Lie Derivatives of the Shape Operator of a Real Hypersurface in a Complex Projective Space

Juan de Dios Pérez and David Pérez-López

Abstract. We consider real hypersurfaces M in complex projective space equipped with both the Levi-Civita and generalized Tanaka–Webster connections. Associated with the generalized Tanaka–Webster connection we can define a differential operator of first order. For any nonnull real number k and any symmetric tensor field of type $(1,1)$ B on M , we can define a tensor field of type $(1,2)$ on M , $B_T^{(k)}$, related to Lie derivative and such a differential operator. We study symmetry and skew symmetry of the tensor $A_T^{(k)}$ associated with the shape operator A of M .

Mathematics Subject Classification. 53C15, 53B25.

Keywords. k th g-Tanaka–Webster connection, complex projective space, real hypersurface, shape operator, Lie derivatives.

1. Introduction

We will denote by $\mathbb{C}P^m$, $m \geq 2$, the complex projective space equipped with the Kählerian structure (J, g) , J being the complex structure and g the Fubini-Study metric with constant holomorphic sectional curvature 4. Take a connected real hypersurface without boundary M in $\mathbb{C}P^m$ whose local normal unit vector field is N . Take $\xi = -JN$. Then ξ is a tangent vector field to M that we call the Reeb vector field (or the structure vector field) on M . For any tangent vector field X on M , we write $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX and $\eta(X) = g(X, \xi)$. Then (ϕ, ξ, η, g) defines on M an almost contact metric structure [1], where g is the induced metric on M .

Takagi, see Refs. [5, 8–10], classified homogeneous real hypersurfaces of $\mathbb{C}P^m$ into six types. All of them are Hopf, that is, their structure vector fields are principal ($A\xi = \alpha\xi$, for a function α on M). Denote by $\mathbb{D}$ the maximal holomorphic distribution on M : at any point $p \in M$, $\mathbb{D}_p = \{X \in T_p M | g(X, \xi_p) = 0\}$. Kimura [5] proved that any Hopf real hypersurface M in $\mathbb{C}P^m$ whose principal curvatures are constant belongs to Takagi's list.

The unique real hypersurfaces in $\mathbb{C}P^m$ with two distinct principal curvatures are geodesic hyperspheres of radius r , $0 < r < \frac{\pi}{2}$, see Ref. [2]. Their principal curvatures are $2\cot(2r)$ with eigenspace $\mathbb{R}[\xi]$ and $\cot(r)$ with eigenspace $\mathbb{D}$.

The canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold was defined, independently, by Tanaka [11], and Webster [13], and it is known as the Tanaka–Webster connection. For contact metric manifolds, Tanno [12] introduced a generalized Tanaka–Webster connection.

For a real hypersurface M of $\mathbb{C}P^m$ and any nonnull real number k , Cho, see [3, 4], generalized Tanno's definition to the concept of k th generalized Tanaka–Webster connection by

$$\hat{\nabla}_X^{(k)} Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \quad (1.1)$$

for any X, Y tangent to M . Then the four elements of the almost contact metric structure on M are parallel for this connection and if the shape operator of the real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the real hypersurface is contact and the k th generalized Tanaka–Webster connection coincides with the Tanaka–Webster connection.

We define the k th Cho operator on M associated with the tangent vector field X by $F_X^{(k)} Y = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any Y tangent to M . The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $T^{(k)}(X, Y) = F_X^{(k)} Y - F_Y^{(k)} X$ for any X, Y tangent to M . We also define the k th torsion operator associated with the tangent vector field X by $T_X^{(k)} Y = T^{(k)}(X, Y)$ for any Y tangent to M .

Let $\mathcal{L}$ denote the Lie derivative on M . Then $\mathcal{L}_X Y = \nabla_X Y - \nabla_Y X$ for any X, Y tangent to M . On M we can also define a differential operator of first order associated with the k th generalized Tanaka–Webster connection by $\mathcal{L}_X^{(k)} Y = \hat{\nabla}_X^{(k)} Y - \hat{\nabla}_Y^{(k)} X = \mathcal{L}_X Y + T_X^{(k)} Y$, for any X, Y tangent to M .

Let now B be a symmetric tensor of type (1,1) defined on M . We can associate with B a tensor field of type (1,2) $B_T^{(k)}$ by $B_T^{(k)}(X, Y) = [T_X^{(k)}, B]Y = T_X^{(k)} BY - BT_X^{(k)} Y$, for any X, Y tangent to M .

Consider the condition $\mathcal{L}^{(k)} B = \mathcal{L} B$ for some nonnull real number k . This means that for any X, Y tangent to M $(\mathcal{L}_X^{(k)} B)Y = (\mathcal{L}_X B)Y$. This is equivalent to having $B_T^{(k)} = 0$.

Generalizing this we can consider that the tensor $B_T^{(k)}$ is symmetric, that is, $B_T^{(k)}(X, Y) = B_T^{(k)}(Y, X)$ for any X, Y tangent to M . This is equivalent to have the following Codazzi-type condition

$$\left((\mathcal{L}_X^{(k)} - \mathcal{L}_X) B \right) Y = \left((\mathcal{L}_Y^{(k)} - \mathcal{L}_Y) B \right) X \quad (1.2)$$

for any X, Y tangent to M .

On the other hand, we can suppose that $B_T^{(k)}$ is skew symmetric, that is, $B_T^{(k)}(X, Y) = -B_T^{(k)}(Y, X)$, for any X, Y tangent to M . This is equivalent to the following Killing-type condition:

$$\left(\left(\mathcal{L}_X^{(k)} - \mathcal{L}_X \right) B \right) Y + \left(\left(\mathcal{L}_Y^{(k)} - \mathcal{L}_Y \right) B \right) X = 0 \quad (1.3)$$

for any X, Y tangent to M .

In the particular case of $B = A$, the shape operator of M , in Ref. [7] the first author proved non-existence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, satisfying $\mathcal{L}^{(k)} A = \mathcal{L} A$, that is, $A_T^{(k)} = 0$, for any nonnull real number k .

The purpose of the present paper is to study real hypersurfaces M in $\mathbb{C}P^m$ such that the shape operator satisfies either (1.2) or (1.3). In fact, we will obtain the following.

Theorem 1. *There does not exist any real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, such that, for some nonnull real number k , $A_T^{(k)}$ is symmetric.*

In the case of $A_T^{(k)}$ being skew symmetric, we have a very different situation given by the

Theorem 2. *Let M be a real hypersurface M in $\mathbb{C}P^m$, $m \geq 3$, and k a nonnull real number. Then the tensor field $A_T^{(k)}$ is skew symmetric if and only if M is locally congruent to a geodesic hypersphere of radius r , $0 < r < \frac{\pi}{2}$, such that $\cot(r) = k$.*

2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc., will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M . Let ∇ be the Levi-Civita connection on M and (J, g) the Kählerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M , we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M , see Ref. [1]. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.1)$$

for any vectors X, Y tangent to M . From (2.1), we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \quad (2.2)$$

From the parallelism of J , we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \quad (2.3)$$

and

$$\nabla_X \xi = \phi AX \quad (2.4)$$

for any X, Y tangent to M , where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equation of Codazzi is given by

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \quad (2.5)$$

for any tangent vector fields X, Y to M . We will call the maximal holomorphic distribution $\mathbb{D}$ on M to the following one: at any $p \in M$, $\mathbb{D}_p = \{X \in$

$T_p M | g(X, \xi_p) = 0 \}$. We will say that M is Hopf if ξ is principal, that is, $A\xi = \alpha\xi$ for a certain function α on M .

In the sequel, we need the following result, which consists of a combination of the Lemmas 2.1, 2.2 and 2.4 in Ref. [6].

Theorem 2.1. *If ξ is a principal curvature vector with corresponding principal curvature α , this is locally constant and if $X \in \mathbb{D}$ is principal with principal curvature λ , then $2\lambda - \alpha \neq 0$ and ϕX is principal with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

3. Proof of Theorem 1

If M satisfies (1.2) for $B = A$, we get $\mathcal{L}_X^{(k)} AY - \mathcal{L}_X AY - A\mathcal{L}_X^{(k)} Y + A\mathcal{L}_X Y = \mathcal{L}_Y^{(k)} AX - \mathcal{L}_Y AX - A\mathcal{L}_Y^{(k)} X + A\mathcal{L}_Y X$ for any X, Y tangent to M . Therefore, we have $F_X^{(k)} AY - F_{AY}^{(k)} X - 2AF_X^{(k)} Y + 2AF_Y^{(k)} X = F_Y^{(k)} AX - F_{AX}^{(k)} Y$, for any X, Y tangent to M . This yields

$$\begin{aligned} & 2g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY - g(\phi A^2 Y, X)\xi \\ & \quad + \eta(X)\phi A^2 Y + k\eta(AY)\phi X \\ & \quad - 2g(\phi AX, Y)A\xi + 2\eta(Y)A\phi AX + 2k\eta(X)A\phi Y + 2g(\phi AY, X)A\xi \\ & \quad - 2\eta(X)A\phi AY - 2k\eta(Y)A\phi X \\ & = -\eta(AX)\phi AY - k\eta(Y)\phi AX - g(\phi A^2 X, Y)\xi \\ & \quad + \eta(Y)\phi A^2 X + k\eta(AX)\phi Y \end{aligned} \quad (3.1)$$

for any X, Y tangent to M . If we suppose that $X, Y \in \mathbb{D}$, (3.1) becomes

$$\begin{aligned} & 2g(\phi AX, AY)\xi - \eta(AY)\phi AX - g(\phi A^2 Y, X)\xi + k\eta(AY)\phi X \\ & \quad - 2g(\phi AX, Y)A\xi + 2g(\phi AY, X)A\xi \\ & = -\eta(AX)\phi AY - g(\phi A^2 X, Y)\xi + k\eta(AX)\phi Y \end{aligned} \quad (3.2)$$

for any $X, Y \in \mathbb{D}$. If M is Hopf, that is $A\xi = \alpha\xi$, then (3.2) gives $2g(\phi AX, AY)\xi - g(\phi A^2 Y, X)\xi - 2\alpha g(\phi AX, Y)\xi + 2\alpha g(\phi AY, X)\xi = -g(\phi A^2 X, Y)\xi$ for any $X, Y \in \mathbb{D}$. This yields $2A\phi AX + A^2\phi X - 2\alpha\phi AX - 2\alpha A\phi X = -\phi A^2 X$ for any $X \in \mathbb{D}$. Let us suppose that $X \in \mathbb{D}$ satisfies $AX = \lambda X$. From Theorem 2.1, we have $A\phi X = \mu\phi X$ with $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Therefore, we obtain $2\lambda\mu + \mu^2 - 2\alpha(\lambda + \mu) + \lambda^2 = 0$. That is, $(\lambda + \mu)(\lambda + \mu - 2\alpha) = 0$.

If $\lambda + \mu = 0$, as $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$, we obtain $\frac{2\lambda^2+2}{2\lambda-\alpha} = 0$. This yields $\lambda^2 + 1 = 0$, which is impossible.

If $\lambda + \mu = 2\alpha$, we should have $\lambda^2 - 2\alpha\lambda + \alpha^2 + 1 = 0$. This gives $\lambda = \alpha \pm \sqrt{-1}$, which is also impossible. Thus, our real hypersurface must be non-Hopf. This means that ξ is not principal. Therefore, we can write $A\xi = \alpha\xi + \beta U$ at least on a neighborhood of a point of M , where U is a unit vector field in $\mathbb{D}$ and β a nonvanishing function on such a neighborhood. From now on, we will denote $\mathbb{D}_U = \{X \in \mathbb{D} | g(X, U) = g(X, \phi U) = 0\}$ and make the calculations on that neighborhood. Then (3.2) becomes

$$2g(\phi AX, AY)\xi - \beta g(Y, U)\phi AX - g(\phi A^2 Y, X)\xi + k\beta g(Y, U)\phi X$$

$$\begin{aligned}
& -2g(\phi AX, Y)A\xi + 2g(\phi AY, X)A\xi \\
& = -\beta g(X, U)\phi AY - g(\phi A^2 X, Y)\xi + k\beta g(X, U)\phi Y
\end{aligned} \quad (3.3)$$

for any $X, Y \in \mathbb{D}$. The scalar product of (3.3) and ϕU yields $-\beta g(Y, U)g(AX, U) + k\beta g(Y, U)g(X, U) = -\beta g(X, U)g(AY, U) + k\beta g(X, U)g(Y, U)$, for any $X, Y \in \mathbb{D}$. As $\beta \neq 0$, we obtain

$$g(Y, U)g(AX, U) = g(X, U)g(AY, U) \quad (3.4)$$

for any $X, Y \in \mathbb{D}$. If in (3.4) we take $X = U$, $Y \in \mathbb{D}_U$ we obtain $g(AU, Y) = 0$ for any $Y \in \mathbb{D}_U$, and if we take $X = U$, $Y = \phi U$ we get $g(AU, \phi U) = 0$. Therefore, we have

$$AU = \beta \xi + \gamma U \quad (3.5)$$

for a certain function γ .

The scalar product of (3.3) and U yields $-\beta g(Y, U)g(\phi AX, U) + k\beta g(Y, U)g(\phi X, U) - 2\beta g(\phi AX, Y) + 2\beta g(\phi AY, X) = -\beta g(X, U)g(\phi AY, U) + k\beta g(X, U)g(\phi Y, U)$, for any $X, Y \in \mathbb{D}$. If $Y = U$ it follows $-\beta g(\phi AX, U) + k\beta g(\phi X, U) - 2\beta g(\phi AX, U) + 2\beta g(\phi AU, X) = 0$ for any $X \in \mathbb{D}$. That is, $3g(A\phi U, X) - k\beta g(\phi U, X) + 2\gamma g(\phi U, X) = 0$. Therefore,

$$A\phi U = \frac{k - 2\gamma}{3}\phi U. \quad (3.6)$$

Take now $X = \xi$, $Y \in \mathbb{D}$ in (3.1). We obtain

$$\begin{aligned}
2\beta g(A\phi U, Y)\xi - \beta \eta(AY)\phi U - k\phi AY + \phi A^2 Y - 2\beta g(\phi U, Y)A\xi + 2kA\phi Y \\
- 2A\phi AY = -\alpha\phi AY + \alpha\beta g(U, \phi Y)\xi + \beta g(AU, \phi Y)\xi + k\alpha\phi Y
\end{aligned} \quad (3.7)$$

for any $Y \in \mathbb{D}$. Its scalar product with ξ gives, being $\beta \neq 0$, $2g(A\phi U, Y) - 2\alpha g(\phi U, Y) + 2k\beta g(\phi Y, U) - 2g(\phi AY, U) = \alpha g(U, \phi Y) + \gamma g(U, \phi Y)$, for any $Y \in \mathbb{D}$. Therefore, we have $4A\phi U = (\alpha + 2k - \gamma)\phi U$, which is equivalent to

$$A\phi U = \frac{\alpha + 2k - \gamma}{4}\phi U. \quad (3.8)$$

From (3.6) and (3.8), it follows

$$3\alpha + 2k + 5\gamma = 0. \quad (3.9)$$

If we take $Y = U$ in (3.7), it follows $-\beta^2\phi U - k\phi AU + \phi A^2 U + 2kA\phi U - 2A\phi AU = -\alpha\phi AU + k\alpha\phi U$. That is, $-\beta^2 - k\gamma + \beta^2 + \gamma^2 + 2k\gamma' - 2\gamma\gamma' = -\alpha\gamma + k\alpha$, where $\gamma' = \frac{\alpha + 2k - \gamma}{4}$. This yields $\gamma^2 - (2\gamma' - \alpha + k)\gamma + k(2\gamma' - \alpha) = 0$. Therefore, $\gamma = \frac{2\gamma' - \alpha + k \pm \sqrt{(2\gamma' - \alpha - k)^2}}{2}$. From this, either $\gamma = k$ or $\gamma = 2\gamma' - \alpha$.

Suppose now $X, Y \in \mathbb{D}_U$. Then (3.2) yields $2g(\phi AX, AY)\xi - g(\phi A^2 Y, X)\xi - 2g(\phi AX, Y)A\xi + 2g(\phi AY, X)A\xi = -g(\phi A^2 Y, X)\xi$. Its scalar product with U gives $-2\beta g(\phi AX, Y) + 2\beta g(\phi AY, X) = 0$. Then $g((\phi A + A\phi)X, Y) = 0$ for any $X, Y \in \mathbb{D}_U$. This implies that $(\phi A + A\phi)X = 0$ for any $X \in \mathbb{D}_U$. If we suppose that $AX = \lambda X$ we obtain that $A\phi X = -\lambda\phi X$. If we take such an X in (3.7) we get $-k\phi AX + \phi A^2 X + 2kA\phi X - 2A\phi AX = -\alpha\phi AX + k\alpha\phi X$. Therefore, $-k\lambda + \lambda^2 - 2k\lambda + 2\lambda^2 = -\alpha\lambda + k\alpha$. This yields $3\lambda^2 - (3k - \alpha)\lambda - k\alpha = (\lambda - k)(\lambda + \frac{\alpha}{3}) = 0$. Thus, either $\lambda = k$ or $\lambda = -\frac{\alpha}{3}$.

From (3.6) and (3.9) if $\gamma = k$, a constant, $\gamma' = -\frac{k}{3}$ is constant and $\alpha = -\frac{7}{3}$ is also constant. Furthermore, all principal curvatures on $\mathbb{D}_U$ are also constant.

If $\gamma = 2\gamma' - \alpha = \frac{2k-4\gamma}{3} - \alpha$, we obtain $3\gamma = 2k - 4\gamma - 3\alpha$. Then $2k - 7\gamma - 3\alpha = 0$ and from (3.9) $4k - 2\gamma = 0$. This yields $\gamma = 2k$ is constant, $\alpha = -4k$ and $\gamma' = -k$ are also constant. As above, all principal curvatures in $\mathbb{D}_U$ are constant.

Take a unit $X \in \mathbb{D}_U$ such that $AX = \lambda X$. The Codazzi equation gives $(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X$. That is, $\nabla_X(\alpha\xi + \beta U) - A\phi AX - \nabla_\xi(\lambda X) + A\nabla_\xi X = -\phi X$. Therefore, $\alpha\lambda\phi X + X(\beta)U + \beta\nabla_X U + \lambda^2\phi X - \lambda\nabla_\xi X + A\nabla_\xi X = -\phi X$. If we take ϕX instead of X , we have similarly $\alpha\lambda X + (\phi X)(\beta)U + \beta\nabla_{\phi X} U - \lambda^2 X + \lambda\nabla_\xi \phi X + A\nabla_\xi \phi X = X$. In both cases, taking the scalar product with ξ , we have

$$g(\nabla_\xi X, U) = g(\nabla_\xi \phi X, U) = 0. \quad (3.10)$$

The scalar product with U of the expression for X yields $X(\beta) - \lambda g(\nabla_\xi X, U) + g(\nabla_\xi X, \beta\xi + \gamma U) = 0$. In the case of ϕX we obtain $(\phi X)(\beta) + \lambda g(\nabla_\xi \phi X, U) + g(\nabla_\xi \phi X, \beta\xi + \gamma U) = 0$. Bearing in mind (3.10), we conclude that

$$Z(\beta) = 0 \quad (3.11)$$

for any $Z \in \mathbb{D}$.

On the other hand, we have $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = U$. That is, $\nabla_{\phi U}(\alpha\xi + \beta U) - A\phi A\phi U - \nabla_\xi(\gamma'\phi U) + A\nabla_\xi \phi U = U$. This implies

$$\alpha\phi A\phi U + \beta\nabla_{\phi U} U + (\phi U)(\beta)U + \gamma'AU - \gamma'\nabla_\xi \phi U + A\nabla_\xi \phi U = U. \quad (3.12)$$

Its scalar product with ξ gives $\beta g(A\phi U, \phi U) + \beta\gamma' + \gamma'g(A\xi, U) + g(\nabla_\xi \phi U, \alpha\xi + \beta U) = 0$. That is, $3\beta\gamma' - \alpha\beta + \beta g(\nabla_\xi \phi U, U) = 0$. Therefore,

$$g(\nabla_\xi \phi U, U) = -3\gamma' + \alpha. \quad (3.13)$$

The scalar product of (3.12) with U gives $-\alpha g(A\phi U, \phi U) + (\phi U)(\beta) + \gamma\gamma' - \gamma'g(\nabla_\xi \phi U, U) + g(\nabla_\xi \phi U, \beta\xi + \gamma U) = 1$. From (3.13) this yields $-\alpha\gamma' + (\phi U)(\beta) + \gamma\gamma' - \gamma'(-3\gamma' + \alpha) - \beta^2 - 3\gamma\gamma' + \alpha\gamma = 1$. Thus, we obtain

$$(\phi U)(\beta) = 1 + 2\alpha\gamma' + 2\gamma\gamma' - 3\gamma'^2 + \beta^2 - \alpha\gamma. \quad (3.14)$$

Now $(\nabla_U A)\xi - (\nabla_\xi A)U = -\phi U$ implies

$$\alpha\phi AU + U(\beta)U + \beta\nabla_U U - \gamma\gamma'\phi U - \xi(\beta)\xi - \beta\phi A\xi - \gamma\nabla_\xi U + A\nabla_\xi U = -\phi U. \quad (3.15)$$

Its scalar product with ξ gives $-\beta g(U, \phi AU) - \xi(\beta) + \gamma g(U, \phi A\xi) + g(\nabla_\xi U, \alpha\xi + \beta U) = 0$. From this we have

$$\xi(\beta) = 0. \quad (3.16)$$

The scalar product of (3.15) with U yields $U(\beta) + g(\nabla_\xi U, \beta\xi + \gamma U) = 0$. This gives

$$U(\beta) = 0. \quad (3.17)$$

From (3.11), (3.14), (3.16) and (3.17) we obtain

$$\text{grad}(\beta) = (\beta^2 + 1 + 2\alpha\gamma' + 2\gamma\gamma' - 3\gamma'^2 - \alpha\gamma)\phi U. \quad (3.18)$$

We will call $\omega = \beta^2 + 1 + 2\alpha\gamma' + 2\gamma\gamma' - 3\gamma'^2 - \alpha\gamma$. We know that $g(\nabla_X \text{grad}(\beta), Y) = g(\nabla_Y \text{grad}(\beta), X)$ for any X, Y tangent to M . In our case we have $X(\omega)g(\phi U, Y) + \omega g(\nabla_X \phi U, Y) = Y(\omega)g(\phi U, X) + \omega g(\nabla_Y \phi U, X)$. If we take $X = \xi$, from (3.16) and the fact that the all the elements different from β appearing in ω are constant, we have $\xi(\omega) = 0$. Thus, we get $-\omega g(U, AY) = \omega g(\nabla_\xi \phi U, Y)$ for any Y tangent to M . Taking now $Y = U$, bearing in mind (3.13) we arrive to $-\omega\gamma = \omega(-3\gamma' + \alpha)$. If we suppose $\omega \neq 0$ it follows $-\gamma = -3\gamma' + \alpha$. If $\gamma = k$, $\gamma' = -\frac{k}{3}$ and $\alpha = -\frac{7k}{3}$. Therefore, $-k = k - \frac{7k}{3}$ implies $k = 0$, which is impossible. In the other possible case $\gamma' = -k$, $\gamma = 2k$ and $\alpha = -4k$. Then $-2k = 3k - 4k$ gives also a contradiction.

Thus, we have proved that $\omega = 0$. Then $1 + 2\alpha\gamma' + 2\gamma\gamma' - 3\gamma'^2 - \alpha\gamma + \beta^2 = 0$. If $\gamma = k$, $\gamma' = -\frac{k}{3}$ and $\alpha = -\frac{7k}{3}$. This yields $\beta^2 + 1 + \frac{4}{3}k^2 = 0$, which is impossible. Then $\gamma = 2k$, $\alpha = -4k$ and $\gamma' = -k$. Thus $\beta^2 + 9k^2 + 1 = 0$, also impossible, and we have finished the proof. $\square$

4. Proof of Theorem 2

If M satisfies (1.3) for $B = A$ and any X, Y tangent to M we obtain

$$\begin{aligned} & -\eta(AY)\phi AX - k\eta(X)\phi AY - g(\phi A^2Y, X)\xi + \eta(X)\phi A^2Y + k\eta(AY)\phi X \\ & -\eta(AX)\phi AY - k\eta(Y)\phi AX - g(\phi A^2X, Y)\xi + \eta(Y)\phi A^2X + k\eta(AX)\phi Y = 0. \end{aligned} \quad (4.1)$$

for any X, Y tangent to M . If $X, Y \in \mathbb{D}$ (4.1) becomes

$$\begin{aligned} & -\eta(AY)\phi AX - g(\phi A^2Y, X)\xi + k\eta(AY)\phi X - \eta(AX)\phi AY \\ & -g(\phi A^2X, Y)\xi + k\eta(AX)\phi Y = 0 \end{aligned} \quad (4.2)$$

for any $X, Y \in \mathbb{D}$. If in (4.1) we take $X = \xi$, $Y \in \mathbb{D}$, we obtain

$$-\eta(AY)\phi A\xi - k\phi AY + \phi A^2Y - \eta(A\xi)\phi AY - g(\phi A^2\xi, Y)\xi + k\eta(A\xi)\phi Y = 0 \quad (4.3)$$

for any $Y \in \mathbb{D}$.

Let us suppose that M is Hopf and $A\xi = \alpha\xi$. From (4.2) we obtain $-g(\phi A^2Y, X)\xi - g(\phi A^2X, Y)\xi = 0$ for any $X, Y \in \mathbb{D}$. Therefore, $A^2\phi X = \phi A^2X$ for any $X \in \mathbb{D}$. If $X \in \mathbb{D}$ satisfies $AX = \lambda X$, we know that $A\phi X = \mu\phi X$ with $\mu = \frac{\alpha\lambda+2}{2\lambda-\alpha}$. Thus, we have $\lambda^2 = \mu^2$ and either $\lambda = \mu$ or $\mu = -\lambda$.

If $\frac{\alpha\lambda+2}{2\lambda-\alpha} = -\lambda$ we obtain $\alpha\lambda + 2 = -2\lambda^2 + \lambda\alpha$. This implies $\lambda^2 + 1 = 0$, which is impossible. Therefore, $\lambda = \mu$. Taking such a Y in (4.3), we have $-k\phi AY + \phi A^2Y - \alpha\phi AY + k\alpha\phi Y = 0$. That is, $-k\lambda + \lambda^2 - \alpha\lambda + k\alpha = 0$. This gives $\lambda^2 - (\alpha + k)\lambda + k\alpha = (\lambda - k)(\lambda - \alpha) = 0$, and the possible solutions are either $\lambda = k$ or $\lambda = \alpha$. Then M has two distinct constant principal curvatures and from Ref. [2] M must be locally congruent to a geodesic hypersphere whose principal curvature on $\mathbb{D}$ is $\cot(r) = k$.

If M is non-Hopf, as in the previous section, we write $A\xi = \alpha\xi + \beta U$, with the same conditions. From (4.2), it follows

$$\begin{aligned} & -\beta g(U, Y)\phi AX - g(\phi A^2Y, X)\xi + k\beta g(U, Y)\phi X \\ & -\beta g(U, X)\phi AY - g(\phi A^2X, Y)\xi + k\beta g(U, X)\phi Y = 0 \end{aligned} \quad (4.4)$$

for any $X, Y \in \mathbb{D}$. Its scalar product with U yields

$$\begin{aligned} \beta g(U, Y)g(A\phi U, X) - k\beta g(U, Y)g(\phi U, X) + \beta g(U, X)g(A\phi U, Y) \\ - k\beta g(U, X)g(\phi U, Y) = 0 \end{aligned} \quad (4.5)$$

for any $X, Y \in \mathbb{D}$. If in (4.5) we take $X \in \mathbb{D}_U$, $\beta g(U, Y)g(A\phi U, X) = 0$. As $\beta \neq 0$, if $Y = U$ we obtain $g(A\phi U, X) = 0$ for any $X \in \mathbb{D}_U$. If in (4.5) we take $X = Y = U$ we have $2\beta g(A\phi U, U) = 0$. This implies $g(A\phi U, U) = 0$. Finally, taking $Y = \phi U$ in (4.5) we get $\beta g(U, X)g(A\phi U, \phi U) - k\beta g(U, X) = 0$ for any $X \in \mathbb{D}$. For $X = U$ we obtain $g(A\phi U, \phi U) = k$. Therefore, we have seen that

$$A\phi U = k\phi U. \quad (4.6)$$

The scalar product of (4.4) and ϕU gives $-\beta g(Y, U)g(AU, X) + k\beta g(U, Y)g(U, X) - \beta g(U, X)g(AU, Y) + k\beta g(U, X)g(U, Y) = 0$ for any $X, Y \in \mathbb{D}$. Taking $X = Y = U$ we have $-2\beta g(AU, U) + 2k\beta = 0$ and then $g(AU, U) = k$. On the other hand, (4.3) yields $-\beta^2 g(Y, U)\phi U - k\phi AY + \phi A^2 Y - \alpha\phi AY + \alpha g(A\xi, \phi Y)\xi + \beta g(AU, \phi Y)\xi + k\alpha\phi Y = 0$ for any $Y \in \mathbb{D}$. Its scalar product with ξ implies

$$\alpha g(A\xi, \phi Y) + \beta g(AU, \phi Y) = 0 \quad (4.7)$$

for any $Y \in \mathbb{D}$. If $Y = \phi X$, $X \in \mathbb{D}_U$, we obtain $\beta g(AU, X) = 0$ for any $X \in \mathbb{D}_U$ and if $Y = \phi U$ in (4.7) it follows $-\alpha\beta - \beta g(AU, U) = 0$. Therefore, $g(AU, U) = -\alpha$ and we get

$$\alpha = -k \quad (4.8)$$

and

$$AU = \beta\xi + kU. \quad (4.9)$$

Let $X, Y \in \mathbb{D}_U$. From (4.4) we have $-g(\phi A^2 Y, X) - g(\phi A^2 X, Y) = 0$. From (4.6) and (4.9) $\mathbb{D}_U$ is A -invariant and we obtain $\phi A^2 X = A^2 \phi X$ for any $X \in \mathbb{D}_U$. Let us suppose that $Y \in \mathbb{D}_U$ satisfies $AY = \lambda Y$. From (4.3) we get $-k\lambda\phi Y + \lambda^2\phi Y - \alpha\lambda\phi Y + k\alpha\phi Y = 0$. From (4.8) it follows $\lambda^2\phi Y - k^2\phi Y = 0$. Thus, $\lambda^2 = k^2$ and λ is constant.

For such a $Y \in \mathbb{D}_U$ the Codazzi equation gives $(\nabla_Y A)\xi - (\nabla_\xi A)Y = -\phi Y$. Therefore, $\nabla_Y(-k\xi + \beta U) - A\phi AY - \nabla_\xi(\lambda Y) + A\nabla_\xi Y = -\phi Y$. Then $-k\phi AY + Y(\beta)U + \beta\nabla_Y U - A\phi AY - \lambda\nabla_\xi Y + A\nabla_\xi Y = -\phi Y$. Its scalar product with U yields $Y(\beta) - \lambda g(\nabla_\xi Y, U) + g(\nabla_\xi Y, \beta\xi + kU) = 0$ and

$$Y(\beta) = (\lambda - k)g(\nabla_\xi Y, U). \quad (4.10)$$

On the other hand, $(\nabla_Y A)U - (\nabla_U A)Y = 0$. From this we obtain $\nabla_Y(\beta\xi + kU) - A\nabla_Y U - \nabla_U(\lambda Y) + A\nabla_U Y = 0$. That is, $Y(\beta)\xi + \beta\phi AY + k\nabla_Y U - A\nabla_Y U - \lambda\nabla_U Y + A\nabla_U Y = 0$. Its scalar product with ξ gives $Y(\beta) - kg(U, \phi AY) - g(\nabla_Y U, \alpha\xi) + \lambda g(Y, \phi AU) + g(\nabla_U Y, \alpha\xi + \beta U) = 0$. Then

$$Y(\beta) = -\beta g(\nabla_U Y, U). \quad (4.11)$$

Its scalar product with U implies $-g(\nabla_Y U, \beta\xi) - \lambda g(\nabla_U Y, U) + g(\nabla_U Y, \beta\xi + kU) = 0$. This yields $(\lambda - k)g(\nabla_Y U, Y) = 0$. If $g(\nabla_U Y, U) = 0$ from (4.11) we get $Y(\beta) = 0$. If $g(\nabla_U Y, U) \neq 0$, $\lambda = k$ and from (4.10) again

$$Y(\beta) = 0 \quad (4.12)$$

for any $Y \in \mathbb{D}_U$.

Moreover $(\nabla_U A)\xi - (\nabla_\xi A)U = -\phi U$ implies $\nabla_U(-k\xi + \beta U) - A\phi AU - \nabla_\xi(\beta\xi + kU) + A\nabla_\xi U = -\phi U$. Then $-k\phi AU + U(\beta)U + \beta\nabla_U U - A\phi AU -$

$\xi(\beta)\xi - \beta\phi A\xi - k\nabla_\xi U + A\nabla_\xi U = -\phi U$ and its scalar product with ξ gives $-\beta g(U, \phi AU) - \xi(\beta) + kg(U, \phi A\xi) + g(\nabla_\xi U, \alpha\xi + \beta U) = 0$. Therefore,

$$\xi(\beta) = 0 \quad (4.13)$$

and its scalar product with U yields $U(\beta) + g(\nabla_\xi U, \beta\xi) = 0$. That is

$$U(\beta) = 0. \quad (4.14)$$

Now we develop $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = U$. Then $\nabla_{\phi U}(-k\xi + \beta U) - A\phi A\phi U - \nabla_\xi(k\phi U) + A\nabla_\xi\phi U = U$ that implies $-k\phi A\phi U + (\phi U)(\beta)U + \beta\nabla_{\phi U}U - A\phi A\phi U - k\nabla_\xi\phi U + A\nabla_\xi\phi U = U$. Its scalar product with U yields

$$(\phi U)(\beta) = 1 + \beta^2 - 2k^2. \quad (4.15)$$

Its scalar product with ξ gives $\beta g(A\phi U, \phi U) + \beta g(A\phi U, \phi U) + kg(\phi U, \phi A\xi) + g(\nabla_\xi\phi U, -k\xi + \beta U) = 0$. Therefore,

$$g(\nabla_\xi\phi U, U) = -4k \quad (4.16)$$

and

$$\text{grad}(\beta) = \omega\phi U \quad (4.17)$$

where $\omega = 1 + \beta^2 - 2k^2$. Now, as in previous section, $g(\nabla_X(\omega\phi U), Y) = g(\nabla_Y(\omega\phi U), X)$ for any X, Y tangent to M . This yields $X(\omega)g(\phi U, Y) + \omega g(\nabla_X\phi U, Y) = Y(\omega)g(\phi U, X) + \omega g(\nabla_Y\phi U, X)$. If $X = \xi$ we get $\omega g(\nabla_\xi\phi U, Y) = \omega g(\nabla_Y\phi U, \xi) = -\omega g(\phi U, \phi AY) = -\omega g(U, AY)$. Take $Y = U$. Then $\omega g(\nabla_\xi\phi U, U) = k\omega$. This and (4.16) give $\omega = 0$ and, therefore, β is constant and equals $2k^2 - 1$.

Now $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$. Then $\nabla_U(k\phi U) - A\nabla_U\phi U - \nabla_{\phi U}(\beta\xi + kU) + A\nabla_{\phi U}U = -2\xi$, that is, $k\nabla_U\phi U - A\nabla_U\phi U - \beta\phi A\phi U - k\nabla_{\phi U}U + A\nabla_{\phi U}U = -2\xi$. If we take its scalar product with U we obtain $3k\beta = 0$, which is impossible and finishes the proof. $\square$

Acknowledgements

This work was supported by MINECO-FEDER Project MTM 2016-78807-C2-1-P.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature. Universidad de Granada/CBUA.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Blair, D.E.: Riemannian geometry of contact and symplectic manifolds. *Prog. Math.* **203** (2002). Birkhäuser Boston, Inc., Boston, MA
- [2] Cecil, T.E., Ryan, P.J.: Focal sets and real hypersurfaces in complex projective space. *Trans. Am. Math. Soc.* **269**, 481–499 (1982)
- [3] Cho, J.T.: CR-structures on real hypersurfaces of a complex space form. *Publ. Math. Debrecen* **54**, 473–487 (1999)
- [4] Cho, J.T.: Pseudo-Einstein CR-structures on real hypersurfaces in a complex space form. *Hokkaido Math. J.* **37**, 1–17 (2008)
- [5] Kimura, M.: Real hypersurfaces and complex submanifolds in complex projective space. *Trans. A.M.S.* **296**, 137–149 (1986)
- [6] Maeda, Y.: On real hypersurfaces of a complex projective space. *J. Math. Soc. Japan* **28**, 529–540 (1976)
- [7] Pérez, J.D.: Comparing Lie derivatives on real hypersurfaces in complex projective spaces. *Mediterr. J. Math.* **13**, 2161–2169 (2016)
- [8] Takagi, R.: On homogeneous real hypersurfaces in a complex projective space. *Osaka J. Math.* **10**, 495–506 (1973)
- [9] Takagi, R.: Real hypersurfaces in complex projective space with constant principal curvatures. *J. Math. Soc. Japan* **27**, 43–53 (1975)
- [10] Takagi, R.: Real hypersurfaces in complex projective space with constant principal curvatures II. *J. Math. Soc. Japan* **27**, 507–516 (1975)
- [11] Tanaka, N.: On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections. *Japan. J. Math.* **2**, 131–190 (1976)
- [12] Tanno, S.: Variational problems on contact Riemannian manifolds. *Trans. A.M.S.* **314**, 349–379 (1989)
- [13] Webster, S.M.: Pseudohermitian structures on a real hypersurface. *J. Diff. Geom.* **13**, 25–41 (1978)

Juan de Dios Pérez and David Pérez-López
Departamento de Geometría y Topología
Universidad de Granada
18071 Granada
Spain
e-mail: jdperez@ugr.es

David Pérez-López
e-mail: davidp1109@correo.ugr.es

Received: June 5, 2020.

Revised: December 24, 2020.

Accepted: August 3, 2021.