Gamma}}) = 3$. By Proposition 4.9, the Hilbert series of $R^{\bar{\Gamma}}$ is the following:

$$\text{HS}(R^{\bar{\Gamma}}, z) = \frac{9z^2 + 6z + 1}{(1 - z)^4}.$$

- (iii) Let $\Gamma = \langle M_{6,0,1,2,3} \rangle \subset \text{GL}(4, \mathbb{K})$ be a cyclic group of order 6. The ideal I_6 is generated by 16 monomial invariants of Γ of degree 6 and the ideal $\text{relint}(I_6) = (x_0^3 x_1 x_2 x_3, x_0^2 x_1^4 x_2^4 x_3^2, x_0^2 x_1^3 x_2^6 x_3, x_0 x_1^6 x_2^3 x_3^2, x_0 x_1^5 x_2^5 x_3, x_0 x_1 x_2 x_3^3)$ (see Examples 3.8 and 4.3). Therefore, $R^{\bar{\Gamma}}$ has 12 secondary invariants of degree d ; it has 2 secondary invariants of degree $3d$ and by Theorem 4.10 or Example 4.3, $\text{reg}(R^{\bar{\Gamma}}) = 4$. Hence, from Proposition 4.9, we can deduce immediately that the Hilbert series of the ring $R^{\bar{\Gamma}}$ is the following:

$$\text{HS}(R^{\bar{\Gamma}}, z) = \frac{2z^3 + 21z^2 + 12z + 1}{(1 - z)^3}.$$

- (iv) Fix integers $2 \leq n$ and $d = n + 1$ with n even, and fix $\Gamma = \langle M_{t(n+1);0,1,2,\dots,n} \rangle \subset \text{GL}(n + 1, \mathbb{K})$. By Corollary 4.4, $R^{\bar{\Gamma}}$ is Gorenstein, and by Theorem 4.10, we obtain that $\text{reg}(\bar{\Gamma}) = n + 1$.

5. Cohomology of Normal Bundles of RL-Varieties

In this section, we introduce a new family of smooth rational monomial projections of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ which naturally arises from *level* GT -varieties (see Definition 5.1) and we study their canonical modules. We called them RL -varieties to stress the link with the notions of the relative interior and levelness. We devote the rest of this work to determine the cohomology of the normal bundle of any RL -variety (see Theorem 5.9). Both, the coordinate ring and the canonical module of GT -varieties, play an important role on our computations and the proof of Theorem 5.9 is inspired by [3].

In Sect. 4, we have seen that the canonical module ω_{X_d} of a GT -variety X_d with group a finite cyclic group $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \text{GL}(n + 1, \mathbb{K})$ of order d is identified with the ideal $\text{relint}(I_d) = (x_0^{a_0} \dots x_n^{a_n} \in R^{\bar{\Gamma}} \mid 0 \neq a_0 \dots a_n)$

and we have proved that $\text{relint}(I_d)$ is generated by monomials of degree d and $2d$. We begin with the following definition.

Definition 5.1. Let X_d be a GT -variety with group a finite cyclic group $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ of order d and associated GT -system I_d . We say that X_d is a *level GT -variety* if $R^{\bar{\Gamma}}$ is a level ring and, in addition, $\text{reg}(R^{\bar{\Gamma}}) = n+1$. Equivalently, if its canonical module $\text{relint}(I_d) = \langle \mathcal{C}_{d,1} \rangle$ (see Proposition 4.1 and Theorem 4.2).

Let us see some examples of level GT -varieties of any dimension $2 \leq n$, and next a necessary condition on the matrices $M_{d;\alpha_0,\dots,\alpha_n}$ for X_d be level.

Example 5.2. (i) All GT -surfaces are level (see [6, Corollary 4.13]).

(ii) Fix integers $4 \leq n, 1 \leq k$ with n even. For $d := k(n+1)$ and for the finite cyclic group $\Gamma = \langle M_{d;0,1,2,\dots,n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ of order d , the associated GT -variety X_d is level (see Corollary 4.6).

(iii) Fix integers $3 \leq n, 1 \leq t$ with n odd and fix a finite cyclic group $\Gamma_t = \langle M_{t(n+1);0,1,\dots,1,2} \rangle$ of order $d = t(n+1)$. Then, $X_{t(n+1)}$ is a level GT -variety.

Proof. We fix $t \geq 1$. The monomials $x_0^{a_0} \dots x_n^{a_n}$ of degree d in $R^{\bar{\Gamma}_1}$ are the $\mathbb{Z}_{\geq 0}^{n+1}$ -solutions of the systems

$$(*)_r \begin{cases} y_0 + y_1 + \dots + y_{n-1} + y_n = d \\ y_1 + \dots + y_{n-1} + 2y_n = rd, \end{cases} \quad r = 0, 1, 2.$$

We consider new variables $\alpha = y_0, \beta = y_1 + \dots + y_{n-1}$ and $\gamma = y_n$ and the integer systems

$$(**)_r \begin{cases} \alpha + \beta + \gamma = d \\ \beta + 2\gamma = rd, \end{cases} \quad r = 0, 1, 2.$$

Any solution $(a_0, \dots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ of $(*)_r$ gives rise to a solution $(a_0, a_1 + \dots + a_{n-1}, a_n) \in \mathbb{Z}_{\geq 0}^3$ of $(**)_r$. Conversely, a solution $(\alpha, \beta, \gamma) \in \mathbb{Z}_{\geq 0}^3$ of $(**)_r$ gives rise to $\binom{\beta+n-2}{n-2}$ solutions $(\alpha, a_1, \dots, a_{n-1}, \gamma) \in \mathbb{Z}_{\geq 0}^{n+1}$ of $(*)_r$, where $a_1 + \dots + a_{n-1} = \beta$. The solutions $(\alpha, \beta, \gamma) \in \mathbb{Z}_{\geq 0}^3$ of one of the systems $(**)_r$ are $(d, 0, 0)$, $(0, 0, d)$ and $\{(\gamma, d - 2\gamma, \gamma) \mid \gamma = 0, \dots, \frac{d}{2}\}$. Therefore, we have

$$\mu_d = 2 + \sum_{\gamma=0}^{\frac{d}{2}} \binom{d - 2\gamma + n - 2}{n - 2}.$$

Since $x_0 \dots x_n$ is an invariant of $\Gamma_1 \subset \text{GL}(n+1, \mathbb{K})$, as in the proof of Proposition 4.4, it follows that $R^{\bar{\Gamma}_1}$ is Gorenstein; by Proposition 4.5, we only have to check that $\mu_d \leq \binom{n-1+d}{n-1}$. For $n \geq 3$, it holds that

$$\begin{aligned} \mu_d &= 2 + \sum_{\substack{\gamma=n-1 \\ \gamma \text{ odd}}}^{d+n-2} \binom{\gamma}{n-2} \leq 2 + \sum_{\tilde{\gamma}=n-2}^{d+n-2} \binom{\tilde{\gamma}}{n-2} - (n-1) \\ &\leq \sum_{\tilde{\gamma}=n-2}^{d+n-2} \binom{\tilde{\gamma}}{n-2} = \binom{n-1+d}{n-1}. \end{aligned}$$

□

Proposition 5.3. *Fix integers $2 \leq n < d$, a finite cyclic group $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ of order d and set I_d the ideal generated by all monomial invariants of Γ of degree d . If $\mathcal{C}_{d,1} \neq \emptyset$, then there are at least three indices two by two distinct.*

Proof. By contradiction, we assume $M_{d;\alpha_0,\dots,\alpha_n} = M_{d;0,\dots,1,0,a,n,\dots,a}$ with $0 < a < d$, such that $\mathrm{GCD}(a, d) = 1$. Therefore, for any monomial $m \in R^\Gamma$ of degree d , it holds that $\mathrm{supp}(m) \in \{x_0, \dots, x_l\}$ or $\mathrm{supp}(m) \in \{x_{l+1}, \dots, x_n\}$. In other words, $\mathcal{C}_{d,1} = \emptyset$ which contradicts our hypothesis. $\square$

Remark 5.4. Let $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ be a finite cyclic group of order d , and I_d the monomial artinian ideal generated by the set of all monomials $\{m_1, \dots, m_{\mu_d}\} \subset R^\Gamma$ of degree d . Definition 5.1 can be extended to any variety $X_d = \varphi_{I_d}(\mathbb{P}^n) \subset \mathbb{P}^{\mu_d-1}$ even if it is not a GT -variety, or equivalently, I_d is not a Togliatti system, i.e., the condition $\mu_d \leq \binom{n-1+d}{n-1}$ is not satisfied. Many notions and results of this section remain true in this more general setup.

From now onwards, we fix a level GT -variety X_d with group a finite cyclic group $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n} \rangle \subset \mathrm{GL}(n+1, \mathbb{K})$ of order d and we denote I_d its associated GT -system. We denote $\eta_d = |\mathcal{C}_{d,1}|$, i.e., the number of monomials of degree d in $\mathrm{relint}(I_d)$ and we set $N_d := \binom{n+d}{d} - \eta_d - 1$. By $f_d: \mathbb{P}^n \rightarrow \mathbb{P}^{N_d}$, we denote the morphism induced by the inverse system $\mathrm{relint}(I_d)^{-1}$. We denote $\mathcal{X}_d = f_d(\mathbb{P}^n) \subset \mathbb{P}^{N_d}$ and we call $\mathcal{X}_d$ the RL -variety associated with X_d . We have the following.

Proposition 5.5. *$\mathcal{X}_d$ is a smooth rational variety and f_d is an embedding.*

Proof. $\mathcal{X}_d$ is a toric variety parametrized by all monomials of degree d in $\mathrm{relint}(I_d)^{-1}$. It is straightforward to check that $\mathcal{X}_d$ satisfies the smoothness criterion for toric varieties [12, Chapter 5—Corollary 3.2]. In particular, $\mathrm{relint}(I_d)^{-1}$ contains all monomials $x_i^{d-1}x_j$ for all $i, j \in \{0, \dots, n\}$, which is a sufficient condition for f_d to be an embedding. $\square$

In [3], the authors develop a new method to compute the cohomology of the normal bundle of smooth rational projections of the Veronese variety $\nu_{n,d}(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ for which the parametrization is an embedding. Let $\mathcal{X}_d$ be an RL -variety associated with a level GT -variety X_d with group Γ . By Proposition 5.5, any RL -variety $\mathcal{X}_d$ is of this kind. Furthermore, the relation between $\mathcal{X}_d$ and $\mathrm{relint}(I_d) \subset R^\Gamma$ allows us to apply this approach to any RL -variety $\mathcal{X}_d$. In this setting, we have the following presentation of the normal bundle $\mathcal{N}_{\mathcal{X}_d}$ of the RL -variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ (see [3, (3.3)]):

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d) \rightarrow \mathcal{N}_{\mathcal{X}_d} \rightarrow 0. \quad (4)$$

Taking the long sequence of cohomology for (4), we determine the cohomology $\mathbb{K}$ -vector spaces $H^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ in most cases, as the following result shows.

Proposition 5.6. *Let $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ be an RL -variety of dimension $n \geq 2$, and we have:*

- (i) for all $0 < i < n - 1$ and $k \in \mathbb{Z}$, $H^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$.
(ii)

$$h^0(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (N_d + 1) \binom{n+d-k}{n} - (n+1) \binom{n+1-k}{n} & k \leq 1 \\ (N_d + 1) \binom{n+d-k}{n} & 1 < k \leq d \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We fix $k \in \mathbb{Z}$. We twist (4) by $\mathcal{O}_{\mathbb{P}^n}(-k)$, and then, we consider the long exact sequence of cohomology. For any i and k , we obtain

$$\rightarrow H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \rightarrow H^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \rightarrow H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)) \rightarrow . \quad (5)$$

From the additivity of the cohomology, it follows the vanishing $H^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$ for all $0 < i < n - 1$. In addition, we obtain the presentation

$$0 \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)) \rightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \rightarrow H^0(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \rightarrow 0.$$

The result follows from the Bott formulas for the cohomology of $\mathbb{P}^n$ (see [23]). $\square$

Thus far, we have determined the dimension of $H^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ for any i and k except

$$H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \text{ and } H^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)).$$

To compute them, we apply Proposition 5.6(i) to the long exact sequence of cohomology (5). For any k , we obtain the exact sequence

$$\begin{aligned} 0 \rightarrow H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) &\rightarrow H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)) \\ &\rightarrow H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \rightarrow H^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \rightarrow 0. \end{aligned} \quad (6)$$

As an immediate result, we get that for all $k < d + n + 1$

$$H^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0 \quad \text{and} \quad H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \cong H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)). \quad (7)$$

Thus

$$h^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (n+1) \binom{k-2}{n} & n+2 \leq k < d+n+1 \\ 0 & k \leq n+1. \end{cases} \quad (8)$$

We focus on computing $H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ for $k \geq d + n + 1$. We need some preparation. By $\partial_{x_0}, \dots, \partial_{x_n}$, we denote the linear operators acting on R as partial derivatives. Let $m \in R_l$ be a monomial and we write $m = x_0^{a_0} \dots x_n^{a_n}$. We denote ∂_m the composition of linear operators $\partial_{x_0}^{a_0} \dots \partial_{x_n}^{a_n}$.

Lemma 5.7. *Let m be a monomial of degree $k - d - n - 1$, let q and q' be monomials of degree $k - n - 1$, such that m divides both q and q' , and let $0 \leq i \neq j \leq n$ be integers. Then, $x_i \partial_m q$ and $x_j \partial_m q'$ are linearly independent if and only if for any monomial $m' \neq m$ of degree $k - d - n - 1$ which divides q and q' , $x_i \partial_{m'} q$ and $x_j \partial_{m'} q'$ are linearly independent.*

Proof. We write $m' = x_0^{b_0} \dots x_n^{b_n}$, $m = x_0^{c_0} \dots x_n^{c_n}$, $q = x_0^{a_0} \dots x_n^{a_n}$, $q' = x_0^{a'_0} \dots x_n^{a'_n}$. Assume that $x_i \partial_m q$ and $x_j \partial_m q'$ are linearly independent and there is $m' \neq m$, which divides q and q' , such that $x_i \partial_{m'} q$ and $x_j \partial_{m'} q'$ are linearly dependent. Therefore, we have the equality

$$x_0^{a_0-b_0} \dots x_i^{a_i-b_i+1} \dots x_n^{a_n-b_n} = x_0^{a'_0-b_0} \dots x_j^{a'_j-b_j+1} \dots x_n^{a'_n-b_n},$$

which implies $a_l = a'_l$, $0 \leq l \neq i, j \leq n$, $a_i = a'_i - 1$ and $a_j = a'_j + 1$. Then, we obtain

$$\begin{aligned} x_i \partial_m q &= A x_0^{a_0-c_0} \dots x_j^{a_j-c_j} \dots x_i^{a_i-c_i+1} \dots x_n^{a_n-c_n} \\ x_j \partial_m q' &= B x_0^{a_0-c_0} \dots x_j^{a_j-1-c_j+1} \dots x_i^{a_i+1-c_i} \dots x_n^{a_n-c_n} \end{aligned}$$

for some $A, B \in \mathbb{K} \setminus \{0\}$, which is a contradiction. $\square$

An RL -variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ of dimension $n \geq 2$ is a smooth rational variety embedded in $\mathbb{P}^{N_d}$. In [3], the authors introduce a new method to compute the cohomology of the normal bundle of varieties of this kind. With the notation of [3], we write the embedding $f_d : \mathbb{P}(U) \rightarrow \mathbb{P}^{N_d}$ with $U = R_1^\vee$. The RL -variety $\mathcal{X}_d = f_d(\mathbb{P}(U))$ is the projection in $\mathbb{P}^{N_d}$ of the Veronese variety $\nu_d(\mathbb{P}(U)) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ from the projective space $\mathbb{P}(T)$ of dimension $\binom{n+d}{n} - N_d$ where $T^\vee$ is identified with the $\mathbb{K}$ -vector subspace of R_d generated by all the monomials of degree d in $\text{relint}(I_d) = (x_0^{a_0} \dots x_n^{a_n} \in R^\Gamma \mid 0 \neq a_0 \dots a_n)$. Let $0 \leq i \neq j \leq n$, $l \geq 1$ and $t \geq 1$ be integers. We denote $D_{i,j} : S^l U \otimes S^t U \rightarrow S^{l-1} U \otimes S^{t-1} U$ the linear map $\partial_{x_i} \otimes \partial_{x_j} - \partial_{x_j} \otimes \partial_{x_i}$.

Proposition 5.8. *Let $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ be an RL -variety of dimension $n \geq 2$ associated with a level GT -variety X_d with group $\Gamma = \langle M_{d;\alpha_0, \dots, \alpha_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$. Then*

$$h^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} \eta_d + \frac{n(d-1)}{d} \binom{n+d-1}{n} & k = d+n+1 \\ (n+1)\eta_d & k = d+n+2 \\ 0 & k \geq d+n+3. \end{cases}$$

Proof. By [3, Theorem 2], we obtain $h^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-d-n-1)) = \dim(\mu^{-1}(T))$, where $\mu : U \otimes S^{d-1} U \rightarrow S^d U$ is the multiplication map, and for all $k \geq d+n+2$

$$H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = (S^{k-d-n-1} U \otimes T) \cap \left(\bigcap_{0 \leq i, j, r, s \leq n} (\ker(D_{i,j} \circ D_{r,s})) \right).$$

In particular, for $k = d+n+1, d+n+2$, we can conclude that

$$h^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-d-n-1)) = \eta_d + \frac{n(d-1)}{d} \binom{n+d-1}{n}$$

and $H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-d-n-2)) = U \otimes T$. Moreover, for $k \geq d+n+3$, we have

$$H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \cong \{x_0 \otimes q_0 + \dots + x_n \otimes q_n \in R_1 \otimes R_{k-n-2} \mid x_0 \partial_m(q_0) + \dots + x_n \partial_m(q_n) \in \text{relint}(I_d)_1 \text{ for all monomial } m \in R_{k-d-n-1}\},$$

where $\text{relint}(I_d)_1$ denotes the $\mathbb{K}$ -vector subspace of R_d generated by the set $\mathcal{C}_{d,1} \subset \text{relint}(I_d)$ of monomials of degree d . We want to prove that $H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$ for all $k \geq d + n + 3$. Assume that there exist $q_0, \dots, q_n \in R_{k-n-2}$ and a monomial $m \in R_{k-d-n-1}$, such that $0 \neq u_m := x_0 \partial_m(q_0) + \dots + x_n \partial_m(q_n) \in \text{relint}(I_d)_1$. Therefore, any monomial that appears in u_m belongs to $\text{relint}(I_d) \subset \bar{\Gamma}$. Let $q \in R_{k-n-2}$ be a monomial, such that $0 \neq x_i \partial_m q$ is a monomial that occurs in u_m . By Lemma 5.7, we have that if $x_0 \otimes q_0 + \dots + x_n \otimes q_n \in H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$, then for any monomial $m' \in R_{k-d-n-1}$, $x_i \partial_{m'} q \in \text{relint}(I_d) \subset R_1^\Gamma = R_d^\Gamma$. We will show that there always exists a monomial $m' \in R_{k-d-n-1}$ dividing q , such that $x_i \partial_{m'} q \notin R_1^\Gamma$. Thus, it concludes $H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$ for all $k \geq d + n + 3$. Notice that, by Proposition 5.3, $M_{d;\alpha_0, \dots, \alpha_n}$ has three indices $\alpha_i, \alpha_j, \alpha_l$ two by two distinct. We consider monomials $q = x_0^{a_0} \dots x_n^{a_n} \in R_{k-n-2}$ and $m = x_0^{b_0} \dots x_n^{b_n} \in R_{k-d-n-1}$, such that m divides q and $x_i \partial_m q \in \text{relint}(I_d)$. In particular, we have that $b_j < a_j$ for all $0 \leq j \neq i \leq n$ and $b_i \leq a_i - 1$. By assumption, $x_i \partial_m q := x_0^{c_0} \dots x_n^{c_n} = x_0^{a_0-b_0} \dots x_i^{a_i-b_i+1} \dots x_n^{a_n-b_n} \in R_1^\Gamma$. We distinguish two cases.

Case 1: $0 < b_i$. If $\alpha_i = 0$ or $\alpha_i > 0$ and $2\alpha_i - \alpha_j \not\equiv 0 \pmod{d}$, we define the monomial $m' = x_0^{b_0} \dots x_l^{b_l+1} \dots x_i^{b_i-1} \dots x_n^{b_n}$. Then, $x_i \partial_{m'} q = x_0^{c_0} \dots x_j^{c_j-1} \dots x_i^{c_i+1} \dots x_n^{c_n}$. Otherwise, $2\alpha_i - \alpha_l \not\equiv 0 \pmod{d}$ and we define $m' = x_0^{b_0} \dots x_l^{b_l+1} \dots x_i^{b_i-1} \dots x_n^{b_n}$. Then, $x_i \partial_{m'} q = x_0^{c_0} \dots x_l^{c_l-1} \dots x_i^{c_i+1} \dots x_n^{c_n}$ and its associated point does not verify the linear congruence equation $\alpha_0 y_0 + \dots + \alpha_n y_n \equiv 0 \pmod{d}$.

Case 2: $b_i = 0$. We take $0 < b_h$, and we can assume that α_h, α_j are different pair-wise. We define $m' = x_0^{b_0} \dots x_j^{b_j+1} \dots x_h^{b_h-1} \dots x_n^{b_n}$. Then, $x_i \partial_{m'} q = x_0^{c_0} \dots x_j^{c_j-1} \dots x_i^{c_i} \dots x_h^{c_h+1} \dots x_n^{c_n}$ and its associated point does not verify the linear congruence equation $\alpha_0 y_0 + \dots + \alpha_n y_n \equiv 0 \pmod{d}$.

In any case, we have constructed a monomial $m' \in R_{k-d-n-1}$ dividing q , such that $x_i \partial_{m'} q \notin R_1^\Gamma$ and the proposition follows. $\square$

Directly from (6) and Proposition 5.8, we obtain $h^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k))$ for all $k \geq d + n + 1$. We summarize all the computations and establish the main result of this section.

Theorem 5.9. *Fix a level GT-variety X_d with group a finite cyclic group $\Gamma = \langle M_{d;\alpha_0, \dots, \alpha_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ of order d and associated GT-system I_d . Set $\eta_d := |\mathcal{C}_{d,1}|$ the number of monomials of $\text{relint}(I_d)$ of degree d and $N_d := \binom{n+d}{d} - \eta_d - 1$. Let $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ be the RL-variety of dimension $n \geq 2$ associated with X_d . It holds*

- (i) for $0 < i < n - 1$ and for all $k \in \mathbb{Z}$, $h^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0$.
- (ii)

$$h^0(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (N_d + 1) \binom{n+d-k}{n} - (n+1) \binom{n+1-k}{n} & k \leq 1 \\ (N_d + 1) \binom{n+d-k}{n} & 1 < k \leq d \\ 0 & \text{otherwise.} \end{cases}$$

(iii)

$$h^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (n+1) \binom{k-2}{n} & n+2 \leq k < d+n+1 \\ \eta_d + \frac{n(d-1)}{d} \binom{n+d-1}{n} & k = d+n+1 \\ (n+1)\eta_d & k = d+n+2 \\ 0 & k \leq n+1 \text{ or } k \geq d+n+3. \end{cases}$$

(iv)

$$h^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (N_d+1) \binom{k-d-1}{n} - (n+1) \binom{k-2}{n} & k \geq d+n+3 \\ 0 & \text{otherwise.} \end{cases}$$

We end this work by showing two examples pointing out Theorem 5.9. All the computations have been made with the software Macaulay2 [17].

Example 5.10. (i) We fix $d = 5$ and $\Gamma = \langle M_{5;0,1,2} \rangle \subset \mathrm{GL}(3, \mathbb{K})$ a cyclic group of order 5. The ideal $I_5 = (x_0^5, x_0^2 x_1 x_2^2, x_0 x_1^3 x_2, x_1^5, x_2^5) \subset \mathbb{K}[x_0, x_1, x_2]$ is the GT -system generated by all monomial invariants of Γ of degree 5. Its associated GT -variety X_5 is level with $\mathrm{relint}(I_5) = (x_0^2 x_1 x_2^2, x_0 x_1^3 x_2)$ and $\eta_5 = 2$. We present the cohomology table from degree -10 to 0 of the normal bundle $\mathcal{N}_{\mathcal{X}_5}$ of the smooth rational variety $\mathcal{X}_5$ parametrized by the inverse system $\mathrm{relint}(I_5)^{-1}$

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
2 :	150	82	30	.	.	.	.	.	.	.	.
1 :	.	.	6	26	30	18	9	3	.	.	.
0 :	.	.	.	.	.	19	57	114	190	282	390

(ii) We fix $d = 4$ and $\Gamma = \langle M_{4;0,1,1,2} \rangle \subset \mathrm{GL}(4, \mathbb{K})$ a cyclic group of order 4. The associated GT -system $I_4 = (x_0^4, x_0^2 x_3^2, x_0 x_1^2 x_3, x_0 x_1 x_2 x_3, x_0 x_2^2 x_3, x_1^4, x_1^3 x_2, x_1^2 x_2^2, x_1 x_2^3, x_2^4, x_3^4) \subset \mathbb{K}[x_0, x_1, x_2, x_3]$ and its associated GT -threefold X_4 is level with $\mathrm{relint}(I_4) = (x_0 x_1 x_2 x_3)$ (see Example 5.2(iii)). We present the cohomology table from degree -9 to 0 of the normal bundle $\mathcal{N}_{\mathcal{X}_4}$ of the smooth rational variety $\mathcal{X}_4$ parametrized by the inverse system $\mathrm{relint}(I_4)^{-1}$

	-9	-8	-7	-6	-5	-4	-3	-2	-1	0
3 :	710	344	116	.	.	.	.	.	.	.
2 :	.	.	4	46	40	16	4	.	.	.
1 :	.	.	.	.	.	.	.	.	.	.
0 :	.	.	.	.	.	34	136	340	676	1174

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References

- [1] Almeida, C., Andrade, A.V., Miró-Roig, R.M.: Gaps in the number of generators of monomial Togliatti systems. *J. Pure Appl. Algebra* **223**, 1817–1831 (2019)
- [2] Alzati, A., Re, R., Tortora, A.: An algorithm for determining the normal bundle of rational monomial curves. *Rend. Circ. Mat. Palermo II Ser.* **67**(2), 291–306 (2018)
- [3] Alzati, A., Re, R.: Cohomology of normal bundles of special rational varieties. *Commun. Algorithm* **48**(6), 2492–2516 (2020)
- [4] Bruns, W., Herzog, J.: *Cohen-Macaulay Rings*. Cambridge University Press, Cambridge (1993)
- [5] Chardin, M.: Applications of some properties of the canonical module in computational projective algebraic geometry. *J. Symbol. Comput.* **29**(4–5), 527–544 (2000)
- [6] Colarte, L., Mezzetti, E., Miró-Roig, R.M.: On the arithmetic Cohen–Macaulayness of varieties parameterized by Togliatti systems. *Ann. Mat. Pura Appl.* (2021). <https://doi.org/10.1007/s10231-020-01058-2>
- [7] Colarte, L., Mezzetti, E., Miró-Roig, R.M., Salat, M.: On the coefficients of the permanent and the determinant of a circulant matrix. *Appl. Proc. AMS.* **147**, 547–558 (2019)
- [8] Colarte, L., Mezzetti, E., Miró-Roig, R.M., Salat, M.: Togliatti systems associated to the dihedral group and the weak Lefschetz property. *Isr. J. Math.* (to appear)
- [9] Colarte, L., Miró-Roig, R.M.: Minimal set of binomial generators for certain Veronese 3-fold projections. *J. Pure Appl. Algebra* **224**, 768–788 (2020)
- [10] Eisenbud, D., Van de Ven, A.: On the normal bundles of smooth rational space curves. *Math. Ann.* **256**(4), 453–463 (1981)
- [11] Erdős, P., Ginzburg, A., Ziv, A.: Theorem in the additive number theory. *Bull. Res. Council Isr. Sec. F Math. Phys.* **10**, F1 (1961)
- [12] Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: *Discriminants, Resultants and Multidimensional Discriminants*. Springer, New York (1994)

- [13] Gröbner, W.: Über Veronessche Varietäten und deren Projektionen. *Arch. Math.* **16**, 257–264 (1965)
- [14] Hochster, M.: Rings of Invariants of Tori, Cohen–Macaulay rings generated by monomials, and polytopes. *Ann. Math.* **96**, 318–337 (1972)
- [15] Hochster, M., Eagon, J.A.: Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci. *Am. J. Math.* **93**, 1020–1058 (1971)
- [16] Kreuzer, M.: On the canonical module of a 0-dimensional scheme. *Can. J. Math.* **46**(2), 357–379 (1994)
- [17] Grayson, D.R., Stillman, M.E.: Macaulay2, a software system for research in algebraic geometry, <http://www.math.uiuc.edu/Macaulay2/>
- [18] Mezzetti, E., Miró-Roig, R.M.: The minimal number of generators of a Togliatti system. *Ann. Math. Pura Appl.* **195**, 2077–2098 (2016)
- [19] Mezzetti, E., Miró-Roig, R.M.: Togliatti systems and Galois coverings. *J. Algebra* **509**, 263–291 (2018)
- [20] Mezzetti, E., Miró-Roig, R.M., Ottaviani, G.: Laplace equations and the weak Lefschetz property. *Can. J. Math.* **65**, 634–654 (2013)
- [21] Michałek, M., Miró-Roig, R.M.: Smooth monomial Togliatti systems of cubics. *J. Combin. Theory Ser. A* **143**, 66–87 (2016)
- [22] Miró-Roig, R.M., Salat, M.: On the classification of Togliatti systems. *Commun. Alg.* **46**, 2459–2475 (2018)
- [23] Okonek, C., Schneider, M., Spindler, H.: *Vector Bundles on Complex Projective Spaces*. With an Appendix by S. I. Gelfand. Birkhäuser, Basel (2011)
- [24] Serre, J.P.: *Groupes algébriques et Corps de Classes*. Hermann, Paris (1957)
- [25] Sacchiero, G.: Normal bundles of rational curves in projective space. *Ann. Univ. Ferrara Sez. VII* **26**, 33–40 (1981)
- [26] Stanley, R.P.: Invariants of finite groups and their application to combinatorics. *Bull. Am. Math. Soc. (N.S.)* **1**, 475–511 (1979)
- [27] Sturmfels, B.: *Algorithms in Invariant Theory*, 2nd edn. Springer, Berlin (2008)
- [28] Togliatti, E.: Alcuni esempi di superficie algebriche degli iperspazi che rappresentano un’equazione di Laplace. *Commun. Math. Helvetici* **1**, 255–272 (1929)
- [29] Togliatti, E.: Alcune osservazioni sulle superficie razionali che rappresentano equazioni di Laplace. *Ann. Mat. Pura Appl. (4)* **25**, 325–339 (1946)

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