The Lagrange–Charpit Theory of the Hamilton–Jacobi Problem

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Abstract. The Lagrange–Charpit theory is a geometric method of determining a complete integral by means of a constant of the motion of a vector field defined on a phase space associated to a nonlinear PDE of first order. In this article, we establish this theory on the symplectic structure of the cotangent bundle T^*Q of the configuration manifold Q . In particular, we use it to calculate explicitly isotropic submanifolds associated with a Hamilton–Jacobi equation.

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PDEs are a heritage of Humanity. The concern to solve problems posed by physical phenomena led to the first works of the greatest men: Euler, Lagrange, Cauchy, Hamilton, Jacobi, etc.

Classically, several families of solutions were established for nonlinear PDE: general, singular, complete, . . . solutions. In particular, the concept of complete integral is linked to the method of characteristics of Cauchy, which provides an integral hypersurface with uniqueness of solution. In this context, the Lagrange–Charpit method was a significant step, in simplification and depth, in the search for a complete solution (see [10]). These ideas underwent a revival in a geometric context in the modern geometric theory (see [14]) opening applications in physics and engineering.

Given a first-order PDE:

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0,$$

the differential geometric Lagrange–Charpit method consists of considering the exterior differential system $(dz - p dx - q dy, dF)$. There is essentially a single vector field X_F (called then, the characteristic vector field) orthogonal with both 1-forms, that leaves this system stable by Lie derivative.

Considering a first integral G of X_F , the new system $(dz - p dx - q dy, dF, dG)$ has both X_F and X_G as characteristic vector fields. The coincidence between the orthogonal and the characteristic module (since we are in

dimension 5), defines M as completely integrable system (see Sect. 1), and this is the key geometric fact that allows us to find a complete integral of the equation $F = 0$.

This will be our inspiration point to work with the Liouville form w_Q on T^*Q . The power of the Poisson algebra of the canonical symplectic form $\Omega_Q = -dw_Q$ will now be what allows us to build (in any dimension) a completely integrable differential system (see Sect. 2) that we will turn into a symplectic tool on the symplectic space (T^*Q, Ω_Q) . In particular, we will use it to determine isotropic submanifolds associated with a Hamilton–Jacobi problem on T^*Q .

1. “Les systèmes différentiels extérieurs” of Elie Cartan

First of all, we bring this already classic theme, from the work of Cartan. Thus, in [5, (Chapter 3)], his essential objective is the reduction in the number of variables of an exterior differential system. This matter, in turn, is at the base of our procedure.

Let Q be a manifold of dimension n . An exterior differential system of rank k on Q is a choice of a k -dimensional subspace $\mathcal{W}_q$ of T_q^*Q for every $q \in Q$, in such a way that there is a neighborhood U of q and k linearly independent 1-forms $w_1, \dots, w_k$ on U which form a basis of $\mathcal{W}_x$ for every $x \in U$. We say that U is a trivializing neighborhood for $\mathcal{W}$ and that $w_1, \dots, w_k$ are sections of $\mathcal{W}$ on U ; we shall denote by $\mathcal{W}(U)$ the submodule of $\wedge^1(U)$ generated by these sections.

In a similar manner, a distribution $\mathcal{D}$ of rank r on M choose an r -dimensional subspace $\mathcal{D}_q$ of T_qQ for each $q \in Q$, in such a way that in some neighborhood U there exists sections $X_1, \dots, X_r \in \mathcal{D}(U)$ which form a basis of $\mathcal{D}_q$ for every $q \in U$.

We shall call orthogonal distribution of the exterior differential system $\mathcal{W}$, denoted by $\mathcal{W}^0$, to the distribution on Q defined by:

$$(\mathcal{W}^0)_q = (\mathcal{W}_q)^0 = \{D \in T_qQ : w(D) = 0 \text{ if } w \in \mathcal{W}_q\}, \text{ for every } q \in Q.$$

We shall call characteristic distribution of the exterior differential system $\mathcal{W}$ to the distribution $\mathcal{W}_0$ of vector fields on Q ,:

$$\mathcal{W}_0 = \{D : \mathcal{W}^0(U) : L_D w \in \mathcal{W}(U) \text{ if } w \in \mathcal{W}(U)\}$$

for every trivializing neighborhood U of $\mathcal{W}$, where L stands for the Lie derivative.

It is then obvious that:

$$\mathcal{W}_0 \subseteq \mathcal{W}^0.$$

Definition 1. The exterior differential system $\mathcal{W}$ of rank k is said to be completely integrable, if for every point $q \in Q$, there exists a submanifold $N \subset Q$ through q such that:

$$\begin{aligned} \mathcal{W}|_N &= 0 \\ k + \dim N &= n. \end{aligned}$$

Theorem 2. *Let Q be a manifold of dimension n and $\mathcal{W}$ an exterior differential system of rank k on Q . The following conditions are equivalent:*

- (i) $\mathcal{W}$ is completely integrable.
- (ii) $d(\mathcal{W}(U)) \subset \mathcal{W}(U) \wedge^1(U)$ for every trivializing neighborhood U of $\mathcal{W}$.
- (iii) The characteristic distribution $\mathcal{W}_0$ coincide with the orthogonal distribution $\mathcal{W}^0$.
- (iv) The orthogonal distribution $\mathcal{W}^0$ is stable under Lie bracket.
- (v) Every $q \in Q$ has a neighborhood U and functions $f_1, \dots, f_k \in C^\infty(U)$ such that:

$$df_1, \dots, df_k$$

generate $\mathcal{W}(U)$ as $C^\infty(U)$ -module.

One may also consult [6] or [8] .

2. The Lagrange–Charpit Theory on T^*Q

The good fit of the Lagrange–Charpit method of PDE integration with the different geometric aspects of the symplectic structure on the cotangent bundle, allows to make a presentation in this context, thus allowing a brief, effective and self-contained writing.

2.1. Let us consider the classical phase space of a mechanical system as the cotangent bundle T^*Q of the configuration manifold Q and let $\pi_Q : T^*Q \rightarrow Q$ be the canonical projection. Let U be a neighborhood in Q coordinated by $(q_1, \dots, q_n)$ and let us consider the corresponding coordinates $(q_1, \dots, q_n; p_1, \dots, p_n)$ on $\pi_Q^{-1}(U)$.

The Liouville form is defined locally by $w_Q = \sum_i p_i dq_i$. In this way, setting $\Omega_Q = -dw_Q$, we obtain the local expression $\Omega_Q = \sum_i dq_i \wedge dp_i$ of the canonical symplectic form on T^*Q .

For every $F \in C^\infty(T^*Q)$, its Hamiltonian vector field is the unique vector field X_F on T^*Q such that $i_{X_F}\Omega_Q = dF$. In this way, the Poisson bracket on $C^\infty(T^*Q)$ is defined by:

$$\{F, G\} = X_F G = -X_G F = -w(X_F, X_G), \quad F, G \in C^\infty(T^*Q).$$

The local expression of the Hamiltonian vector field X_F is:

$$X_F = \sum \left(F_{p_i} \frac{\partial}{\partial q_i} - F_{q_i} \frac{\partial}{\partial p_i} \right).$$

When $\{F, G\} = 0$, we say that F and G are in involution. A characteristic result, classically attributed to Jacobi and Lie, settles that if $F_1, \dots, F_r \in C^\infty(T^*Q)$ is a system of functionally independent functions in involution, then $r \leq n$ and we can extend it to a system $\{F_1, \dots, F_n, G_1, \dots, G_n\}$ in such a way:

$$\{F_i, F_j\} = \{G_i, G_j\} = 0, \quad \{F_i, G_j\} = \delta_{ij}.$$

This set is then called a Darboux coordinate system, and we can write:

$$\Omega_Q = \sum_{i=1}^n dF_i \wedge dG_i.$$

Let N be a submanifold of T^*Q of dimension m . For every $p \in N$ we denote by $(T_p N)^\perp$ the orthogonal complement of $T_p N$ with respect to Ω_Q . We say that N is isotropic if $\Omega_Q|_N = 0$, that is, if $T_p N \subseteq (T_p N)^\perp$ for every $p \in N$. Then $m \leq 2n - m$ and hence $m \leq n$. We say that N is coisotropic if $(T_p N)^\perp \subseteq T_p N$ for every $p \in N$. It holds $2n - m \leq m$ so $n \leq m$. When N is both isotropic and coisotropic we say that N is a Lagrangian submanifold of T^*Q .

The geometric structure of the Hamilton–Jacobi theory on the n -dimensional configuration space Q is the symplectic phase space of momenta (T^*Q, Ω_Q) . In this way, for a given Hamiltonian function $H \in C^\infty(T^*Q)$, there exists a vector field X_H determined by equation:

$$i_{X_H} \Omega_Q = dH$$

so that its integral curves are the trajectories of the system (v.g.r. see [7]). The classic formulation of the Hamilton–Jacobi problem is to find a function $S(t, q)$ satisfying:

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0.$$

If we put $S(t, q) = W - tE$ for a constant E , then W satisfies:

$$H\left(t, q, \frac{\partial W}{\partial q}\right) = E.$$

In our geometric context, we can write $(dW)^* H = E$, where dW is understood as a section of T^*Q . In this way, as a closed form is locally exact, we seek for closed 1-forms $\alpha: Q \rightarrow T^*Q$ such that:

$$H|_{\text{Im } \alpha} = E.$$

Thus, our objective is to find submanifolds $V \subset T^*Q$ on which H is constant.

Theorem 3 (Hamilton–Jacobi for isotropic submanifolds). *Let us consider the Hamiltonian dynamical equation:*

$$(a) \quad i_{X_H} \Omega_Q = dH$$

and let V be an isotropic submanifold of T^*Q .

- (i) If $X_H \in \mathfrak{X}(V)$ then $H|_V$ is constant.
- (ii) If $H|_V$ is constant then $X_H \in \mathfrak{X}(V)^\perp$.

Proof. (i) If $X_H \in \mathfrak{X}(V)$ as $\Omega_Q|_V = 0$ we have that $H|_V$ is constant.

(ii) If $H|_V$ is constant, then the restriction of the left-hand side of (a) to V vanishes, hence $X_H \in \mathfrak{X}(V)^\perp$. $\square$

For the Lagrangian counterpart see [4] or the smart discussion [7].

2.2. Our purpose now is to construct a completely integrable exterior differential system in such a way that, roughly, a generator system of exact differentials parameterizes isotropic submanifolds on (T^*Q, Ω_Q) . In this way, to establish and solve our problem, we need to extend the previous phase space to $T^*Q \times \mathbb{R}$. In this new frame, the consideration of contact geometry will be essential in our purpose

Thus, a contact manifold $\mathcal{C}$ is a $(2n+1)$ -manifold endowed with a 1-form α such that:

$$\ker(\alpha) \cap \ker(d\alpha) = \{0\}$$

in such a way that $d\alpha$ is nondegenerate on $\ker(\alpha)$.

The Reeb vector field is the unique $\mathcal{R} \in \mathfrak{X}(\mathcal{C})$ such that $\alpha(\mathcal{R}) = 1$, $i_{\mathcal{R}}d\alpha = 0$.

Every differentiable function $H : \mathcal{C} \rightarrow \mathbb{R}$ has associated a vector field X^H , called the Hamiltonian vector field defined by the relations:

$$L_{X_H}\alpha = f\alpha, \quad f \in C^\infty(\mathcal{C}), \quad \alpha(X_H) = -H,$$

We can take local Darboux coordinates (q_i, p_j, z) such that:

$$\alpha = dz - \sum_{i=1}^n p_i dq_i.$$

Then $\mathcal{R} = \partial/\partial z$ and the vector field X^H has the expression:

$$(b) \quad X^H = \sum_{i=1}^n (p_i H_{p_i} - H) \frac{\partial}{\partial z} - \sum_{i=1}^n (p_i H_z + H_{q_i}) \frac{\partial}{\partial p_i} + \sum_{i=1}^n H_{p_i} \frac{\partial}{\partial q_i}$$

(see, for example, [9] or [1]). Instead of X^H we will consider here the reduced Hamiltonian vector field X_{red}^H ,

$$(c) \quad X_{red}^H = X^H + H\mathcal{R}$$

which, as we will see, has interesting geometric properties for us.

On the $(2n+1)$ -dimensional manifold $T^*Q \times \mathbb{R}$, where $\mathbb{R}$ is supposed to be coordinated by the variable z , on which we define the extended Liouville form:

$$\widehat{w}_Q = dz - w_Q.$$

The pair $(T^*Q \times \mathbb{R}, \widehat{w}_Q)$ is a contact manifold.

Given a function $H \in C^\infty(T^*Q) \hookrightarrow C^\infty(T^*Q \times \mathbb{R})$, from (b) and (c), we have

$$X_{red}^H = \sum_{i=1}^n p_i H_{p_i} \frac{\partial}{\partial z} - \sum_{i=1}^n H_{q_i} \frac{\partial}{\partial p_i} + \sum_{i=1}^n H_{p_i} \frac{\partial}{\partial q_i}.$$

Let us denote by j the inclusion $j : T^*M \hookrightarrow T^*M \times \mathbb{R} : p_q \mapsto (p_q, 0)$.

Theorem 4. *Let us consider the Hamilton equation on T^*Q*

$$i_{X_H}\Omega_Q = dH.$$

Let $F_1, \dots, F_{n-1}$ be functionally independent functions defined on an open domain $U \subset T^*Q$ verifying:

$$(d) \quad [H, F_i] = 0, \quad [F_i, F_j] = 0, \quad (1 \leq i, j \leq n-1).$$

Then, the exterior differentiable system on $U \times \mathbb{R} \subset T^*Q \times \mathbb{R}$

$$\mathcal{W} = \{\widehat{w}_Q, dH, dF_1, \dots, dF_{n-1}\}$$

is completely integrable. Accordingly, there is a local basis of exact differentials for $\mathcal{W}$ and hence there are functions $g_0, \dots, g_n, F_n$ defined on a local domain (which we denote in the same way) such that:

$$\widehat{w}_Q = g_0 dH + g_1 dF_1 + \dots + g_{n-1} dF_{n-1} + g_n F_n.$$

Proof. The argument relies on two key facts. The first one is that the orthogonal distribution $\mathcal{W}^\perp$ is (locally) generated by the vector fields:

$$(e) \quad X_{red}^H, X_{red}^{F_1}, \dots, X_{red}^{F_{n-1}}.$$

The second one is that both, the orthogonal and the characteristic distribution coincide, that is $\mathcal{W}^0 = \mathcal{W}_0$.

In this way, is it easy to see that:

$$(f) \quad \widehat{w}_Q(X_{red}^H) = 0, \quad \widehat{w}_Q(X_{red}^{F_i}) = 0, \quad (1 \leq i \leq n-1).$$

$$(g) \quad dF_i(X_{red}^H) = 0, \quad dF_i(X_{red}^{F_j}) = 0, \quad (1 \leq i, j \leq n-1).$$

Hence, as $\text{rank}(\mathcal{W}^0) = (2n+1) - \text{rank}(\mathcal{W}) = n$, we obtain that the local basis in (e) generate $\mathcal{W}^0$.

Having disposed of this first step, a brief computation gives us:

$$\begin{aligned} L_{X_{red}^{F_i}}(\widehat{w}_Q) &= dF_i, \quad (1 \leq i \leq n-1), \\ L_{X_{red}^H}(\widehat{w}_Q) &= dH. \end{aligned}$$

Thus, from (f) and (g) we obtain $\mathcal{W}^0 \subseteq \mathcal{W}_0$, which is precisely the assertion of the Theorem. $\square$

Corollary 5. *With the previous notations, the equations:*

$$H = a_0, \quad F_1 = a_1, \dots, \quad j^*(F_n) = a_n$$

for $a_0, \dots, a_n$ constants, determine an isotropic submanifold in $U \subset T^*Q$ on which H is constant.

Example 6. Let us consider the canonical coordinates $(q_1, q_2; p_1, p_2)$ on $T^*(\mathbb{R}^2)$. The extended Liouville form on $T^*(\mathbb{R}^2) \times \mathbb{R}$ is $\widehat{w} = p_1 dq_1 + p_2 dq_2 - dz$. Let us consider the Hamilton–Jacobi equation $i_{X_H} \Omega = dH$ with $H = pq$. Then

$$X_{red}^H = p_2 \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial q_2} + 2p_1 p_2 \frac{\partial}{\partial z}.$$

As a first integral of X_{red}^H is $F_1 = p_1$, we take the following exterior differential system on $T^*(\mathbb{R}^2) \times \mathbb{R}$:

$$\mathcal{W} = \{\widehat{w}, dH, dF_1\}.$$

Thus, there are functions F_2, g_0, g_1, g_2 such that:

$$\widehat{w} = g_0 dH + g_1 dF_1 + g_2 dF_2.$$

By restricting to $\{H = a, F_1 = b\}$ (for a, b constants), we get:

$$\widehat{w} = g_2 dF_2$$

In this way, as $\widehat{w} = bdq_1 + (a/b)dq_2 - dz = d(bq_1 + (a/b)q_2 - z)$ and $j^*(bq_1 + (a/b)q_2 - z) = bq_1 + (a/b)q_2$, we obtain that:

$$bq_1 + \frac{a}{b}q_2 = c, \quad p_1 = b, \quad p_1 p_2 = a$$

are the equations of an isotropic submanifold of $T^*(\mathbb{R}^2)$ on which H is constant.

3. Appendix: Liouville vs. Reeb Dynamics

The objective of this section is to indicate some research perspectives, which in parallel to the lines of this article, have gained a great recent boom and they share, as an essential starting point, some of the concepts used in this work.

In the previous section we have searched for isotropic submanifolds of T^*Q as solutions of $\{w_Q = 0\}$. This sufficient condition is not, of course, necessary. However, there are situations in which both conditions coincide.

Definition 7. We shall call Liouville vector field on T^*Q to the unique vector field satisfying:

$$i_{\Delta}\Omega_Q = -w_Q.$$

Proposition 8. We have, (i) $L_{\Delta}\Omega_Q = \Omega_Q$.

(ii) The Lie derivative, L_{Δ} preserve the set of Hamiltonian vector fields.

Proof. (i) is trivial. For (ii), if

$$i_{X_f}\Omega_Q = df,$$

then

$$\begin{aligned} i_{[\Delta, X_f]}\Omega_Q &= L_{\Delta}(i_{X_f}\Omega_Q) - i_{X_f}(L_{\Delta}\Omega_Q) \\ &= L_{\Delta}(df) - df \\ &= d(\Delta f - f). \end{aligned}$$

□

We call $M \subset T^*Q$ a Liouville submanifold if $\Delta \in \mathfrak{X}(M)$.

We can state

Proposition 9. A Liouville submanifold $N \subset T^*Q$ is isotropic if and only if $w_Q|_N = 0$.

Proposition 10. *Let M be an Liouville coisotropic submanifold of T^*Q of codimension $m \leq n - 1$. Then $\Delta \notin \mathfrak{X}(M)^\perp$. Moreover, if $f_1, \dots, f_m$ are local involutive generators of N then Δ is not a linear combination of the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_m}$.*

Proof. If $\Delta \in \mathfrak{X}(M)^\perp$ then $w_Q|_M = 0$ and M would be isotropic which contradicts that $\dim(M) \geq n + 1$. For the last statement it is enough to take into account that the orthogonal subspace $\mathfrak{X}(M)^\perp$ is generated by $X_{f_1}, \dots, X_{f_m}$ (see, by example, [8]). $\square$

Much beyond symplectic geometry, the concept of Liouville dynamics has attracted a lot of recent interest, paradoxically due to the great and rich variety of applications of contact geometry in Mathematical Physics (see, by example, [11, 12, 15]).

In particular, the consideration of invariant measures, has involved the construction of a powerful tool with interest not only in itself, but also in its applications (in integration, in the development of Hamiltonian Monte Carlo methods, in the construction of algorithms in statistical physics, etc.) dragging with it exact symplectic structures induced by the contact geometry in such a way that, surprisingly, the Liouville dynamics of this structures reproduce the primitive contact flow (see [2, 3]).

As an example of the confluence between the Reeb and the Liouville dynamics, let:

$$H : T^*Q \rightarrow \mathbb{R}$$

be a smooth function such that 0 is a regular value and let X_H be its Hamiltonian vector field,

$$i_{X_H}\Omega_Q = dH$$

Let us assume that Y is transverse to the submanifold $\mathcal{C} = H^{-1}(0)$ and that $\Delta(H) \neq 0$ on C . Then

$$C \hookrightarrow T^*Q$$

is a coisotropic submanifold and has a contact structure induced by the restriction the Liouville 1-form w_Q such that $T(C)^\perp$ is generated by the Reeb vector field ζ of $(C, w_Q|_C)$, which is given by the restriction to C of the vector field

$$\frac{X_H}{\Delta(H)}$$

(see [3, 13]).

Despite the strongly non-geometric character of many of the applications of the Liouville–Reeb dynamics, it is worth noting the new rise of the now classic Weinstein manifolds theory: that is, exact symplectic manifolds whose Liouville vector fields are complete and gradient-like for global smooth functions (see, for example, [16]).

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