Mediterr. J. Math. (2021) 18:212
https://doi.org/10.1007/s00009-021-01850-1
1660-5446/21/050001-18
published online September 7, 2021

Mediterranean Journal of Mathematics

Dynamics of Weighted Composition Operators on Spaces of Entire Functions of Exponential and Infraexponential Type

María J. Beltrán-Meneu and Enrique Jordá

Abstract. Given an affine symbol φ and a multiplier w, we focus on the weighted composition operator $C_{w,\varphi}$ acting on the spaces Exp and Exp^0 of entire functions of exponential and of infraexponential type, respectively. We characterize the continuity of the operator and, for w the product of a polynomial by an exponential function, we completely characterize power boundedness and (uniform) mean ergodicity. In the case of multiples of composition operators, we also obtain the spectrum and characterize hypercyclicity.

Mathematics Subject Classification. 30D15, 47A16, 47A35, 47B38.

Keywords. Weighted composition operator, spaces of entire functions of (infra) exponential type, power bounded operator, mean ergodic operator, hypercyclic operator.

1. Introduction and Outline of the Paper

The purpose of this paper was to study the dynamics of the weighted composition operator $C_{w,\varphi} : f \to w(f \circ \varphi)$ on the space Exp *of entire functions of exponential type* and on the space Exp^0 *of entire functions of infraexponential type*. The first space is formed by the entire functions which are of exponential type α for some $\alpha > 0$, endowed with its natural locally convex topology which makes it an (LB)-space, and the second one by all the entire functions which are of exponential type for each $\alpha > 0$, endowed with its natural locally convex topology which makes it a Fréchet space. Here we continue the research of the first author in [10], where the dynamics of the operator is studied on weighted Banach spaces of entire functions H_{v^α}, $H^0_{v^\alpha}$, defined by weights of exponential type $v^\alpha(z) = e^{-\alpha z}$, $\alpha > 0$, $z \in \mathbb{C}$. We refer to the next section for the precise notation and definitions.

In Sect. 3 we characterize the continuity of the operator when the symbol φ is an affine function, that is, when $\varphi(z) = az + b$, $a, b \in \mathbb{C}$, and we show it is never compact. In the setting of the Banach spaces H_{v^α} and $H^0_{v^\alpha}$, the operator $C_{w,\varphi}$ is not continuous if $|a| > 1$, or if $|a| = 1$ and the multiplier w is not constant [10, Theorem 8]. On the spaces Exp and Exp^0, for every $a \in \mathbb{C}$ we obtain continuity for multipliers of the form $w(z) = p(z)e^{\beta z}$, $\beta \in \mathbb{C}$, in the case of Exp and $w(z) = p(z)$ in the case of Exp^0, p being a polynomial. These weighted composition operators are *natural* in the following sense: if φ is an entire function and $w(z) = p(z)e^{\beta z}$ (resp. $w(z) = p(z)$), $\beta \in \mathbb{C}$, p a polynomial, then $C_{w,\varphi}$ is continuous in Exp (resp. in Exp^0) if and only if $\varphi(z) = az + b$, $a, b \in \mathbb{C}$. Moreover, if $\varphi(z) = az + b$, $a, b \in \mathbb{C}$ and $w(z) = p(z)e^{q(z)}$, p, q being polynomials, then $C_{w,\varphi}$ is continuous in Exp (resp. in Exp^0) if and only if $q(z) = \beta z$, $\beta \in \mathbb{C}$ (resp. $q \equiv 0$).

The most relevant results we present are given in Sect. 4. Since the pioneer work of Bonet and Domański [18], the study of ergodic properties of composition operators and weighted composition operators in Banach and Fréchet spaces of analytic functions has become a very active area of research in mathematical analysis, see [7,8,10,12,13,23,28]. For natural weighted composition operators we completely characterize when $C_{w,\varphi}$ is power bounded and (uniformly) mean ergodic on Exp and on Exp^0. Here, contrary to what happens in the Banach space $H^0_{v^\alpha}$, $\alpha > 0$ [10, Theorem 16 b)], power boundedness is equivalent to (uniform) mean ergodicity.

Theorem ME-*Exp*. Let $\varphi(z) = az + b$, $a, b \in \mathbb{C}$ and $w(z) = p(z)e^{\beta z}$, p being a polynomial and $\beta \in \mathbb{C}$. The operator $C_{w,\varphi}$ is (uniformly) mean ergodic on Exp if and only if it is power bounded if and only if one of the following conditions occurs:

(i) $|a| < 1$ and $\left|w\left(\frac{b}{1-a}\right)\right| \le 1$.
(ii) $|a| = 1$, $a \ne 1$, $w(z) = \lambda e^{\beta z}$, $|\lambda| \le \left|e^{-\beta \frac{b}{1-a}}\right|$, $\beta \in \mathbb{C}$.
(iii) $a = 1$, $b = 0$, $w(z) \equiv \lambda$, $|\lambda| \le 1$ (multiplication operator case).

Theorem ME-*Exp*$^{\mathbf{0}}$. Let $\varphi(z) = az + b$, $a, b \in \mathbb{C}$ and let $w(z)$ be a polynomial. The operator $C_{w,\varphi}$ is mean ergodic on Exp^0 if and only if it is power bounded and if and only if one of the following conditions occur:

(i) $|a| < 1$ and $\left|w\left(\frac{b}{1-a}\right)\right| \le 1$.
(ii) $|a| = 1$, $a \ne 1$, $w(z) \equiv \lambda$, $|\lambda| \le 1$.
(iii) $a = 1$, $b = 0$, $w(z) \equiv \lambda$, $|\lambda| \le 1$ (multiplication operator case).
(iv) $a = 1$, $b \ne 0$, $w(z) \equiv \lambda$, $|\lambda| < 1$.

From our results it follows that $C_{w,\varphi}$ is (uniformly) mean ergodic on Exp (Exp^0) if and only if there exists $\alpha_0 > 0$ such that $C_{w,\varphi} : H^0_{v^\alpha} \to H^0_{v^\alpha}$ is power bounded and mean ergodic for each $\alpha > \alpha_0$ ($\alpha > 0$) except in the case of Theorem ME-*Exp* (ii), where if $\beta \ne 0$, there is no $\alpha > 0$ such that $C_{w,\varphi}(H^0_{v^\alpha}) \subseteq H^0_{v^\alpha}$, as w is not constant (see [10, Theorem 8]). We point out that the theorems in Sect. 4 which are valid for symbols $\varphi(z) = az + b$, $a \ne 1$, are stated only for $\varphi(z) = az$, since the general case follows immediately from this reduction.

In Sect. 5 we focus our study on the case when w is a constant function, that is, on multiples of composition operators. In this case we give a complete description of the spectrum, and also completely characterize the hyperciclicity of the operator on Exp and on Exp^0. More precisely, we get that if $\varphi = az + b$ with $a \neq 1$, then λC_φ cannot be hypercyclic (even weakly supercyclic). In fact, this is satisfied for general weighted composition operators. For multiples of the translation operator we get the following characterization:

Theorem Hyp. The weighted translation operator $\lambda T_b : f(z) \to \lambda f(z + b)$, $\lambda, b \in \mathbb{C}$, $b \neq 0$, satisfies:

(i) It is hypercyclic, topologically mixing and chaotic on Exp for every $\lambda \in \mathbb{C}$.

(ii) It is topologically transitive on Exp^0 if and only if $|\lambda| = 1$. In this case, it is hypercyclic and topologically mixing.

Finally, we include an appendix where we improve some results of [10] for weighted composition operators defined on Banach spaces.

2. Notation and Preliminaries

Our notation is standard. We denote by $H(\mathbb{C})$ the space of entire functions endowed with the compact open topology τ_{co} of uniform convergence on the compact subsets of $\mathbb{C}$, and by $\mathbb{D}$ the open unit disc centered at zero. Given two entire functions w and φ, the *weighted composition operator* $C_{w,\varphi}$ on $H(\mathbb{C})$ is defined by

$$C_{w,\varphi}(f) = w(f \circ \varphi), \ \ f \in H(\mathbb{C}).$$

The function φ is called *symbol* and w is called *multiplier*. $C_{w,\varphi}$ combines the composition operator $C_\varphi : f \mapsto f \circ \varphi$ with the pointwise multiplication operator $M_w : f \mapsto w \cdot f$.

We say that $v : \mathbb{C} \to]0, \infty[$ is a *weight* if it is continuous, decreasing and radial, that is, $v(z) = v(|z|)$ for every $z \in \mathbb{C}$. It is *rapidly decreasing* if $\lim\limits_{r\to\infty} r^k v(r) = 0$ for all $k \in \mathbb{N}$.

For an arbitrary weight v on $\mathbb{C}$, the weighted Banach spaces of entire functions with O- and o-growth conditions are defined as

$$H_v = \left\{ f \in H(\mathbb{C}) : \|f\|_v := \sup_{z\in\mathbb{C}} v(z)|f(z)| < \infty \right\},$$
$$H_v^0 = \left\{ f \in H(\mathbb{C}) : \lim_{|z|\to\infty} v(z)|f(z)| = 0 \right\}.$$

$(H_v, \| \ \|_v)$ and $(H_v^0, \| \ \|_v)$ are Banach spaces, and $(H_v^0, \| \ \|_v) \hookrightarrow (H_v, \| \ \|_v) \hookrightarrow (H(\mathbb{C}), \tau_{co})$ with continuous inclusions. If we assume v is rapidly decreasing, then H_v^0 and H_v contain the polynomials. We denote by B_v and B_v^0 the closed unit balls of H_v and H_v^0, respectively. B_v is compact with respect to τ_{co}.

Given the exponential weight $v(z) = e^{-|z|}$, $z \in \mathbb{C}$, consider the decreasing sequence $(v^n)_n$ and the increasing sequence of weights $(v^{1/n})_n$. The inclusions $H_{v^n} \hookrightarrow H^0_{v^{n+1}}$ and $H_{v^{1/(n+1)}} \hookrightarrow H^0_{v^{1/n}}$, $n \in \mathbb{N}$, are compact. We are interested in the *space of entire functions of exponential type*

$$Exp := ind_{n\in\mathbb{N}} H_{v^n} = ind_{n\in\mathbb{N}} H^0_{v^n} = \{f \in H(\mathbb{C}) : \ \exists n \in \mathbb{N} \text{ such that } f \in H^0_{v^n}\}$$

and *of infraexponential type*

$$\begin{aligned} Exp^0 &:= proj_{n\in\mathbb{N}} H_{v^{1/n}} = proj_{n\in\mathbb{N}} H^0_{v^{1/n}} \\ &= \{f \in H(\mathbb{C}) : f \in H^0_{v^{1/n}} \quad \text{for every } n \in \mathbb{N}\}. \end{aligned}$$

The space Exp is an inductive limit of Banach spaces, that is, the increasing union of the spaces H_{v^n} with the strongest locally convex topology for which all the injections $H_{v^n} \hookrightarrow Exp$, $n \in \mathbb{N}$, become continuous. It is a (DFN)-algebra [24]. The space Exp^0 is a projective limit of Banach spaces, that is, the decreasing intersection of the spaces $H_{v^{1/n}}$, $n \in \mathbb{N}$, whose topology is defined by the sequence of norms $\| \cdot \|_{v^{1/n}}$. It is a nuclear Fréchet algebra [25]. Clearly $Exp^0 \subset Exp$ and the polynomials are contained and dense in both spaces, so Exp^0 is dense in Exp. The inductive limit Exp is *boundedly retractive*, that is, for every bounded subset B of Exp, there exists $n \in \mathbb{N}$ such that B is bounded in $H^0_{v^n}$ and the topologies of Exp and $H^0_{v^n}$ coincide on B. This is a stronger condition than being *regular*, i.e. for every bounded subset B of Exp, there exists $n \in \mathbb{N}$ such that B is bounded in $H^0_{v^n}$. Weighted algebras of entire functions have been considered by many authors; see, e.g. [14,16,24,25,29] and the references therein.

Our notation for locally convex spaces and functional analysis is standard [26]. For a locally convex space E, cs(E) denotes a system of continuous seminorms determining the topology of E and the space of all continuous linear operators on E is denoted by $L(E)$. Given $T \in L(E)$ we say that $x_0 \in E$ is a *fixed point* of T if $T(x_0) = x_0$, and that it is *periodic* if there exists $n \in \mathbb{N}$ such that $T^n(x_0) = x_0$, where $T^n := T \circ \overset{n)}{\cdots} \circ T$. A continuous linear operator T from a locally convex space E into itself is called *hypercyclic* if there is a vector x (which is called a hypercyclic vector) in E such that its orbit $(x, Tx, T^2x, \dots)$ is dense in E. Every hypercyclic operator T on E is *topologically transitive* in the sense of dynamical systems, that is, for every pair of non-empty open subsets U and V of E there is n such that $T^n(U)$ meets V. The operator T is *topologically mixing* if for every pair of non-empty open subsets U and V of E there is n_0 such that for each $n \geq n_0$, $T^n(U)$ meets V. T is *chaotic* if it is topologically transitive and has a dense set of periodic points.

An operator $T \in L(E)$ is said to be *power bounded* if $(T^n)_n$ is an equicontinuous set of $L(E)$. The spaces we are considering are barrelled; therefore, an operator $T \in L(E)$ is power bounded if and only if, for each $x \in E$, its orbit $(x, Tx, T^2x, \dots)$ is bounded in E. T is called *mean ergodic* if the Cesàro means $(T_{[n]}x)_n$, $T_{[n]}x := \frac{1}{n}\sum_{j=1}^n T^j x$, $x \in E$, converge in E. If the sequence of the Cesàro means of the iterates of T converges in $L(E)$ endowed with the topology τ_b of uniform convergence on bounded sets, the operator T is called *uniformly mean ergodic*. Mean ergodic operators in barrelled locally convex spaces have been considered, e.g. in [1–6,11]. As Exp and Exp^0 are Montel spaces, according to [2, Proposition 2.4], they are uniformly mean ergodic, that is, each power bounded operator on them is automatically uniformly mean ergodic. Clearly, if T is mean ergodic, then $(T^n x/n)_n$ converges to 0 for each $x \in E$, and if it is uniformly mean ergodic, $(T^n/n)_n$ converges to 0 in τ_b.

For a good exposition of ergodic theory we refer the reader to the monograph [27], and for the subject of linear dynamics, to the monographs by Bayart and Matheron [9] and by Grosse-Erdmann and Peris [21].

For $T \in L(E)$, the *resolvent set* $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $(\lambda I - T)^{-1}$ exists in $L(E)$. The set $\sigma(T) := \mathbb{C}\backslash\rho(T)$ is called the *spectrum* of T. The *point spectrum* $\sigma_p(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not injective. If we need to stress the space E, then we write $\sigma(T, E)$ and $\sigma_p(T, E)$.

3. Continuity and Compactness

An operator $T : E \to E$ on a locally convex space E is said to be compact (resp. bounded) if there exists a 0-neighbourhood U in E such that $T(U)$ is a relatively compact (resp. bounded) subset of E. Every bounded operator is continuous. If the bounded subsets of E are relatively compact, as it happens in Exp and Exp^0, then bounded and compact operators coincide. T is said to be Montel if it maps bounded sets into relatively compact sets. So, every continuous operator on Exp and on Exp^0 is Montel. If E is a Banach space, T is bounded (resp. Montel) if and only if it is continuous (resp. compact). This is not satisfied on the spaces under consideration.

In order to study the continuity and compactness, first we need the following lemmata for inductive and projective limits of Banach spaces, respectively:

Lemma 1 [6, Lemma 4.1]. *Let $E = ind_n E_n$ and $F = ind_m F_m$ be two (LB)-spaces which are increasing unions of Banach spaces $E = \cup_n E_n$ and $F = \cup_m F_m$. Let $T : E \to F$ be a linear map.*

- (i) *T is continuous if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T(E_n) \subseteq F_m$ and the restriction $T_{n,m} : E_n \to F_m$ is continuous.*
- (ii) *Assume that F is a regular (LB)-space. Then T is bounded if and only if there exists $m \in \mathbb{N}$ such that $T(E_n) \subseteq F_m$ and $T : E_n \to F_m$ is continuous for all $n \in \mathbb{N}$.*

Lemma 2 [5, Lemma 25]. *Let $E := proj_m E_m$ and $F := proj_n F_n$ be Fréchet spaces such that $E = \cap_m E_m$ with each $(E_m, \| \|_m)$ a Banach space and $F = \cap_n F_n$ with each $(F_n, \| \|_n)$ a Banach space. Moreover, assume E is dense in E_m and that $E_{m+1} \subseteq E_m$ with a continuous inclusion for each $m \in \mathbb{N}$ (resp. $F_{n+1} \subseteq F_n$ with a continuous inclusion for each $n \in \mathbb{N}$). Let $T : E \to F$ be a linear operator.*

- (i) *T is continuous if and only if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that T has a unique continuous linear extension $T_{m,n} : E_m \to F_n$.*

(ii) *Assume that T is continuous. Then T is bounded if and only if there exists $m \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, the operator T has a unique continuous linear extension $T_{m,n} : E_m \to F_n$.*

We are interested in the study of the dynamics of $C_{w,\varphi}$ when the multiplier is of the type $w(z) = p_N(z)e^{\beta z}$, p_N a polynomial of degree N, and $\beta \in \mathbb{C}$. The next result (see [10, Proposition 5] and [19, Proposition 3.1]) yields that for such multipliers, in order to have the continuity of $C_{w,\varphi}$ we must reduce to affine symbols, i.e. symbols of the form $\varphi(z) = az+b$ for some $a, b \in \mathbb{C}$.

Proposition 3. *Consider φ and w entire functions such that $C_{w,\varphi} : H_{v^\alpha} \to H_{v^\gamma}$, $0 < \alpha < \gamma$, is continuous. If there exists $\lambda > 1$ such that $\lim_{|z|\to\infty} |w(z)| e^{|z|(\lambda\alpha-\gamma)} = \infty$, then φ must be affine. In particular, this happens when considering the multiplier $w(z) = p_N(z)e^{\beta z}$, $\beta \in \mathbb{C}$ in the case of Exp, and $w(z) = p_N(z)$ in the case of Exp^0.*

Accordingly, in the rest of the paper we only consider affine symbols. In the next proposition we obtain that, unlike what happens when the operator acts on the Banach space H_{v^α}, $\alpha > 0$, the operator can be continuous on Exp and Exp^0 if $|a| > 1$ or if $|a| = 1$ and the multiplier w is not constant (see [10, Theorem 8]).

Proposition 4. *For $\varphi(z) = az + b$, $a, b \in \mathbb{C}$, the operator $C_{w,\varphi}$ is continuous on Exp (resp. Exp^0) if and only if $w \in Exp$ (resp. $w \in Exp^0$).*

Proof. As $C_{w,\varphi}(1) = w$, it is trivial that w must belong to the corresponding space if the operator is well defined. Let us see the converse. Assume first $w \in Exp$. Then there exists $s \in \mathbb{N}$ such that $w \in H^0_{v^s}$. Observe that given $n \in \mathbb{N}$, we can find $m \in \mathbb{N}$, $m > n|a| + s$ such that

$$\begin{aligned}\lim_{|z|\to\infty} |w(z)|e^{n|az+b|-m|z|} &\leq \lim_{|z|\to\infty} |w(z)|e^{|z|(n|a|-m)}e^{n|b|}\\ &\leq \lim_{|z|\to\infty} |w(z)|e^{-s|z|}e^{n|b|} = 0.\end{aligned}$$

The continuity follows now by Lemma 1(i) and [10, Lemma 1]. By [10, Lemma 3], we also have that $C_{w,\varphi} : H^0_{v_n} \to H^0_{v_m}$ is compact.

Assume now $w \in Exp^0$. Given $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $s \in \mathbb{N}$ such that $0 < \frac{1}{s} < \frac{1}{n} - \frac{1}{m}|a|$. Therefore,

$$\begin{aligned}\lim_{|z|\to\infty} |w(z)|e^{\frac{1}{m}|az+b|-\frac{1}{n}|z|} &\leq \lim_{|z|\to\infty} |w(z)|e^{-|z|(\frac{1}{n}-\frac{1}{m}|a|)}e^{\frac{|b|}{m}}\\ &\leq \lim_{|z|\to\infty} |w(z)|e^{-\frac{1}{s}|z|}e^{\frac{|b|}{m}} = 0.\end{aligned}$$

The continuity follows now by Lemma 2(i) and [10, Lemma 1]. By [10, Lemma 3], we get that $C_{w,\varphi} : H^0_{v^{1/m}} \to H^0_{v^{1/n}}$ is compact. □

Proposition 5. *For $\varphi(z) = az+b$, $a, b \in \mathbb{C}$, $w \neq 0$, the operator $C_{w,\varphi}$ is never compact on Exp and on Exp^0.*

Proof. Let us study first the case $C_{w,\varphi} : Exp \to Exp$. Assume the operator is compact, i.e. bounded. By Lemma 1(ii) and [10, Lemma 1], there exists $m \in \mathbb{N}$ such that $\sup_{z\in\mathbb{C}} |w(z)|e^{n|az+b|-m|z|} < \infty$ for all $n \in \mathbb{N}$. So, there exists $C_n > 0$ such that, for every $z \in \mathbb{C}$,

$$|w(z)| \le C_n e^{m|z|-n|az+b|} \le C_n e^{n|b|} e^{|z|(m-n|a|)}.$$

Thus, taking $n > m/|a|$ we get that the entire function w converges to 0 as $|z| \to \infty$, a contradiction if $w \not\equiv 0$.

The case $C_{w,\varphi} : Exp^0 \to Exp^0$ is analogous using Lemma 2(ii) and [10, Lemma 1]. □

In the next result we characterize the continuity of the operator for a type of multipliers.

Proposition 6. *Consider $\varphi(z) = az + b$, $a, b \in \mathbb{C}$ and $w(z) = p_N(z)e^{q_M(z)}$ with p_N and q_M polynomials of degrees N, $M \in \mathbb{N}_0$, respectively.*

(i) *$C_{w,\varphi} : Exp \to Exp$ is continuous if and only if $M \le 1$, that is, if $w(z) = p_N(z)e^{\beta z}$ for some $\beta \in \mathbb{C}$.*

(ii) *$C_{w,\varphi} : Exp^0 \to Exp^0$ is continuous if and only if $w(z) = p_N(z)$.*

Proof. By Proposition 4, $C_{w,\varphi}$ will be continuous if and only if w belongs to the corresponding space. Given $p_N(z) = \sum_{j=0}^N a_j z^j$, $a_j \in \mathbb{C}$, and $q_M(z) = \sum_{j=0}^M b_j z^j$, $b_j \in \mathbb{C}$, consider $\widetilde{p}_N(z) = \sum_{j=0}^N |a_j| z^j$ and $\widetilde{q}_M(z) = \sum_{j=0}^M |b_j| z^j$. For $z \in \mathbb{C}$ and $\alpha > 0$ we have

$$|w(z)|e^{-\alpha|z|} = |p_N(z)e^{q_M(z)}|e^{-\alpha|z|} \le \widetilde{p}_N(|z|)e^{\widetilde{q}_M(|z|)-\alpha|z|}.$$

So, if $M \le 1$ there exists $n \in \mathbb{N}$ such that $w \in H_{v^n} \subseteq Exp$. If $M = 0$, $w \in H_{v^{1/n}}$ for every $n \in \mathbb{N}$, i.e. $w \in Exp^0$, and the continuity holds. Conversely, if $M \ge 2$ observe that $w \notin H_{v^\alpha}$ for each $\alpha > 0$, hence the operator can not be continuous on Exp neither on Exp^0. In the case of Exp^0, if $q_M(z) = b_1 z$, $b_1 \ne 0$, we can find $c \in \mathbb{C}$, $|c| = 1$, and $n \in \mathbb{N}$ such that

$$\sup_{z\in\mathbb{C}} |w(z)|e^{-\frac{1}{n}|z|} = \sup_{z\in\mathbb{C}} |p_N(z)e^{b_1 z}|e^{-\frac{1}{n}|z|} \ge \sup_{r\ge 0} |p_N(cr)|e^{r(|b_1|-\frac{1}{n})} = \infty.$$

Thus, as $w \notin H_{v^{1/n}}$ the operator $C_{w,\varphi}$ can not be continuous on Exp^0. □

4. Power Boundedness and Mean Ergodicity

In this section, consider $w \in Exp$ (resp. $w \in Exp^0$) and $\varphi(z) = az + b$, $a, b \in \mathbb{C}$. We have seen that $C_{w,\varphi} : Exp \to Exp$ (resp. $C_{w,\varphi} : Exp^0 \to Exp^0$) is continuous. Its iterates have the following expression:

$$C_{w,\varphi}^k f = w_{[k]}(f \circ \varphi^k),\ k \in \mathbb{N},\ f \in Exp\ (Exp^0),$$

where $w_{[k]}(z) := \prod_{j=0}^{k-1} w(\varphi^j(z))$, $z \in \mathbb{C}$. Observe that the symbol φ has a fixed point $z_0 = \frac{b}{1-a}$ if and only if $a \ne 1$, and for $k \in \mathbb{N}$, we get

$$\varphi^k(z) = \begin{cases} a^k z + b\frac{1-a^k}{1-a} = a^k(z - \frac{b}{1-a}) + \frac{b}{1-a} & \text{if } a \ne 1 \\ z + bk & \text{if } a = 1. \end{cases}$$

If we take $f \equiv 1$, we get a necessary condition for the operator to be power bounded or mean ergodic. Recall that Exp is boundedly retractive, i.e. each convergent sequence is convergent in some Banach space of the inductive limit.

Proposition 7. (i) *If $C_{w,\varphi} : Exp \to Exp$ is power bounded (resp. mean ergodic), then there exists $m \in \mathbb{N}$ such that $\sup_k \|w_{[k]}\|_{v^m} < \infty$ (resp. $\lim_k \frac{\|w_{[k]}\|_{v^m}}{k} = 0$).*
(ii) *If $C_{w,\varphi} : Exp^0 \to Exp^0$ is power bounded (resp. mean ergodic), then $\sup_k \|w_{[k]}\|_{v^{1/m}} < \infty$ (resp. $\lim_k \frac{\|w_{[k]}\|_{v^{1/m}}}{k} = 0$) for every $m \in \mathbb{N}$.*

In the next result we prove that the necessary condition for power boundedness in Proposition 7 is also sufficient when C_φ is power bounded on each H_{v^α}, $\alpha > 0$. This situation occurs for $|a| \le 1$, $a \ne 1$ (see [10, Theorem 22]), and obviously when $a = 1$ and $b = 0$. However, in general it is not sufficient. For instance, in Corollary 21 we obtain that the composition operator C_φ is never mean ergodic on Exp and on Exp^0 if $|a| > 1$ or if $a = 1$ and $b \ne 0$.

Proposition 8. *Let $|a| \le 1$, $a \ne 1$ or $a = 1$, $b = 0$. The following is satisfied:*
(i) *$C_{w,\varphi} : Exp \to Exp$ is power bounded if and only if there exists $m \in \mathbb{N}$ such that $\sup_k \|w_{[k]}\|_{v^m} < \infty$.*
(ii) *$C_{w,\varphi} : Exp^0 \to Exp^0$ is power bounded if and only if $\sup_k \|w_{[k]}\|_{v^{1/n}} < \infty$ for every $n \in \mathbb{N}$.*

Proof. Under the hypothesis we are considering, $C_\varphi : H_{v^\alpha} \to H_{v^\alpha}$ is power bounded for every $\alpha > 0$ (see [10, Theorem 22] for the non trivial case $|a| \le 1$, $a \ne 1$, $b \ne 0$).

(i) Assume there exist $m \in \mathbb{N}$ and $C > 0$ such that $\|w_{[k]}\|_{v^m} \le C$ for every $k \in \mathbb{N}$. Given $f \in H_{v^n}$, we get

$$\|C^k_{w,\varphi} f\|_{v^{n+m}} = \sup_{z\in\mathbb{C}} \prod_{j=0}^{k-1} |w(\varphi^j(z))| e^{-m|z|} |f(\varphi^k(z))| e^{-n|z|} \le C \|C^k_\varphi f\|_{v^n}$$

for every $k \in \mathbb{N}$, so the power boundedness holds. The converse follows by Proposition 7. For (ii) proceed analogously in order to get that for every $f \in Exp^0$ and every $n \in \mathbb{N}$ there exists $D > 0$ such that $\|C^k_{w,\varphi} f\|_{v^{\frac{1}{n}}} \le D \|f\|_{v^{\frac{1}{2n}}}$ for every $k \in \mathbb{N}$. □

Since each non constant entire function w is unbounded, it is satisfied that there exists $a \in \mathbb{C}$ such that the sequence $\left(\frac{|w(a)|^k}{k}\right)_k$ is unbounded. As an immediate consequence, the next result on multiplication operators is satisfied as follows:

Corollary 9. *Let $w \in Exp$ (resp. Exp^0). The multiplication operator $M_w : f \mapsto w \cdot f$ is power bounded and (uniformly) mean ergodic on Exp (resp. Exp^0) if and only if $w \equiv \lambda$ with $|\lambda| \le 1$.*

In the rest of the paper, for $a \neq 1$ we can assume without loss of generality that $\varphi(z) = az$. Indeed, in the next lemma we prove that for $a \neq 1$, every weighted composition operator is conjugated to another one with symbol $\varphi(z) = az$. So, the study of power boundedness and (uniform) mean ergodicity can be reduced to this case.

Lemma 10. *If $a \neq 1$, $b \in \mathbb{C}$ and $X = Exp$ or $X = Exp^0$, the dynamical systems $C_{w,\varphi} : X \to X$, $f \mapsto w(z)f(az+b)$ and $C_{w\circ\tau,\varphi_a} : X \to X$, $f \mapsto w(z-\frac{b}{1-a})f(az)$, where $\tau(z) = z-\frac{b}{1-a}$ and $\varphi_a(z) = az$, $z \in \mathbb{C}$, are conjugated.*

Proof. It is easy to see that the conjugacy holds through the homeomorphism $C_\tau : X \to X$, $f(z) \mapsto f(z - \frac{b}{1-a})$. Indeed, $C_\tau^{-1} \circ C_{w\circ\tau,\varphi_a} \circ C_\tau = C_{w,\varphi}$. □

Proposition 11. *Consider $\varphi(z) = az$, $a \in \mathbb{C}$, if $a \neq 1$ and $\varphi(z) = z + b$, $b \in \mathbb{C}\backslash\{0\}$ otherwise. The operator $C_{w,\varphi}$ is not mean ergodic and thus not power bounded on Exp and on Exp^0 if any of the following conditions is satisfied:*

(i) *$a \neq 1$ and $|w(0)| > 1$.*
(ii) *$a = 1$ and there exist $n_0 \in \mathbb{N}$ and $C > 1$ such that $|w(jb)| > C$ for every $j \in \mathbb{N}, j \geq n_0$.*
(iii) *$|a| > 1$ and there exist $n_0 \in \mathbb{N}$ and $C > 0$ satisfying $|w(a^j)| > C$ for every $j \in \mathbb{N}, j \geq n_0$.*
(iv) *$a = 1$ or $|a| > 1$ and there exists an increasing subsequence $(k_s)_s \subseteq \mathbb{N}$ satisfying $\prod_{j=0}^{k_s-1} |w(\varphi^j(z_0))| > C$ for some $z_0 \in \mathbb{C}$, $C > 0$, and every $s \in \mathbb{N}$.*

Proof. (i) and (ii) follow by Proposition 7, as $\|w_{[k]}\|_{v^\alpha} \geq \prod_{j=0}^{k-1} \left|w\left(\varphi^j(z_0)\right)\right| e^{-\alpha|z_0|}$ tends to infinity for $z_0 = 0$ and $z_0 = n_0 b$, respectively.

(iii) Take $M \in \mathbb{N}$ such that $|a|^M > 1/C$ and consider $f(z) = z^M$. Evaluating at the point $z_0 = a^{n_0}$, for every $\alpha > 0$ we get

$$\frac{\|C_{w,\varphi}^k f\|_{v^\alpha}}{k} \geq \frac{1}{k}\prod_{j=0}^{k-1} |w(a^{j+n_0})||f(a^{k+n_0})|e^{-\alpha|z_0|} > |a|^{n_0 M} e^{-\alpha|z_0|}\frac{(C|a|^M)^k}{k}.$$

So, the operator can be mean ergodic neither on Exp, nor on Exp^0.

(iv) For $f(z) = z$ and $\alpha > 0$ we get

$$\frac{\|C_{w,\varphi}^{k_s} f\|_{v^\alpha}}{k_s} \geq \frac{1}{k_s}\prod_{j=0}^{k_s-1} |w(\varphi^j(z_0))||\varphi^{k_s}(z_0)|e^{-\alpha|z_0|} > Ce^{-\alpha|z_0|}\frac{|\varphi^{k_s}(z_0)|}{k_s}.$$

As $\varphi^{k_s}(z_0) = z_0 a^{k_s}$ if $|a| > 1$ and $\varphi^{k_s}(z_0) = z_0 + k_s b$ if $a = 1$, the conclusion holds.

□

We have seen above that a necessary condition for the power boundedness and mean ergodicity of $C_{w,\varphi}$ is that the products of the iterates of the composition operator applied on the multiplier are bounded. In what follows, we consider multipliers of the form $w(z) = p_N(z)e^{\beta z}$, $N \in \mathbb{N}_0$, $\beta \in \mathbb{C}$ when considering the space Exp and $w(z) = p_N(z)$, $N \in \mathbb{N}_0$, when considering

Exp^0, where p_N is a polynomial of degree N. For $\alpha > 0$ we get the following: If $a \neq 1$,

$$\|w_{[k]}\|_{v^\alpha} = \sup_{z\in\mathbb{C}} \left(\prod_{j=0}^{k-1} |p_N(\varphi^j(z))| \right) |e^{\beta \frac{1-a^k}{1-a} z}| e^{-\alpha|z|} \tag{4.1}$$

If $a = 1$, $b \neq 0$,

$$\|w_{[k]}\|_{v^\alpha} = |e^{\beta b(k-1)/2}|^k \sup_{z\in\mathbb{C}} \left(\prod_{j=0}^{k-1} |p_N(z + jb)| \right) |e^{\beta k z}| e^{-\alpha|z|} \tag{4.2}$$

4.1. Case $|a| < 1$

Lemma 12. *Let $s, t > 0$ and $f_s(x) := x^s e^{-tx}$, $x \in [0, \infty)$. The maximum of f_s is attained at $x_s = s/t$ and $f_s(x_s) = (\frac{s}{et})^s$ satisfies that $\lim_{s\to\infty} x_s = \infty$, $\lim_{s\to\infty} M^s f_s(x_s) = \infty$ and $\lim_{s\to\infty} a^{s^2} M^s f_s(x_s) = 0$ for each $0 < a < 1$ and $M > 0$.*

Theorem 13. *Let $\varphi(z) = az, |a| < 1$. If $w(z) = p_N(z)e^{\beta z}$, $N \in \mathbb{N}_0$, $\beta \in \mathbb{C}$ when considering $C_{w,\varphi} : Exp \to Exp$ and $\beta = 0$ for $C_{w,\varphi} : Exp^0 \to Exp^0$, then $C_{w,\varphi}$ is power bounded and (uniformly) mean ergodic if and only if $|w(0)| \leq 1$.*

Proof. If $|w(0)| > 1$, the operator cannot be mean ergodic by Proposition 11(i).

If $|w(0)| \leq 1$, [13, Theorem 3.10(i)] yields that for $r > 0$, there exists $C > 1$ such that, for every $|z| \leq r$ and $k \in \mathbb{N}$, then $\prod_{j=0}^{k} |w(a^j z)| < C$. On the other hand, observe that there exists $M > 0$ such that $|p_N(z)| \leq M|z|^N$ for every $|z| \geq r$. Fix $z_0 \in \mathbb{C}$ and take $j_0 \in \mathbb{N}$ such that $|a^j z_0| \leq r$ for every $j \in \mathbb{N}, j \geq j_0$. Therefore, for $k \geq j_0$ we have

$$\begin{aligned}\prod_{j=0}^{k-1} |p_N(a^j z_0)e^{\beta a^j z_0}| &\leq \prod_{j=0}^{j_0-1} M \left| a^j z_0 \right|^N e^{|\beta||z_0| \frac{1-|a|^{j_0}}{1-|a|}} \prod_{j=j_0}^{k} |w(a^j z_0)| \\ &\leq C M^{j_0} |z_0|^{N j_0} |a|^{N j_0 (j_0-1)/2} e^{\frac{|\beta||z_0|}{1-|a|}}.\end{aligned}$$

Consider $\alpha > \frac{|\beta|}{1-|a|}$ and $k \in \mathbb{N}$. For $s = \min(j_0, k)$, we get the following:

$$\prod_{j=0}^{k-1} |w(a^j z_0)| e^{-\alpha|z_0|} \leq C M^s |a|^{Ns(s-1)/2} \sup_{z\in\mathbb{C}} |z|^{Ns} e^{-(\alpha - \frac{|\beta|}{1-|a|})|z|}$$

By Lemma 12, there exists $D > 0$, independent of z_0, such that

$$\prod_{j=0}^{k-1} |w(a^j z_0)| e^{-\alpha|z_0|} \leq D$$

for every $k \in \mathbb{N}$. So, the conclusion holds by Proposition 8. $\square$

4.2. Case $|a| = 1$, $a \neq 1$

Theorem 14. *Given $\varphi(z) = az, |a| = 1,\ a \neq 1,$ we get the following:*

(i) *If $w(z) = p_N(z)e^{\beta z}$, $N \neq 0$, $\beta \in \mathbb{C}$ when considering $C_{w,\varphi} : Exp \to Exp$ and $\beta = 0$ for $C_{w,\varphi} : Exp^0 \to Exp^0$, the operator is not mean ergodic, hence not power bounded.*

(ii) *If $w(z) = \lambda e^{\beta z}$, $\lambda, \beta \in \mathbb{C}$ when considering $C_{w,\varphi} : Exp \to Exp$ and $\beta = 0$ for $C_{w,\varphi} : Exp^0 \to Exp^0$, the operator is power bounded, thus uniformly mean ergodic if and only if $|\lambda| \leq 1$.*

Proof. (i) Observe that there exist $R > 0$ and $M > 0$ such that $|p_N(z)| > M|z|^N$ for every $|z| > R$. If $N \neq 0$ and $k \in \mathbb{N}$, then

$$\|w_{[k]}\|_{v^\alpha} > M^k \sup_{|z|>R} |z|^{Nk}|e^{\beta \frac{1-a^k}{1-a} z}|e^{-\alpha|z|} \geq M^k \sup_{r>R} r^{Nk} e^{-(\alpha - |\beta| \frac{|1-a^k|}{|1-a|})r}$$
$$\geq M^k \sup_{r>R} r^{Nk} e^{-\alpha r}.$$

By Lemma 12, if k is big enough, then

$$\sup_{r>R} r^{Nk} e^{-\alpha r} = \sup_{r>0} r^{Nk} e^{-\alpha r}.$$

Thus, the operator cannot be mean ergodic and power bounded by applying again Lemma 12 and Proposition 8.

(ii) The case $|\lambda| > 1$ follows by Proposition 11(i), since $|w(0)| > 1$. If $|\lambda| \leq 1$, for every $\alpha \geq \frac{2|\beta|}{|1-a|}$ we get

$$\|w_{[k]}\|_{v^\alpha} \leq \sup_{z\in\mathbb{C}} e^{-(\alpha - \frac{2|\beta|}{|1-a|})|z|} = 1.$$

Now, the conclusion follows by Proposition 8.

□

4.3. Case $a = 1$

Theorem 15. *Let $\varphi(z) = z + b,\ b \in \mathbb{C}\backslash\{0\}$.*

(i) *If $w(z) = p_N(z)e^{\beta z}$, $\beta \in \mathbb{C}$, $N \in \mathbb{N}_0$, then the operator $C_{w,\varphi}$ is not mean ergodic and thus not power bounded on Exp.*

(ii) *If $w(z) = p_N(z)$, $N \in \mathbb{N}_0$, the operator $C_{w,\varphi}$ is power bounded and mean ergodic on Exp^0 if and only if $w \equiv \lambda$, $|\lambda| < 1$.*

Proof. If $\beta \neq 0$, given $\alpha > 0$ we can find $k \in \mathbb{N}$ such that $|\beta|k > \alpha$ and $\theta_k \in \mathbb{R}$ such that the supremum in (4.2) is bigger than

$$\sup_{r\geq 0} \prod_{j=0}^{k-1} |p_N(re^{i\theta_k} + jb)|e^{r(|\beta|k-\alpha)} = \infty.$$

Therefore, $w_{[k]} \notin H_{v^\alpha}$ and the operator is not mean ergodic and not power bounded on Exp by Proposition 7. If $\beta = 0$ and $N \neq 0$, there exist n_0 and $C > 1$ such that $|p_N(jb)| > C$ for $j > n_0$. Now, the conclusion holds by Proposition 11(ii).

Finally, let us study the case $w \equiv \lambda$, $\lambda \in \mathbb{C}$. For $\lambda C_\varphi : Exp \to Exp$, take $c \in \mathbb{C}$ such that $cb = |c||b|$ and $|\lambda| > e^{-|cb|}$. The function $e^{cz} \in Exp$ and it satisfies $(\lambda C_\varphi)^k e^{cz} = (\lambda e^{cb})^k e^{cz}$. So, as $|\lambda e^{cb}| > 1$, we cannot find $n \in \mathbb{N}$ such that $\frac{(\lambda C_\varphi)^k(e^{cz})}{k}$ is bounded on H_{v^n}; therefore, the operator is not mean ergodic on Exp.

In the case of $\lambda C_\varphi : Exp^0 \to Exp^0$, the operator is neither power bounded, nor mean ergodic if $|\lambda| > 1$ by Proposition 7(ii). If $|\lambda| = 1$, take $f(z) = z \in Exp^0$ in order to see that, evaluating at $z = 0$,

$$\frac{\|(\lambda C_\varphi)^k f\|_{v^{1/n}}}{k} = \frac{1}{k} \sup_{z\in\mathbb{C}} |z + kb| e^{-\frac{|z|}{n}} \geq |b|.$$

Therefore, the operator cannot be mean ergodic, neither power bounded. If $|\lambda| < 1$, there exists $n_0 \in \mathbb{N}$ such that $|\lambda| < e^{-\frac{|b|}{n}}$ for every $n \geq n_0$. [10, Theorem 19(i)] implies the operator $\lambda C_\varphi : H_{v^{1/n}} \to H_{v^{1/n}}$ is power bounded for every $n \geq n_0$, thus it is power bounded and uniformly mean ergodic on Exp^0. □

4.4. Case $|a| > 1$

Theorem 16. *Given $\varphi(z) = az, |a| > 1$, and $w(z) = p_N(z)e^{\beta z}$, $\beta \in \mathbb{C}$, $N \in \mathbb{N}_0$, the operator $C_{w,\varphi}$ is not mean ergodic and thus not power bounded on Exp and on Exp^0 (when considering Exp^0, put $\beta = 0$).*

Proof. Take n_0 such that $|p_N(a^k)| > 1$ for every $k \geq n_0$, $z_0 = a^{n_0}$ and consider $(k_s)_s \subseteq \mathbb{N}$ increasing such that $\mathrm{Arg}(\beta a^{n_0} \frac{1-a^{k_s}}{1-a})$ $=\mathrm{Arg}(\frac{\beta a^{n_0}}{1-a})+\mathrm{Arg}(1 - a^{k_s}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, for every $s \in \mathbb{N}$,

$$\prod_{j=0}^{k_s-1} |w(a^{j+n_0})| = \prod_{j=0}^{k_s-1} |p_N(a^{j+n_0})||e^{\beta a^{j+n_0}}| > |e^{\beta a^{n_0} \frac{1-a^{k_s}}{1-a}}|$$
$$= e^{\mathrm{Re}(\beta a^{n_0} \frac{1-a^{k_s}}{1-a})} \geq 1.$$

Then, Proposition 11(iv) yields the following conclusion: □

Remark 17. Theorems 15(i) and 16 when $w \equiv \lambda$, $|\lambda| < 1$, show that Proposition 8 does not hold in general.

5. Multiples of Composition Operators

In this section we focus on multiples of composition operators, that is, operators of the form λC_φ, $\lambda \in \mathbb{C}$, $\varphi(z) = az + b$, $a, b \in \mathbb{C}$. A complete characterization of power boundedness and mean ergodicity of these operators acting on weighted Banach spaces of entire functions can be found in [10], where hypercyclicity and the spectrum are also studied. Here we consider the study on the spaces Exp and Exp^0.

5.1. Spectrum

Regarding the invertibility of the weighted composition operator $C_{w,\varphi}$, $\varphi(z) = az + b$, on Exp and Exp^0, we observe important differences from the results obtained on the Banach spaces H_{v^α}, $\alpha > 0$. Notice that $C_{w,\varphi}^{-1} = C_{\frac{1}{w\circ\varphi^{-1}},\varphi^{-1}}$, with $\varphi^{-1}(z) = \frac{1}{a}z - \frac{b}{a}$, $z \in \mathbb{C}$. As $C_{w,\varphi} : H_{v^\alpha} \to H_{v^\alpha}$ can only be continuous if $|a| \leq 1$ (see [10, Theorem 8]), it has no inverse, i.e. $0 \in \sigma(C_{w,\varphi}, H_{v^\alpha})$ if $|a| < 1$. By contrast, Proposition 4 yields the operator can be invertible when acting on Exp and Exp^0 :

Proposition 18. *Let $\varphi(z) = az+b$, $a, b \in \mathbb{C}$. The operator $C_{w,\varphi} : Exp \to Exp$ (resp. $C_{w,\varphi} : Exp^0 \to Exp^0$) is invertible, i.e. $0 \notin \sigma(C_{w,\varphi})$ if and only if w and $\frac{1}{w}$ belong to Exp (resp. Exp^0). In particular, $0 \notin \sigma(\lambda C_\varphi), \lambda \in \mathbb{C}$.*

Lemma 19. *The entire function $f(z) = \sum_{j=0}^{\infty} a_j z^j, a_j, z \in \mathbb{C}$, satisfies the following:*

- (i) *It belongs to Exp if and only if there exists $B, C > 0$ such that $|a_j| \leq C\frac{B^j}{j!}$ for every $j \in \mathbb{N}$.*
- (ii) *It belongs to Exp^0 if and only if, for every $B > 0$ there exists $C > 0$ such that $|a_j| \leq C\frac{1}{B^j j!}$ for every $j \in \mathbb{N}$.*

Proof. (i) is proved in [21, Lemma 4.18] and (ii) follows analogously. □

The study of the spectrum of multiples of composition operators reduces to the study of the spectrum of C_φ. In the next proposition we determine it. Compare the result to [10, Proposition 13] and [20, Corollary 8 and Theorem 5].

Proposition 20. *For the composition operator C_φ, $\varphi(z) = az+b$, $a, b \in \mathbb{C}$, we get the follows:*

- (i) *If $a = 1$, then $\sigma(C_\varphi, Exp) = \sigma_p(C_\varphi, Exp) = \mathbb{C}\backslash\{0\}$ and $\sigma(C_\varphi, Exp^0) = \sigma_p(C_\varphi, Exp^0) = \{1\}$.*
- (ii) *If $|a| \neq 1$ or $a \neq 1$ is a root of unity, then $\sigma_p(C_\varphi) = \sigma(C_\varphi) = \{a^j, j = 0, 1, \dots\}$.*
- (iii) *if $|a| = 1$ and a is not a root of unity, then $\sigma_p(C_\varphi) = \{a^j, j = 0, 1, \dots\}$ and $\mu \in \sigma(C_\varphi)$ if and only if for all $k > 0$ and $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|a^n - \mu| < \varepsilon k^{-n}$.*

Proof. (i) Consider $a = 1$ and $C_\varphi : Exp \to Exp$. It is easy to see that the function $e^{\alpha z}$ is an eigenvector associated with the eigenvalue $e^{\alpha b}$ for every $\alpha \in \mathbb{C}$. Thus, $\mathbb{C}\backslash\{0\} \subseteq \sigma_p(C_\varphi, Exp) \subseteq \sigma(C_\varphi, Exp)$ and the conclusion is satisfied by Proposition 18. When considering $C_\varphi : Exp^0 \to Exp^0$, as $Exp^0 = \operatorname{proj}_{n \geq m} H_{v^{1/n}}$ for every $m \in \mathbb{N}$, [3, Lemma 2.1] and [10, Proposition 13] yield $\sigma(C_\varphi, Exp^0) \subseteq \cup_{n \geq m} \sigma(C_\varphi, H_{v^{1/n}}) = \{e^\delta, |\delta| \leq \frac{|b|}{m}\}$ for every $m \in \mathbb{N}$. Thus, $\sigma(C_\varphi, Exp^0) \subseteq \{1\}$. The assertion follows because 1 is an eigenvalue. Indeed, constant functions are fixed points of the operator.

By Lemma 10, in (ii) and (iii) we can assume without loss of generality that $b = 0$. The point spectrum in these cases follows from the fact that, if $a \neq 1$, then $C_\varphi z^j = a^j z^j$, $j \in \mathbb{N}_0$, and the proof of [22, Proposition 3.3(i)]. Let us analyse the spectrum. First, observe that if $|a| \leq 1$, $a \neq 1$, then $\{a^j, j = 0, 1, \dots\} \subseteq \sigma(C_\varphi) \subseteq \overline{\{a^j, j = 0, 1, \dots\}}$ by [3, Lemma 2.1], [4, Lemma 5.2] and [10, Proposition 13].

(ii) By the inclusions above, if $a \neq 1$ is a root of unity the conclusion is trivial, and if $|a| < 1$, the assertion holds because $0 \notin \sigma(C_\varphi)$ by Proposition 18. If $|a| > 1$, as $\sigma(C_\varphi) = (\sigma(C_{\varphi^{-1}}))^{-1}$ with $\varphi^{-1}(z) = \frac{1}{a}z$, the previous case yields the following conclusion.

(iii) Assume $|a| = 1$ and a is not a root of unity. We prove that $\mu \in \rho(C_\varphi)$ if and only if there exists $k > 0$ and $\varepsilon > 0$ such that $|a^j - \mu| \geq \varepsilon k^{-j}$ for each $j \in \mathbb{N}$.

Let us analyse first $\sigma(C_\varphi, Exp)$. We have seen above that, if the condition is satisfied, $C_\varphi - \mu I$ is injective. We prove it is also surjective. Given $g(z) = \sum_{j=0}^\infty a_j z^j \in Exp$, by Lemma 19(i) there exists $B, C > 0$ such that $|a_j| \leq C\frac{B^j}{j!}$ for every $j \in \mathbb{N}$. It is easy to see that the function $f(z) := \sum_{j=0}^\infty \frac{a_j}{a^j - \mu} z^j, z \in \mathbb{C}$, satisfies $(C_\varphi - \mu I)f = g$ and that there exists $D > 0$ such that

$$\frac{|a_j|}{|a^j - \mu|} \leq C\frac{B^j k^j}{j!\varepsilon} \leq D\frac{(Bk)^j}{j!}.$$

Again by Lemma 19, we get $f \in Exp$, and so, $\mu \in \rho(C_\varphi, Exp)$. For the converse, assume $\mu \in \rho(C_\varphi, Exp)$. Then $(C_\varphi - \mu I)^{-1} : Exp \to Exp$ exists, it is continuous and satisfies

$$(C_\varphi - \mu I)^{-1}\left(\sum_{j=0}^\infty a_j z^j\right) = \sum_{j=0}^\infty \frac{a_j}{a^j - \mu} z^j.$$

The continuity implies (see Lemma 1) that for $n = 1$ there exist $k \in \mathbb{N}$ and $C > 0$ such that, for each monomial $z^j \in Exp$, $j \in \mathbb{N}_0$, we have

$$\sup_{z\in\mathbb{C}} \frac{|z|^j}{|a^j - \mu|} e^{-k|z|} \leq C \sup_{z\in\mathbb{C}} |z|^j e^{-|z|}.$$

By Lemma 12, for every $j \in \mathbb{N}_0$ we get

$$\left(\frac{j}{ke}\right)^j \frac{1}{|a^j - \mu|} \leq C\left(\frac{j}{e}\right)^j, \text{ i.e. } |a^j - \mu| \geq \frac{\varepsilon}{k^j} \text{ for some } \varepsilon > 0.$$

For $\sigma(C_\varphi, Exp^0)$ proceed analogously but using Lemma 19(ii) and Lemma 2.

□

5.2. Dynamics

As an immediate consequence of the theorems in Sect. 4, we get the following:

Corollary 21. *Let $\varphi(z) = az + b$, $a, b \in \mathbb{C}$. The properties of (uniform) mean ergodicity and power boundedness are equivalent for the operator λC_φ, $\lambda \in \mathbb{C}$, acting on Exp and on Exp^0. The following is satisfied:*

(i) *$\lambda C_\varphi : Exp \to Exp$ is mean ergodic (power bounded) if and only if $|a| \leq 1, a \neq 1$ and $|\lambda| \leq 1$.*

(ii) *$\lambda C_\varphi : Exp^0 \to Exp^0$ is mean ergodic (power bounded) if and only if $|a| \leq 1, a \neq 1$ and $|\lambda| \leq 1$, or $a = 1$ and $|\lambda| < 1$.*

The last characterization differs from the one obtained for the Banach space $H^0_{v^\alpha}$, $\alpha > 0$, if $|a| = 1$ and a is not a root of unity or if $a = 1$ (recall that on $H^0_{v^\alpha}$ the continuity of the operator only occurs when $|a| \le 1$). In this setting we get that for $|a| = 1$, a not a root of unity and $|\lambda| = 1$, the multiples of composition operators are power bounded and mean ergodic on $H^0_{v^\alpha}$, but not uniformly mean ergodic. When $a = 1$ and λ is small enough, the operator can be power bounded and uniformly mean ergodic, unlike in Exp. However, we can find λ, $|\lambda| < 1$, such that the operator is neither power bounded nor mean ergodic on the Banach setting, unlike what happens in Exp^0 (see [10]).

We finish this section studying the hypercyclicity of the operator. It is considered only in the case $a = 1$, since otherwise $\varphi(z) = az + b$ has a fixed point and [15, Proposition 2.1] yields $C_{w,\varphi}$ cannot be weakly supercyclic. So, we focus on multiples of translation operators, that is, on operators of the form λT_b, $\lambda, b \in \mathbb{C}$, $\lambda T_b(f)(z) = \lambda f(z+b)$. The result is a consequence of the one obtained for H_{v^α}, $\alpha > 0$ in [10, Theorem 20].

Theorem 22. *The weighted translation operator λT_b, $\lambda, b \in \mathbb{C}$, satisfies the following:*

(i) *It is hypercyclic, topologically mixing and chaotic on Exp for every $\lambda \in \mathbb{C}$.*

(ii) *It is hypercyclic on Exp^0 if and only if $|\lambda| = 1$. In this case, it is topologically mixing.*

Proof. (i) Given $\lambda \in \mathbb{C}$ there exists $m \in \mathbb{N}$ such that $e^{-m|b|} < |\lambda| < e^{m|b|}$. By [10, Theorem 20], $\lambda T_b : H^0_{v_m} \to H^0_{v_m}$ is topologically mixing (thus, hypercyclic) and chaotic. By the comparison principle (see [17, Lemma 3]), the operator is hypercyclic, topologically mixing and chaotic on Exp. This assertion also follows by [11, Proposition 3.3].

(ii) Given $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$, there exists $m \in \mathbb{N}$ such that $|\lambda| < e^{-\frac{|b|}{m}}$ or $e^{\frac{|b|}{m}} < |\lambda|$. By [10, Theorem 20], $\lambda T_b : H^0_{v_{1/m}} \to H^0_{v_{1/m}}$ is not hypercyclic. As Exp^0 is dense on $H^0_{v_{1/m}}$, the comparison principle yields that λT_b cannot be hypercyclic on Exp^0. Consider now $|\lambda| = 1$. Without loss of generality, we can consider $\lambda = 1$ [9, Corollary 3.3]. Now the conclusion follows by [11, Proposition 3.7]. $\square$

Acknowledgements

The research of the first author was supported by the research project MTM2016-76647-P. The research of the second author was partially supported by MTM2016-76647-P and PID2020-119457GB-I00.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

6. Appendix

We finish the paper with two results on mean ergodicity on weighted Banach spaces of entire functions which improve some parts of [10, Theorem 16]. The first one must be compared with [10, Theorem 16 a)].

Theorem 23. *Let* $\varphi(z) = az + b$ $a, b \in \mathbb{C}$, $|a| < 1$ *and* $w(z) = p_N(z)e^{\beta z}$, $N \in \mathbb{N}_0$, $\beta \in \mathbb{C}$. *If* $|w(0)| \leq 1$, *then* $C_{w,\varphi} : H_{v^\alpha} \to H_{v^\alpha}$ *and* $C_{w,\varphi} : H^0_{v^\alpha} \to H^0_{v^\alpha}$ *are power bounded and uniformly mean ergodic for every* $\alpha > \frac{|\beta|}{1-|a|}$.

Proof. Without loss of generality, we restrict to the case $b = 0$ (see Lemma 10). By [10, Theorem 8] and proceeding as in the proof of Theorem 13, we get the operator $C_{w,\varphi}$ is power bounded and compact on H_{v^α} and on $H^0_{v^\alpha}$ for every $\alpha > \frac{|\beta|}{1-|a|}$. As a consequence, $C_{w,\varphi}$ is uniformly mean ergodic on these spaces by Yosida-Kakutani Mean Ergodic Theorem [30]. $\square$

The next result completes [10, Theorem 16(b)]:

Theorem 24. *Let* $\varphi(z) = az + b$, $a, b \in \mathbb{C}$, $|a| < 1$ *and* $w(z) = e^{\beta z}$, $\beta \in \mathbb{C}$. *The operator* $C_{w,\varphi} : H^0_{v^\alpha} \to H^0_{v^\alpha}$, $\alpha = \frac{|\beta|}{1-|a|}$, *is power bounded. It is mean ergodic if and only if* $a \in \mathbb{D}\backslash[0, 1)$.

Proof. By [10, Theorem 16(b)], it only remains to show that for $a \in \mathbb{D}\backslash[0, 1)$, the operator is mean ergodic. Without loss of generality, assume $b = 0$. Observe that for every $f \in H^0_{v^\alpha}$ and every $z \in \mathbb{C}$, $C^k_{w,\varphi}f(z) = e^{\beta z \frac{1-a^k}{1-a}} f(a^k z) \xrightarrow{k\to\infty} e^{\frac{\beta z}{1-a}} f(0) \in H^0_{v^\alpha}$ (observe that $e^{\frac{\beta z}{1-a}} f(0) \notin H^0_{v^\alpha}$ if $a \in (0, 1)$). Proceeding as in the proof of [23, Lemma 3.4(1)], we get $C^k_{w,\varphi}f$ converges to $e^{\frac{\beta z}{1-a}}\delta_0(f)$ as k tends to infinity in the weak topology. As the operator is power bounded, the operator is mean ergodicity by Yosida's theorem [27, Theorem 1.3, p.26]. $\square$

References

[1] Albanese, A.A., Bonet, J., Ricker, W.J.: Mean ergodic operators in Fréchet spaces. Ann. Acad. Sci. Fenn. Math. **34**, 401–436 (2009)

[2] Albanese, A.A., Bonet, J., Ricker, W.J.: On mean ergodic operators. Oper. Theory Adv. Appl. **201**, 1–20 (2010)

[3] Albanese, A.A., Bonet, J., Ricker, W.J.: The Cesàro operator in the Fréchet spaces ℓ^{p+} and L^{p-}. Glasg. Math. J. **59**, 273–287 (2017)

[4] Albanese, A.A., Bonet, J., Ricker, W.J.: The Cesàro operator on Korenblum type spaces of analytic functions. Collect. Math. **69**(2), 263–281 (2018)

[5] Albanese, A.A., Bonet, J., Ricker, W.J.: Operators on the Fréchet sequence spaces $ces(p+)$, $1 \leq p<\infty$, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113**(2), 1533–1556 (2019)

[6] Albanese, A.A., Bonet, J., Ricker, W.J.: Linear operators on the (LB)-sequence spaces $ces(p-)$, $1<p \leq \infty$. In: Ferrando, J. (eds.) Descriptive Topology and Functional Analysis II. TFA 2018, vol. 286, pp. 43–67. Springer Proc. Math. Stat., Springer, Cham

[7] Arendt, W., Chalendar, I., Kumar, M., Srivastava, M.S.: Asymptotic behaviour of the powers of composition operators on Banach spaces of holomorphic functions. Indiana Univ. Math. J. **67**(4), 1571–1595 (2018)

[8] Arendt, W., Chalendar, I., Kumar, M., Srivastava, M.S.: Powers of composition operators: asymptotic behaviour on Bergman, Dirichlet and Bloch spaces. J. Aust. Math. Soc. **108**(3), 289–320 (2020)

[9] Bayart, F., Matheron, E.: Dynamics of linear operators. Cambridge Tracts in Mathematics, vol. 179. Cambridge University Press, Cambridge (2009)

[10] Beltrán-Meneu, M.J.: Dynamics of weighted composition operators on weighted Banach spaces of entire functions. J. Math. Anal. Appl. **492**(1), 124422 (2020)

[11] Beltrán, M.J., Bonet, J., Fernández, C.: Classical Operators on the Hörmander Algebras. Discrete Cont. Dyn. Syst. **35**(2), 637–652 (2015)

[12] Beltrán-Meneu, M.J., Gómez-Collado, M.C., Jordá, E., Jornet, D.: Mean ergodic composition operators on Banach spaces of holomorphic functions. J. Funct. Anal. **270**(12), 4369–4385 (2016)

[13] Beltrán-Meneu, M.J., Gómez-Collado, M.C., Jordá, E., Jornet, D.: Mean ergodicity of weighted composition operators on spaces of holomorphic functions. J. Math. Anal. Appl. **444**, 1640–1651 (2016)

[14] Berenstein, C.A., Gay, R.: Complex Analysis and Special Topics in Harmonic Analysis. Springer, New York (1995)

[15] Bès, J.: Dynamics of weighted composition operators. Complex Anal. Oper. Theory **8**, 159–176 (2014)

[16] Bierstedt, K.D., Meise, R., Summers, W.H.: A projective description of weighted inductive limits. Trans. Am. Math. Soc. **272**, 107–160 (1982)

[17] Bonet, J.: Hypercyclic and chaotic convolution operators. J. Lond. Math. Soc. **62**, 253–262 (2000)

[18] Bonet, J., Domański, P.: A note on mean ergodic composition operators on spaces of holomorphic functions. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **105**(2), 389–396 (2011)

[19] Bonet, J., Mangino, E.M.: Associated weights for spaces of p-integrable entire functions. Quaest. Math. **43**, 747–760 (2020)

[20] Gómez-Orts, E.: Weighted composition operators on Korenblum type spaces of analytic functions. RACSAM **114**, 199 (2020)

[21] Grosse-Erdmann, K.G., Peris, A.: Linear Chaos. Springer, London (2011)

[22] Guo, K., Izuchi, K.: Composition operators on Fock type spaces. Acta Sci. Math. **74**, 807–828 (2008)

[23] Jordá, E., Rodríguez, A.: Ergodic properties of composition operators on Banach spaces of analytic functions. J. Math. Anal. Appl. **486**(1), 123891 (2020)

[24] Meise, R.: Sequence space representations for (DFN)-algebras of entire functions modulo closed ideals. J. Reine Angew. Math. **363**, 59–95 (1985)

[25] Meise, R., Taylor, B.A.: Sequence space representations for (FN)-algebras of entire functions modulo closed ideals. Studia Math. **85**(3), 203–227 (1987)

[26] Meise, R., Vogt, D.: Introduction to Functional Analysis. The Clarendon Press, Oxford University Press, New York (1997)

[27] Petersen, K.: Ergodic Theory. Cambridge University Press, Cambridge (1983)

[28] Tien, P.T.: The iterates of composition operators on Banach spaces of holomorphic functions. J. Math. Anal. Appl. **487**(1), 123945 (2020)

[29] Wolf, E.: Weighted Fréchet spaces of holomorphic functions. Stud. Math. **174**, 255–275 (2006)

[30] Yosida, K., Kakutani, S.: Operator-theoretical treatment of Markoff's process and mean ergodic theorem. Ann. Math. **42**(1), 188–228 (1941)

María J. Beltrán-Meneu
Departament d'Educació i Didàctiques Específiques
Universitat Jaume I
Av. Vicent Sos Baynat, s/n
12071 Castelló de la Plana
Spain
e-mail: mmeneu@uji.es

Enrique Jordá
Instituto Universitario de Matemática Pura y Aplicada IUMPA
Universitat Politècnica de València
Camino de Vera, s/n
46022 Valencia
Spain

Received: October 21, 2020.
Revised: January 4, 2021.
Accepted: August 5, 2021.