



A Class of Integral Operators that Fix Exponential Functions

Carlo Bardaro, Ilaria Mantellini , Gumrah Uysal and Basar Yilmaz

Abstract. In this paper we introduce a general class of integral operators that fix exponential functions, containing several recent modified operators of Gauss–Weierstrass, or Picard or moment type operators. Pointwise convergence theorems are studied, using a Korovkin-type theorem and a Voronovskaja-type formula is obtained.

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1. Introduction

The classical Bohman–Korovkin theorem (see [10, 22, 23]) is one of the pivotal results of approximation theory and several convergence theorems known in literature employ this basic tool. It states that a sequence of positive linear operators $T_n f$ acting on the set of the continuous functions over a compact interval of the real line converges to the identity operator only if it converges on a finite number of test functions which form a so-called Chebyshev system. A complete treatment of the Korovkin theorem can be found in the monographies [2, 3].

In this respect, if a sequence of operators $T_n f$ is such that $T_n \varphi = \varphi$ for some continuous function φ , then to obtain the convergence appears very simple, if the functions φ belong to a Chebyshev system. Thus, it is of interest to define sequences of operators that have this property. In literature the so-called King type operators, have this property, especially in case of discrete operators (see e.g. [21]). In this paper we define an entire class of

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positive linear integral operators which fix exponential functions. This kind of results were obtained by Agratini, Aral and Deniz [1], by Aral [6] and recently by Yilmaz, Uysal and Aral [26], by considering modifications of specific operators (Picard, Gauss–Weierstrass and moment-type operators). All the operators considered in these papers are special case of our present theory. Our approach includes also all the integral operators having a compactly supported kernel. Thus it applies for example to spline-type operators [11, 24]. Related results can be found in [16, 18, 19] and [17].

For our operators, we apply the Gadjiev version of Bohman–Korovkin theorem (see [14, 15]) in case of unbounded domains, for functions belonging to certain weighted spaces of continuous functions. Setting for every $n \in \mathbb{N}$ and $a > 0$,

$$A_{a,n} := \int_{-\infty}^{\infty} \exp(at)K_n(t)dt, \quad \lambda_n(x) := x - \frac{1}{a} \log A_{a,n},$$

we define

$$(T_n f)(x) = A_{a,n} \int_{-\infty}^{\infty} \exp(-at)f(\lambda_n(x) + t) K_n(t)dt,$$

where $\{K_n\}$ is a family of non-negative functions (kernel) belonging to a suitable function space. Setting for $a > 0$, $\exp_a(x) := e^{ax}$ for $x \in \mathbb{R}$, we show that $(T_n \exp_a)(x) = \exp(ax)$ and $(T_n \exp_{2a})(x) = \exp(2ax)$. Moreover we show that $T_n e_j \rightarrow e_j$ where $e_j(x) = x^j$, $j = 0, 1, 2$, so obtaining two uniform convergence theorems in weighted spaces of continuous functions. Then, using certain moduli of continuity we obtain certain quantitative estimates of the convergence and finally a Voronovskaja-type asymptotic formula. These kinds of asymptotic formulae are very useful also for applications, especially for discrete operators, like e.g. sampling-type operators (see, e.g., [11–13]). For integral operators of Mellin type see [7, 8]). The last section is devoted to several examples.

We recall here, that general approaches to convergence of integral operators was recently given in [9], and more recently in [20] and [25] in the frame of nonlinear operators.

2. Basic Notations

Let us denote by \mathbb{R} the set of real numbers, and by \mathbb{N} the set of the positive integers. By $L^1(\mathbb{R})$ we denote the space comprising all the Lebesgue integrable functions on \mathbb{R} with respect to the Lebesgue measure, and by $L^\infty(\mathbb{R})$ the space comprising all the essentially bounded functions on \mathbb{R} . By $C(\mathbb{R})$ we denote the space of all the continuous functions defined on \mathbb{R} . Finally, for $r \in \mathbb{N}$, we will say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally of class C^r at a point $x \in \mathbb{R}$ if there is a neighbourhood U of x such that f is $(r - 1)$ -fold continuously differentiable in U and $f^{(r)}(x)$ exists.

3. The Class of Integral Operators

For any $a > 0$, we define the function $\exp_a(t) := e^{at}$, for $t \in \mathbb{R}$.

We define now the function space

$$L_{\exp_a}(\mathbb{R}) := \{g : \mathbb{R} \rightarrow \mathbb{R} : g(\cdot)e^{a|\cdot|} \in L^1(\mathbb{R})\}.$$

Let now $\{K_n\}$ be a non negative kernel, that is, for every $n \in \mathbb{N}$, $K_n(t) \geq 0$, for every $t \in \mathbb{R}$, $K_n \in L^1(\mathbb{R})$, and

$$\|K_n\|_1 = \int_{-\infty}^{\infty} K_n(t)dt = 1.$$

In what follows we assume that $K_n \in L_{\exp_a}(\mathbb{R})$, for every sufficiently large $n \in \mathbb{N}$, namely for $n \geq n_0$, with $n_0 \in \mathbb{N}$. We set, for $n \geq n_0$,

$$A_{a,n} := \int_{-\infty}^{\infty} \exp_a(t)K_n(t)dt, \quad \lambda_n(x) := x - \frac{1}{a} \log A_{a,n}.$$

Note that from the assumption $K_n \in L_{\exp_a}(\mathbb{R})$, we have $A_{a,n} < +\infty$, $A_{-a,n} < +\infty$ and if moreover for every $n \in \mathbb{N}$, K_n is an even function, that is $K_n(t) = K_n(-t)$, for every $t \in \mathbb{R}$, one has easily $A_{a,n} = A_{-a,n}$. We introduce now the sequence of integral operators defined by

$$\begin{aligned} (T_n f)(x) &:= A_{a,n} \int_{-\infty}^{\infty} \exp_{-a}(t)f(\lambda_n(x) + t)K_n(t)dt \\ &= A_{a,n} \int_{-\infty}^{\infty} e^{-at} f(\lambda_n(x) + t)K_n(t)dt, \end{aligned} \tag{3.1}$$

for every function f belonging to the domain $D := \bigcap_{n \geq n_0} \text{dom } T_n$, where $\text{dom } T_n$ denotes the set of all the Lebesgue measurable functions f such that $(T_n|f|)(x)$ is convergent for almost all $x \in \mathbb{R}$.

Note that if $f \in L^\infty(\mathbb{R})$, then $f \in D$. Moreover, the functions \exp_a and \exp_{2a} both belong to D . We have the following

Proposition 3.1. *Under the established assumptions on the kernel $\{K_n\}$, we have*

$$(T_n \exp_a)(x) = \exp_a(x), \quad (T_n \exp_{2a})(x) = \exp_{2a}(x). \tag{3.2}$$

Proof. We have

$$\begin{aligned} (T_n \exp_a)(x) &= A_{a,n} \int_{-\infty}^{\infty} e^{-at} e^{a(\lambda_n(x)+t)} K_n(t)dt = A_{a,n} e^{a\lambda_n(x)} \int_{-\infty}^{\infty} K_n(t)dt \\ &= A_{a,n} e^{a\lambda_n(x)} = \exp_a(x). \end{aligned}$$

Analogously, we have

$$\begin{aligned} (T_n \exp_{2a})(x) &= A_{a,n} \int_{-\infty}^{\infty} e^{-at} e^{2a(\lambda_n(x)+t)} K_n(t)dt \\ &= A_{a,n} e^{2a\lambda_n(x)} \int_{-\infty}^{\infty} e^{at} K_n(t)dt \\ &= A_{a,n} \exp_{2a}(x) \frac{A_{a,n}}{(A_{a,n})^2} = \exp_{2a}(x), \end{aligned}$$

that is the assertion □

Remark 3.2. In order to establish a connection with classical integral operators of convolution type of the form

$$(T_n^0 f)(x) := \int_{-\infty}^{\infty} f(x+t)K_n(t)dt,$$

for continuous and bounded functions $f \in D$ we have

$$\lim_{a \rightarrow 0^+} (T_n f)(x) = (T_n^0 f)(x).$$

whenever

$$\lim_{a \rightarrow 0^+} \frac{1}{a} \log A_{a,n} = 0.$$

Indeed, under the above assumptions on the kernel $\{K_n\}$, it is easy to show that $\lim_{a \rightarrow 0^+} A_{a,n} = 1$, and $\lim_{a \rightarrow 0^+} \lambda_n(x) = x$. Thus under the assumptions on the function f the assertion follows by the Lebesgue theorem of dominated convergence.

4. Pointwise and Uniform Convergence

In this section we will study the pointwise convergence of $T_n f$ to f , where f belongs to a suitable weighted space of continuous functions, as introduced in [14], using a Korovkin-type theorem, established in [14] (see also [15]). In order to do that, we introduce the following constants

$$B_{-a,n} := \int_{-\infty}^{\infty} t e^{-at} K_n(t) dt$$

$$C_{-a,n} := \int_{-\infty}^{\infty} t^2 e^{-at} K_n(t) dt.$$

We call these constants the *exponential moments* of orders 1 and 2 respectively. At the same way, we refer to $A_{-a,n}$ as the exponential moment of order 0. We introduce the following subspace of $L_{\exp_a}(\mathbb{R})$, in which the above moments are well-defined

$$L_{\exp_a}^*(\mathbb{R}) := \{g \in L_{\exp_a}(\mathbb{R}) : (c + d|\cdot|)^2 e^{a|\cdot|} g(\cdot) \in L^1(\mathbb{R})\},$$

with c, d positive constants. If $K_n \in L_{\exp_a}^*(\mathbb{R})$, then $B_{-a,n}, C_{-a,n}$ exist finite.

Let us introduce now the test functions $e_0(t) = 1, e_1(t) = t, e_2(t) = t^2, t \in \mathbb{R}$. Obviously, $e_j \in D$, for $j = 0, 1, 2$.

We have the following

Proposition 4.1. *Let $K_n \in L_{\exp_a}^*(\mathbb{R})$, for sufficiently large $n \in \mathbb{N}$. Then we have*

$$(T_n e_0)(x) = A_{a,n} A_{-a,n}$$

$$(T_n e_1)(x) = A_{a,n} A_{-a,n} x - \frac{1}{a} A_{a,n} A_{-a,n} \log A_{a,n} + A_{a,n} B_{-a,n}$$

and

$$(T_n e_2)(x) = A_{a,n} A_{-a,n} x^2 + \left(2A_{a,n} B_{-a,n} - 2A_{a,n} A_{-a,n} \frac{1}{a} \log A_{a,n} \right) x + A_{a,n} A_{-a,n} \frac{1}{a^2} \log^2 A_{a,n} + A_{a,n} C_{-a,n} - 2A_{a,n} B_{-a,n} \frac{1}{a} \log A_{a,n}.$$

Proof. We have

$$(T_n e_0)(x) = A_{a,n} \int_{-\infty}^{\infty} e^{-at} K_n(t) dt = A_{a,n} A_{-a,n}.$$

Next,

$$\begin{aligned} (T_n e_1)(x) &= A_{a,n} \int_{-\infty}^{\infty} e^{-at} (\lambda_n(x) + t) K_n(t) dt \\ &= A_{a,n} \lambda_n(x) \int_{-\infty}^{\infty} e^{-at} K_n(t) dt + A_{a,n} \int_{-\infty}^{\infty} t e^{-at} K_n(t) dt \\ &= A_{a,n} A_{-a,n} x - A_{a,n} A_{-a,n} \frac{1}{a} \log A_{a,n} + A_{a,n} B_{-a,n}. \end{aligned}$$

Finally,

$$\begin{aligned} (T_n e_2)(x) &= A_{a,n} \int_{-\infty}^{\infty} e^{-at} (\lambda_n(x) + t)^2 K_n(t) dt \\ &= A_{a,n} (\lambda_n(x))^2 A_{-a,n} + A_{a,n} C_{-a,n} + 2A_{a,n} B_{-a,n} \lambda_n(x) \\ &= A_{a,n} A_{-a,n} x^2 + 2A_{a,n} \left(B_{-a,n} - \frac{1}{a} A_{-a,n} \log A_{a,n} \right) x \\ &\quad + A_{a,n} \left(A_{-a,n} \frac{1}{a^2} \log^2 A_{a,n} + C_{-a,n} - \frac{2}{a} B_{-a,n} \log A_{a,n} \right). \end{aligned}$$

The proof is completed □

Corollary 4.2. *Under the assumptions of Proposition 4.1, if moreover*

$$\lim_{n \rightarrow +\infty} A_{a,n} = \lim_{n \rightarrow +\infty} A_{-a,n} = 1, \tag{4.1}$$

and

$$\lim_{n \rightarrow +\infty} B_{-a,n} = \lim_{n \rightarrow +\infty} C_{-a,n} = 0,$$

then

$$\lim_{n \rightarrow +\infty} (T_n e_j)(x) = e_j(x) \quad (j = 0, 1, 2).$$

Proof. It is an immediate consequence of Proposition 4.1 □

Using Proposition 4.1 and Corollary 4.2, we now give a uniform convergence result for the operators $(T_n f)$ when f belongs to suitable weighted spaces of continuous functions. The key tool is a Korovkin-type theorem proved in [14] (see also [15]).

Given a continuous, strictly increasing function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, we define the function $\rho(x) := 1 + \varphi^2(x)$, and assume that $\lim_{x \rightarrow \pm\infty} \rho(x) = +\infty$.

We will consider two particular cases: $\varphi_1(x) = x$, and $\varphi_2(x) = \exp_a(x)$, and we set $\rho_1(x) := 1 + x^2$ and $\rho_2(x) := 1 + \exp_{2a}(x)$.

Let us consider the spaces, for $j = 1, 2$

$$C_{\rho_j}^0(\mathbb{R}) := \left\{ f \in C(\mathbb{R}) : \lim_{x \rightarrow \pm\infty} \frac{f(x)}{\rho_j(x)} = \ell_f \in \mathbb{R} \right\}.$$

Note that if $f \in C_{\rho_j}^0(\mathbb{R})$, then $|f(x)| \leq M_f \rho_j(x)$, for a suitable constant $M_f > 0$ and $x \in \mathbb{R}$. We define a norm on the space $C_{\rho_j}^0(\mathbb{R})$ on setting

$$\|f\|_{\rho_j} := \left\| \frac{f}{\rho_j} \right\|_{\infty}.$$

We are ready to prove the main theorem of this section.

Theorem 4.3. *Let $j = 1, 2$ and let $f \in C_{\rho_j}^0(\mathbb{R})$. Then, under the assumptions of Proposition 4.1 and Corollary 4.2, we have*

$$\lim_{n \rightarrow +\infty} \|T_n f - f\|_{\rho_j} = 0.$$

Proof. First, consider the case $j = 1$. The test functions e_j obviously belong to $C_{\rho_1}^0(\mathbb{R})$ and moreover

$$\left\| \frac{T_n e_0 - e_0}{\rho_1} \right\|_{\infty} = \left\| \frac{A_{a,n} A_{-a,n} - 1}{\rho_1} \right\|_{\infty} \leq |A_{a,n} A_{-a,n} - 1|, \tag{4.2}$$

and the last term tends to 0 as $n \rightarrow +\infty$. Next,

$$\begin{aligned} \left\| \frac{T_n e_1 - e_1}{\rho_1} \right\|_{\infty} &\leq |A_{a,n} A_{-a,n} - 1| \sup_{x \in \mathbb{R}} \frac{|x|}{x^2 + 1} \\ &\quad + \frac{1}{a} A_{a,n} A_{-a,n} |\log A_{a,n}| + A_{a,n} |B_{-a,n}|, \end{aligned}$$

and so

$$\lim_{n \rightarrow +\infty} \left\| \frac{T_n e_1 - e_1}{\rho_1} \right\|_{\infty} = 0.$$

Analogously, one can see that

$$\lim_{n \rightarrow +\infty} \left\| \frac{T_n e_2 - e_2}{\rho_1} \right\|_{\infty} = 0.$$

For $j = 2$, taking into account of (4.2) and Proposition 3.1, we obtain again the convergence on the test functions $\exp^k(ax)$, for $k = 0, 1, 2$. Applying Theorem 2 in [14] we obtain the assertion \square

5. Quantitative Estimates

For $K_n \in L_{\exp_a}^*(\mathbb{R})$, we define the *absolute exponential moment of order 1* of K_n as

$$\tilde{B}_{-a,n} := \int_{-\infty}^{\infty} e^{-at} |t| K_n(t) dt.$$

In the spaces $C_{\rho_j}^0(\mathbb{R})$, $j = 1, 2$ we can define various modulus of continuity. We begin with an estimate of the convergence expressed by Theorem 4.3, in terms of the classical modulus of continuity ω defined by

$$\omega(f, \delta) := \sup_{|h| \leq \delta} |f(x+h) - f(x)| \quad (\delta > 0).$$

As it is well-known, for any uniformly continuous function f one has $\lim_{\delta \rightarrow 0} \omega(f, \delta) = 0$. We have the following estimate

Theorem 5.1. *Let $K_n \in L_{\exp_a}^*(\mathbb{R})$, for sufficiently large values of $n \in \mathbb{N}$, and let $f \in C_{\rho_j}^0(\mathbb{R})$. Then for every $\delta > 0$, we have*

$$\begin{aligned} \|T_n f - f\|_{\rho_j} &\leq \omega(f, \delta) A_{a,n} \left(A_{-a,n} + \frac{1}{\delta} \tilde{B}_{-a,n} + \frac{1}{a\delta} A_{-a,n} |\log A_{a,n}| \right) \\ &\quad + \|f\|_{\rho_j} |A_{a,n} A_{-a,n} - 1|. \end{aligned}$$

Proof. For every $x \in \mathbb{R}$ and $\delta > 0$ we have

$$\begin{aligned} |(T_n f)(x) - f(x)| &\leq A_{a,n} \int_{-\infty}^{\infty} e^{-at} \omega \left(f, \left| t - \frac{1}{a} \log A_{a,n} \right| \right) K_n(t) dt \\ &\quad + |f(x)| |A_{a,n} A_{-a,n} - 1| =: I_1 + I_2. \end{aligned}$$

Thus we have to estimate only I_1 . In order to do that, we use the following well-known property of ω (see e.g. [4])

$$\omega(f, \lambda\delta) \leq (1 + \lambda) \omega(f, \delta) \quad (\lambda, \delta > 0).$$

Setting $\lambda := \frac{|t - \frac{1}{a} \log A_{a,n}|}{\delta}$, we have

$$\begin{aligned} I_1 &\leq A_{a,n} \omega(f, \delta) \int_{-\infty}^{\infty} e^{-at} \left(1 + \frac{|t - \frac{1}{a} \log A_{a,n}|}{\delta} \right) K_n(t) dt \\ &\leq \omega(f, \delta) A_{a,n} \left(A_{-a,n} + \frac{1}{\delta} \int_{-\infty}^{\infty} e^{-at} (|t| + a^{-1} |\log A_{a,n}|) K_n(t) dt \right) \\ &\leq \omega(f, \delta) A_{a,n} \left(A_{-a,n} + \frac{1}{\delta} \tilde{B}_{-a,n} + \frac{1}{a\delta} A_{-a,n} |\log A_{a,n}| \right). \end{aligned}$$

Passing to norm, we get the desired result

$$\begin{aligned} \|T_n f - f\|_{\rho_j} &\leq \omega(f, \delta) A_{a,n} \left(A_{-a,n} + \frac{1}{\delta} \tilde{B}_{-a,n} + \frac{1}{a\delta} A_{-a,n} |\log A_{a,n}| \right) \\ &\quad + \|f\|_{\rho_j} |A_{a,n} A_{-a,n} - 1| \end{aligned}$$

□

Corollary 5.2. *Let the assumptions of Theorem 5.1 be satisfied, and (4.1) holds. If moreover there is $\alpha > 0$ such that $(n^\alpha \tilde{B}_{-a,n})_n$, and $(n^\alpha |\log A_{a,n}|)_n$ are bounded sequences, then there exists an absolute constant $M > 0$ such that*

$$\|T_n f - f\|_{\rho_j} \leq M \omega \left(f, \frac{1}{n^\alpha} \right) + |A_{a,n} A_{-a,n} - 1| \|f\|_{\rho_j},$$

for sufficiently large values of $n \in \mathbb{N}$.

Proof. It is a consequence of Theorem 5.1 on setting $\delta = n^{-\alpha}$ □

Remark 5.3. Note that, under the assumptions of Corollary 5.2, if the function f satisfies a Lipschitz condition of order 1, and $|A_{a,n} A_{-a,n} - 1| = \mathcal{O}(n^{-\alpha})$ as $n \rightarrow +\infty$, then

$$\|T_n f - f\|_{\rho_j} = \mathcal{O}(n^{-\alpha}) \quad (n \rightarrow +\infty).$$

Now we state another estimate using a suitable weighted modulus of continuity. In order to do that, we introduce the exponential moment of order 4 of $K_n \in L^*_{\text{exp}_a}(\mathbb{R})$ setting

$$E_{-a,n} := \int_{-\infty}^{\infty} e^{-at} t^4 K_n(t) dt.$$

In order to establish rate of convergence, we will use special kind of modulus of continuity $\tilde{\omega}$ which is compatible with the space $C^0_{\rho_1}(\mathbb{R})$. This weighted modulus of continuity was first introduced in [27] for $f \in C^0_{\rho_1}(\mathbb{R}^+)$, then considered in [5] for $f \in C^0_{\rho_1}(\mathbb{R})$ as follows:

$$\tilde{\omega}(f, \delta) = \sup_{x \in \mathbb{R}, |h| \leq \delta} \frac{|f(x+h) - f(x)|}{1 + (h+x)^2}. \tag{5.1}$$

For any function $f \in C^0_{\rho_1}(\mathbb{R})$, $m \in \mathbb{N}$ and $\lambda, \delta \in \mathbb{R}^+$, $\tilde{\omega}$ has the following properties (see [5, 27]):

1. $\tilde{\omega}(f, \delta)$ is a monotonically increasing function of δ ,
2. $\lim_{\delta \rightarrow 0^+} \tilde{\omega}(f, \delta) = 0$,
3. $\tilde{\omega}(f, m\delta) \leq m\tilde{\omega}(f, \delta)$, for every $m \in \mathbb{N}$,
4. $\tilde{\omega}(f, \lambda\delta) \leq (1 + \lambda)\tilde{\omega}(f, \delta)$ for every $\lambda > 0$.

Theorem 5.4. *Let $K_n \in L^*_{\text{exp}_a}$ be such that $E_{-a,n}$ exists finite. For $f \in C^0_{\rho_1}(\mathbb{R})$, there holds*

$$\begin{aligned} & \|T_n f - f\|_{\rho_1} \\ & \leq 16A_{a,n} \tilde{\omega}\left(f, \sqrt{C_{-a,n}}\right) \\ & \quad \times \left\{ C_{-a,n} + (1 + C_{-a,n}) A_{-a,n} + \sqrt{E_{-a,n} + A_{-a,n} (1 + C^2_{-a,n})} \right\} \\ & \quad + \|f\|_{\rho_1} |A_{a,n} A_{-a,n} - 1| \end{aligned}$$

provided that $|a^{-1} \log A_{a,n}| \leq \sqrt{C_{-a,n}}$ for sufficiently large $n \in \mathbb{N}$.

Proof. In view of Proposition 4.1, we have

$$\begin{aligned} |(T_n f)(x) - f(x)| & \leq A_{a,n} \left| \int_{-\infty}^{\infty} e^{-at} f(\lambda_n(x) + t) K_n(t) dt - f(x) \int_{-\infty}^{\infty} e^{-at} K_n(t) dt \right| \\ & \quad + |f(x)| |(T_n e_0)(x) - 1| \\ & = A_{a,n} \left| \int_{-\infty}^{\infty} [f(\lambda_n(x) + t) - f(x)] e^{-at} K_n(t) dt \right| \\ & \quad + |f(x)| |A_{a,n} A_{-a,n} - 1|. \end{aligned}$$

For any $\delta > 0$, properties of $\tilde{\omega}$ enable us to write

$$|f(\lambda_n(x) + t) - f(x)| = |f(x + t - a^{-1} \log A_{a,n}) - f(x)|$$

$$\begin{aligned} &\leq \left(1 + (x + t - a^{-1} \log A_{a,n})^2\right) \tilde{\omega}(f, |t - a^{-1} \log A_{a,n}|) \\ &\leq \left(1 + (x + t - a^{-1} \log A_{a,n})^2\right) \left(1 + \frac{|t - a^{-1} \log A_{a,n}|}{\delta}\right) \tilde{\omega}(f, \delta). \end{aligned}$$

In view of this observation, there holds

$$\begin{aligned} |(T_n f)(x) - f(x)| &\leq \tilde{\omega}(f, \delta) A_{a,n} \\ &\times \int_{-\infty}^{\infty} \left(1 + (x + t - a^{-1} \log A_{a,n})^2\right) \left(1 + \frac{|t - a^{-1} \log A_{a,n}|}{\delta}\right) e^{-at} K_n(t) dt \\ &+ |f(x)| |A_{a,n} A_{-a,n} - 1| \leq \tilde{\omega}(f, \delta) A_{a,n} \\ &\times \int_{-\infty}^{\infty} \left(1 + \left(x + t - \frac{1}{a} \log A_{a,n}\right)^2\right) \left(1 + \frac{|a^{-1} \log A_{a,n}|}{\delta}\right) e^{-at} K_n(t) dt \\ &+ \tilde{\omega}(f, \delta) A_{a,n} \int_{-\infty}^{\infty} \left(1 + \left(x + t - \frac{1}{a} \log A_{a,n}\right)^2\right) \frac{|t|}{\delta} e^{-at} K_n(t) dt \\ &+ |f(x)| |A_{a,n} A_{-a,n} - 1| := J_1 + J_2 + |f(x)| |A_{a,n} A_{-a,n} - 1|. \end{aligned}$$

For J_1 , using the well-known inequality $|a - b|^p \leq 2^{p-1} (|a|^p + |b|^p)$, $a, b \in \mathbb{R}$, $p \geq 1$, we have

$$\begin{aligned} J_1 &\leq \tilde{\omega}(f, \delta) A_{a,n} \\ &\times \left(1 + \frac{|a^{-1} \log A_{a,n}|}{\delta}\right) \{A_{-a,n} + 2C_{-a,n} + 2(x - a^{-1} \log A_{a,n})^2 A_{-a,n}\} \\ &\leq 4A_{a,n} \tilde{\omega}(f, \delta) \left(1 + \frac{|a^{-1} \log A_{a,n}|}{\delta}\right) \{C_{-a,n} + (1 + x^2 + a^{-2} \log^2 A_{a,n}) A_{-a,n}\}. \end{aligned}$$

By the Cauchy–Schwarz inequality for J_2 , we have

$$\begin{aligned} J_2 &= \tilde{\omega}(f, \delta) A_{a,n} \int_{-\infty}^{\infty} (1 + (x + t - a^{-1} \log A_{a,n})^2) \frac{|t|}{\delta} e^{-at} K_n(t) dt \\ &\leq \tilde{\omega}(f, \delta) A_{a,n} \int_{-\infty}^{\infty} (1 + (x + t - a^{-1} \log A_{a,n})^2)^2 e^{-at} K_n(t) dt)^{1/2} \frac{(C_{-a,n})^{1/2}}{\delta}. \end{aligned}$$

Repeated application of the inequality $|a - b|^p \leq 2^{p-1} (|a|^p + |b|^p)$, $a, b \in \mathbb{R}$, $p \geq 1$ yields

$$\begin{aligned} \left(1 + (x + t - a^{-1} \log A_{a,n})^2\right)^2 &\leq (1 + 2(t^2 + \lambda_n^2(x)))^2 \\ &\leq 2 \left(1 + 4(t^2 + \lambda_n^2(x))^2\right) \leq 2 + 16(t^4 + \lambda_n^4(x)) \\ &= 2 + 16t^4 + 16(x - a^{-1} \log A_{a,n})^4 \\ &\leq 2 + 16t^4 + 128(x^4 + a^{-4} \log^4 A_{a,n}). \end{aligned}$$

Using above inequality, we have

$$J_2 \leq 16 \tilde{\omega}(f, \delta) A_{a,n} (E_{-a,n} + A_{-a,n} (1 + x^4 + a^{-4} \log^4 A_{a,n}))^{1/2} \frac{(C_{-a,n})^{1/2}}{\delta}.$$

Collecting all inequalities we get

$$|(T_n f)(x) - f(x)|$$

$$\begin{aligned} &\leq 4A_{a,n}\tilde{\omega}(f, \delta) \left(1 + \frac{|a^{-1} \log A_{a,n}|}{\delta}\right) \{C_{-a,n} + (1 + x^2 + a^{-2} \log^2 A_{a,n}) A_{-a,n}\} \\ &\quad + 16\tilde{\omega}(f, \delta) A_{a,n} (E_{-a,n} + A_{-a,n} (1 + x^4 + a^{-4} \log^4 A_{a,n}))^{1/2} \frac{(C_{-a,n})^{1/2}}{\delta} \\ &\quad + |f(x)| |A_{a,n}A_{-a,n} - 1|. \end{aligned}$$

Choosing $\delta = (C_{-a,n})^{1/2}$ with $|a^{-1} \log A_{a,n}| \leq \delta$ for sufficiently large n , dividing both sides by $1 + x^2$ and taking supremum over all x , we obtain

$$\begin{aligned} &\|T_n f - f\|_{\rho_1} \\ &\leq 16A_{a,n}\tilde{\omega}\left(f, \sqrt{C_{-a,n}}\right) \{C_{-a,n} + (1 + a^{-2} \log^2 A_{a,n}) A_{-a,n} \\ &\quad + \sqrt{E_{-a,n} + A_{-a,n} (1 + a^{-4} \log^4 A_{a,n})}\} + |f(x)| |A_{a,n}A_{-a,n} - 1| \\ &\leq 16A_{a,n}\tilde{\omega}\left(f, \sqrt{C_{-a,n}}\right) \{C_{-a,n} + (1 + C_{-a,n}) A_{-a,n} \\ &\quad + \sqrt{E_{-a,n} + A_{-a,n} (1 + (C_{-a,n})^2)}\} \\ &\quad + \|f\|_{\rho_1} |A_{a,n}A_{-a,n} - 1| \end{aligned}$$

that is the assertion □

Remark 5.5. The estimate of Theorem 5.4 gives an evaluation of the convergence in terms of the modulus of continuity with parameter $\sqrt{C_{-a,n}}$, under the further assumptions that $C_{-a,n} \rightarrow 0$ and $|A_{a,n}A_{-a,n} - 1| \rightarrow 0$ as $n \rightarrow +\infty$. Thus one can obtain a corresponding corollary, as for Theorem 5.1 (see Corollary 5.2 and Remark 5.3).

6. An Asymptotic Formula

Here we establish two asymptotic formulae of Voronovskaya type for functions $f \in C^0_{\rho_j}(\mathbb{R})$, $j = 1, 2$, which are locally of class C^2 at a point x . These kinds of formulae give an exact evaluation of the order of pointwise convergence. We examine the case $j = 1$.

Theorem 6.1. *Let $f \in C^0_{\rho_1}(\mathbb{R})$ be locally of class C^2 at a point $x \in \mathbb{R}$. Let $K_n \in L^*_{\exp_a}(\mathbb{R})$, for sufficiently large values of $n \in \mathbb{N}$, be a kernel satisfying the assumptions of Corollary 4.2. Assume that there exist $\alpha > 0$ such that the kernel $\{K_n\}$ satisfies further the following conditions:*

- (i) $\lim_{n \rightarrow +\infty} n^\alpha (A_{a,n} A_{-a,n} - 1) = \ell_0$, $\lim_{n \rightarrow +\infty} n^\alpha B_{-a,n} = \ell_1$,
 $\lim_{n \rightarrow +\infty} n^\alpha C_{-a,n} = \ell_2$ and $\lim_{n \rightarrow +\infty} n^\alpha \frac{1}{a} \log A_{a,n} = \ell_3$, with $\ell_j \in \mathbb{R}$,
 for $j = 0, 1, 2, 3$.
- (ii) For every $\eta > 0$,

$$\lim_{n \rightarrow +\infty} n^\alpha \int_{|t| \geq \eta} e^{-at} \left(t - \frac{1}{a} \log A_{a,n}\right)^2 K_n(t) dt = 0.$$

Then,

$$\lim_{n \rightarrow +\infty} n^\alpha [(T_n f)(x) - f(x)] = \ell_0 f(x) + (\ell_1 - \ell_3) f'(x) + \frac{\ell_2}{2} f''(x). \tag{6.1}$$

Proof. Since f has a polynomial growth of order 2, and locally of class C^2 at the point x , using a local Taylor formula of the second order, we can write

$$\begin{aligned} f(\lambda_n(x) + t) &= f\left(x - \frac{1}{a} \log A_{a,n} + t\right) = f(x) + f'(x) \left(t - \frac{1}{a} \log A_{a,n}\right) \\ &\quad + \frac{f''(x)}{2} \left(t - \frac{1}{a} \log A_{a,n}\right)^2 + r_x \left(t - \frac{1}{a} \log A_{a,n}\right) \\ &\quad \times \left(t - \frac{1}{a} \log A_{a,n}\right)^2, \end{aligned} \tag{6.2}$$

where $r_x(y)$ is a bounded function such that $\lim_{y \rightarrow 0} r_x(y) = 0$.

We write

$$\begin{aligned} (T_n f)(x) - f(x) &= A_{a,n} \int_{-\infty}^{\infty} e^{-at} [f(\lambda_n(x) + t) - f(x)] K_n(t) dt \\ &\quad + (A_{a,n} A_{-a,n} - 1) f(x) =: I_1 + I_2. \end{aligned}$$

As to I_2 we have by assumptions that $n^\alpha I_2 \rightarrow \ell_0 f(x)$ as $n \rightarrow +\infty$. Therefore we evaluate now the term I_1 . Inserting (6.2) in I_1 , we can write

$$\begin{aligned} I_1 &= A_{a,n} \left(B_{-a,n} - A_{-a,n} \frac{1}{a} \log A_{a,n} \right) f'(x) \\ &\quad + A_{a,n} \left(C_{-a,n} + \frac{A_{-a,n}}{a^2} \log^2 A_{a,n} - \frac{2}{a} B_{-a,n} \log A_{a,n} \right) \frac{f''(x)}{2} + R_x \\ &=: I_1^1 + I_1^2 + R_x, \end{aligned} \tag{6.3}$$

where

$$R_x := A_{a,n} \int_{-\infty}^{\infty} e^{-at} r_x \left(t - \frac{1}{a} \log A_{a,n} \right) \left(t - \frac{1}{a} \log A_{a,n} \right)^2 K_n(t) dt.$$

Now,

$$\lim_{n \rightarrow +\infty} n^\alpha I_1^1 = (\ell_1 - \ell_3) f'(x), \quad \lim_{n \rightarrow +\infty} n^\alpha I_1^2 = \frac{\ell_2}{2} f''(x).$$

Thus we have to estimate the remainder term R_x .

Since $\lim_{y \rightarrow 0} r_x(y) = 0$, given an arbitrary $\varepsilon > 0$, there is $\delta \in]0, 1[$ such that $|r_x(y)| < \varepsilon$ whenever $|y| \leq \delta$. We take now an index \bar{n} such that for $n \geq \bar{n}$, one has $a^{-1} |\log A_{a,n}| < \delta/2$. Thus for $|t| < \delta/2$, we have also $|t - a^{-1} \log A_{a,n}| < \delta$. Thus

$$\left| r_x \left(t - \frac{1}{a} \log A_{a,n} \right) \right| < \varepsilon \quad (|t| < \delta/2). \tag{6.4}$$

Writing

$$\begin{aligned} R_x &= A_{a,n} \left\{ \int_{-\delta/2}^{\delta/2} + \int_{|t| \geq \delta/2} \right\} e^{-at} r_x \left(t - \frac{1}{a} \log A_{a,n} \right) \left(t - \frac{1}{a} \log A_{a,n} \right)^2 K_n(t) dt \\ &=: R_1 + R_2, \end{aligned}$$

we have, by (6.4),

$$|R_1| \leq A_{a,n} \varepsilon \left(C_{-a,n} + A_{-a,n} \frac{1}{a^2} \log^2 A_{a,n} - \frac{2B_{-a,n}}{a} \log A_{a,n} \right).$$

Therefore using the assumptions (i) we have, for a suitable absolute positive constant M ,

$$\limsup_{n \rightarrow +\infty} n^\alpha |R_1| \leq M\varepsilon.$$

As to the term R_2 , using the boundedness of r_x we can write

$$|R_2| \leq A_{a,n} \|r_x\|_\infty \int_{|t| \geq \delta/2} e^{-at} \left(t - \frac{1}{a} \log A_{a,n} \right)^2 K_n(t) dt,$$

and by (ii) we obtain

$$\lim_{n \rightarrow +\infty} n^\alpha |R_2| = 0.$$

This implies

$$\limsup_{n \rightarrow +\infty} n^\alpha |R_x| \leq M\varepsilon,$$

that is $|R_x| \rightarrow 0$ as $n \rightarrow +\infty$. Thus the theorem is completely proved \square

Theorem 5.4 works perfectly in several particular examples, as we will see in the next section. But in certain situations, as for example, the moment-type operators, formula (6.1) becomes

$$\lim_{n \rightarrow +\infty} n^\alpha [(T_n f)(x) - f(x)] = 0,$$

for a suitable constant α for which all the assumptions are satisfied. In case of the moment-type operator, we have $\alpha = 1$. If we try to take $\alpha = 2$ some of the assumptions (i) of Theorem 5.4 are not satisfied. This result is however interesting, but it gives no exact information about the pointwise order of approximation at a point x . Therefore, we will formulate a slight generalization of the above theorem, in order to include also the case of moment-type operators, by changing assumptions (i). We have the following

Theorem 6.2. *Let $f \in C_{\rho_1}^0(\mathbb{R})$ be locally of class C^2 at a point $x \in \mathbb{R}$. Let $K_n \in L_{\exp_a}^*(\mathbb{R})$, for sufficiently large values of $n \in \mathbb{N}$, be a kernel satisfying the assumptions of Corollary 4.2. Assume that there exist $\alpha > 0$ such that the kernel $\{K_n\}$ satisfies assumption (ii) of Theorem 5.4 and the following conditions:*

- (j) $\lim_{n \rightarrow +\infty} n^\alpha (A_{a,n} A_{-a,n} - 1) = \ell_0,$
 $\lim_{n \rightarrow +\infty} n^\alpha (B_{-a,n} - A_{-a,n} \frac{1}{a} \log A_{a,n}) = \lambda_1,$
- (jj) $\lim_{n \rightarrow +\infty} n^\alpha \left(C_{-a,n} + \frac{A_{-a,n}}{a^2} \log^2 A_{a,n} - \frac{2}{a} B_{-a,n} \log A_{a,n} \right) = \lambda_2$

Then,

$$\lim_{n \rightarrow +\infty} n^\alpha [(T_n f)(x) - f(x)] = \ell_0 f(x) + \lambda_1 f'(x) + \frac{\lambda_2}{2} f''(x). \tag{6.5}$$

Proof. The proof is clearly exactly the same. \square

7. Examples

In this section we discuss some examples of kernels $\{K_n\}$ for which the theory developed can be applied.

(1) *The Gauss–Weierstrass kernel*

For $n \in \mathbb{N}$, let us consider the kernel $\{K_n\}$ with

$$K_n(t) = \sqrt{\frac{n}{\pi}} e^{-nt^2} \quad (t \in \mathbb{R}).$$

This kernel is defined by even functions. First we evaluate the exponential moments of orders 0, 1, 2. In order to do that, we first calculate the coefficients $A_{a,n}$ using a differentiation under the integral. By solving a simple first order linear differential equation, we obtain

$$A_{a,n} = A_{-a,n} = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{at} e^{-nt^2} dt = e^{a^2/(4n)}.$$

Hence $A_{a,n} A_{-a,n} = e^{a^2/(2n)}$. Taking $\alpha = 1$, we obtain

$$\lim_{n \rightarrow +\infty} n(A_{a,n} A_{-a,n} - 1) = \lim_{n \rightarrow +\infty} n(e^{a^2/(2n)} - 1) = \frac{a^2}{2}.$$

Moreover, using partial integrations, one has

$$B_{-a,n} = -\frac{a}{2n} e^{a^2/4n}, \quad C_{-a,n} = \frac{1}{2n} A_{a,n} + \frac{a^2}{4n^2} A_{a,n},$$

and so

$$\lim_{n \rightarrow +\infty} nB_{-a,n} = \frac{1}{2}, \quad \lim_{n \rightarrow +\infty} nC_{-a,n} = \frac{1}{2}.$$

Next, we have

$$\lim_{n \rightarrow +\infty} n \log A_{a,n} = \frac{a^2}{4},$$

therefore we have also

$$\lim_{n \rightarrow +\infty} n \log^2 A_{a,n} = 0.$$

Therefore, $\ell_1 - \ell_3 = -3a/4$, $\ell_2 = 1/2$. Now we prove that the Gauss–Weierstrass kernel satisfies also assumption (ii) of Theorem 5.4. Let $\eta > 0$ be fixed. Let us set

$$J := n \int_{|t| \geq \eta} e^{-at} \left(t - \frac{1}{a} \log A_{a,n} \right)^2 K_n(t) dt.$$

Then

$$\begin{aligned} J &\leq n \int_{|t| \geq \eta} e^{-at} t^2 K_n(t) dt + n \left(\frac{1}{a^2} \log^2 A_{a,n} \right) \int_{|t| \geq \eta} e^{-at} K_n(t) dt \\ &\quad + 2n \frac{1}{a} |\log A_{a,n}| \int_{|t| \geq \eta} e^{-at} |t| K_n(t) dt =: J_1 + J_2 + J_3. \end{aligned}$$

As to J_1 we can write

$$\begin{aligned} J_1 &= \frac{n\sqrt{n}}{\sqrt{\pi}} \int_{|t|\geq\eta} e^{-at^2} e^{-nt^2} dt \\ &= \frac{n\sqrt{n}}{\sqrt{\pi}} \left\{ \int_{\eta}^{\infty} e^{-at^2} e^{-nt^2} dt + \int_{\eta}^{\infty} e^{at^2} e^{-nt^2} dt \right\} \\ &=: J_1^1 + J_1^2. \end{aligned}$$

For J_1^1 we have by a suitable substitution,

$$J_1^1 \leq \frac{e^{-a\eta}}{\sqrt{\pi}} \int_{\sqrt{n}\eta}^{\infty} v^2 e^{-v^2} dv,$$

and so by the absolute continuity of the Lebesgue integral, we obtain

$$\lim_{n \rightarrow +\infty} J_1^1 = 0.$$

Analogously,

$$J_1^2 \leq \frac{1}{\sqrt{\pi}} \int_{\sqrt{n}\eta}^{\infty} e^{av} v^2 e^{-v^2} dv,$$

and again, since the integrand in the right-hand side is Lebesgue integrable, we obtain $J_1^2 \rightarrow 0$ as $n \rightarrow +\infty$. Next, we evaluate J_2 . We have immediately

$$J_2 \leq \frac{n}{a^2} \log^2 A_{a,n} A_{-a,n} \rightarrow 0, \quad (n \rightarrow +\infty).$$

Finally, we evaluate J_3 . Using the same reasonings as for the estimate of J_1 , we have

$$\begin{aligned} J_3 &\leq \frac{a\sqrt{n}}{2\sqrt{\pi}} \left\{ e^{-a\eta} \int_{\eta}^{\infty} t e^{-nt^2} dt + \int_{\eta}^{\infty} e^{at} t e^{-nt^2} dt \right\} \\ &=: J_3^1 + J_3^2. \end{aligned}$$

As for J_1 we obtain easily that $J_3^1 \rightarrow 0, J_3^2 \rightarrow 0$ as $n \rightarrow +\infty$. Concluding, we obtain assumption (ii) of Theorem 5.4. Therefore all the assumptions introduced are satisfied with $\alpha = 1$. The asymptotic formula of Theorem 5.4 reads

$$\lim_{n \rightarrow +\infty} n((T_n f)(x) - f(x)) = \frac{a^2}{2} f(x) - \frac{3a}{4} f'(x) + \frac{1}{4} f''(x),$$

which is a result of [26].

(2) *The Picard Kernel*

For $n \in \mathbb{N}$, let us consider the kernel $\{K_n\}$ with

$$K_n(t) = \frac{\sqrt{n}}{2} e^{-\sqrt{n}|t|} \quad (t \in \mathbb{R}).$$

Also this kernel is defined by even functions. Let us evaluate the exponential moments of order 0, 1, 2. First we have

$$A_{a,n} = A_{-a,n} = \frac{n}{n - a^2}, \text{ and } A_{a,n}^2 = \left(\frac{n}{n - a^2} \right)^2$$

and from sufficiently large values of n , $A_{a,n}$ is positive. Moreover

$$\lim_{n \rightarrow +\infty} n(A_{a,n}A_{-a,n} - 1) = 2a^2, \quad \lim_{n \rightarrow +\infty} n \log A_{a,n} = a^2.$$

Now, we evaluate the moment $B_{-a,n}$. Using elementary calculation, based on partial integration, we can see that

$$B_{-a,n} = -\frac{2an}{(n-a^2)^2}, \quad C_{-a,n} = \sqrt{n} \left(\frac{1}{(a+\sqrt{n})^3} + \frac{1}{(\sqrt{n}-a)^3} \right).$$

Therefore,

$$\lim_{n \rightarrow +\infty} nB_{-a,n} = -2a, \quad \lim_{n \rightarrow +\infty} nC_{-a,n} = 2.$$

Finally we check assumption (ii) of Theorem 5.4. We proceed as in the previous example. Let $\eta > 0$ be fixed and set

$$J := n \int_{|t| \geq \eta} e^{-at} \left(t - \frac{1}{a} \log A_{a,n} \right)^2 K_n(t) dt.$$

We have

$$J \leq \frac{n\sqrt{n}}{2} \left\{ \int_{|t| \geq \eta} e^{-at} t^2 e^{-\sqrt{n}|t|} dt + \frac{1}{a^2} \log^2 \frac{n}{n-a^2} \int_{|t| \geq \eta} e^{-at} e^{-\sqrt{n}|t|} dt + \frac{2}{a} \log \frac{n}{n-a^2} \int_{|t| \geq \eta} e^{-at} |t| e^{-\sqrt{n}|t|} dt \right\} =: J_1 + J_2 + J_3.$$

Using now elementary calculations, based on suitable substitutions, it is easy to see that

$$J_1 \leq \frac{e^{-a\eta}}{2} \int_{\sqrt{n}\eta}^{\infty} u^2 e^{-u/2} du + \frac{1}{2} \int_{\sqrt{n}\eta}^{\infty} u^2 e^{-u/2} du,$$

for every n such that $1 - a/\sqrt{n} > 1/2$. Thus, $J_1 \rightarrow 0$ as $n \rightarrow +\infty$.

Next, let us consider J_2 . We have easily

$$J_2 \leq \frac{n}{a^2} \log^2 \frac{n}{n-a^2} A_{-a,n},$$

and so $J_2 \rightarrow 0$ as $n \rightarrow +\infty$. Finally, as to J_3 we have

$$J_3 \leq \frac{e^{-a\eta}\sqrt{n}}{a} \log \frac{n}{n-a^2} \int_{\sqrt{n}\eta}^{\infty} v e^{-v} dv + \frac{\sqrt{n}}{a} \log \frac{n}{n-a^2} \int_{\sqrt{n}\eta}^{\infty} u e^{-u/2} du,$$

for every n such that $1 - a/\sqrt{n} > 1/2$. Therefore, $J_3 \rightarrow 0$ as $n \rightarrow +\infty$, and assumption (ii) is satisfied. Therefore all the assumptions introduced are satisfied with $\alpha = 1$. The asymptotic formula of Theorem 5.4 reads

$$\lim_{n \rightarrow +\infty} n((T_n f)(x) - f(x)) = 2a^2 f(x) - 3a f'(x) + f''(x)$$

which is a result of [6].

(3) *The moment kernel*

For $n \in \mathbb{N}$ let us consider the kernel $\{K_n\}$ with

$$K_n(t) = n\chi_{[0,1/n]}(t) \quad (t \in \mathbb{R}).$$

This kernel is not even, and the functions K_n have compact support. We calculate now the coefficients $A_{a,n}$, $A_{-a,n}$, $B_{-a,n}$ and $C_{-a,n}$ and some their properties. We have

$$A_{a,n} = n \int_0^{1/n} e^{at} dt = \frac{n}{a}(e^{a/n} - 1), \quad A_{-a,n} = \frac{n}{a}(1 - e^{-a/n}),$$

and so

$$\lim_{n \rightarrow +\infty} A_{a,n} = \lim_{n \rightarrow +\infty} A_{-a,n} = 1, \quad \lim_{n \rightarrow +\infty} \frac{1}{a} n \log A_{a,n} = \frac{1}{2}.$$

Moreover,

$$\lim_{n \rightarrow +\infty} n(A_{a,n}A_{-a,n} - 1) = 0.$$

Next,

$$B_{-a,n} = -\frac{e^{-a/n}}{a} + \frac{n}{a^2}(1 - e^{-a/n}), \quad C_{-a,n} = -\frac{e^{-a/n}}{an} + \frac{2}{a}B_{-a,n},$$

and both tend to zero as $n \rightarrow +\infty$. Now,

$$\lim_{n \rightarrow +\infty} nB_{-a,n} = 1/2, \quad \lim_{n \rightarrow +\infty} nC_{a,n} = 0.$$

Concerning assumption (ii) of Theorem 5.4, since the functions K_n have supports $[0, 1/n]$ for any given $\eta > 0$, one can find an integer n_0 such that for any $n \geq n_0$ the set $\{t : |t| \geq \eta\}$ does not intersect the interval $[0, 1/n]$, and so denoting $H_{n,\eta} := \{t : |t| \geq \eta\} \cap [0, 1/n]$, we have

$$\int_{H_n} e^{-at} \left(t - \frac{1}{a} \log A_{a,n} \right)^2 dt = 0$$

This implies that assumption (ii) is trivially satisfied. Therefore all the assumptions introduced are satisfied with $\alpha = 1$. The asymptotic formula of Theorem 5.4 reads

$$\lim_{n \rightarrow +\infty} n((T_n f)(x) - f(x)) = 0.$$

This result is not fully satisfactory, due to the fact that we have no a precise order of pointwise approximation. Therefore we now employ Theorem 6.2 with $\alpha = 2$. In order to do that, we have to check the assumptions (j) and (jj). As to (j), we first calculate the limit

$$\lim_{n \rightarrow +\infty} n^2 \left(B_{-a,n} - \frac{1}{a} \log A_{a,n} \right),$$

which can be written as

$$\lim_{n \rightarrow +\infty} \frac{n^2}{a^2} \left(-ae^{-a/n} + a\frac{n}{a}(1 - e^{-a/n}) - a \log \left(\frac{n}{a}(e^{a/n} - 1) \right) \right).$$

This can be interpreted as a restriction of the limit

$$\lim_{x \rightarrow 0} \frac{a}{x^2} \left(-e^{-x} + \frac{1 - e^{-x}}{x} - \log \left(\frac{e^x - 1}{x} \right) \right).$$

The above limit can be solved using elementary techniques, based on the L'Hospital rule and its value is given by $-3a/8$. Now, in order to calculate λ_1 we write

$$\begin{aligned} & n^2 \left(B_{-a,n} - A_{-a,n} \frac{1}{a} \log A_{a,n} \right) \\ &= n^2 \left(B_{-a,n} - \frac{1}{a} \log A_{a,n} \right) + n^2 \frac{1}{a} \log A_{a,n} (1 - A_{-a,n}), \end{aligned}$$

thus we consider only the limit

$$\lim_{n \rightarrow +\infty} n^2 \frac{1}{a} \log A_{a,n} (1 - A_{-a,n}).$$

Since as we have seen before $\lim_{n \rightarrow +\infty} (n/a) \log A_{a,n} = 1/2$, we have to calculate only $\lim_{n \rightarrow +\infty} n(1 - A_{-a,n})$ and it is not difficult to see that its value is given by $a/2$. Thus, finally we obtain $\lambda_1 = -a/8$.

Now we proceed to the calculation of λ_2 . At first we calculate the limit

$$\lim_{n \rightarrow +\infty} n^2 C_{-a,n}.$$

Again this limit can be considered as a restriction of the limit

$$\lim_{x \rightarrow 0} \frac{1}{x^2} \left(-xe^{-x} - 2e^{-x} + 2 \frac{1 - e^{-x}}{x} \right),$$

and using an analogous elementary approach, its value is given by $1/3$.

Next, using the previous results, we have immediately

$$\lim_{n \rightarrow +\infty} n^2 \frac{A_{-a,n}}{a^2} \log^2 A_{a,n} = \frac{1}{4},$$

and

$$\lim_{n \rightarrow +\infty} \frac{2}{a} n^2 B_{-a,n} \log A_{a,n} = 1/2.$$

Thus, we have $\lambda_2 = 1/12$.

The last limit $\ell_0 = \lim_{n \rightarrow +\infty} n^2 (A_{a,n} A_{-a,n} - 1)$ can be obtained with the same reasonings as a restriction of the limit

$$\lim_{x \rightarrow 0} a^2 \frac{e^x + e^{-x} - 2 - x^2}{x^4}$$

which is given by $a^2/12$. Concluding, the Voronovskaya formula (6.5) with $\alpha = 2$ for the moment type operator is given by (see also [26])

$$\lim_{n \rightarrow +\infty} n^2 [(T_n f)(x) - f(x)] = \frac{a^2}{12} f(x) - \frac{a}{8} f'(x) + \frac{1}{24} f''(x).$$

(4) *A spline kernel*

For $n \in \mathbb{N}$ let us consider the kernel $\{K_n\}$ with

$$K_n(t) = n(1 - |nt|)\chi_{[-1/n, 1/n]}(t) \quad (t \in \mathbb{R}).$$

For this kernel we have, by elementary calculations

$$A_{a,n} = A_{-a,n} = n \int_{-1/n}^{1/n} e^{at}(1 - n|t|)dt = \frac{2n^2}{a^2}(\cosh(a/n) - 1),$$

and so

$$\lim_{n \rightarrow +\infty} A_{\pm a,n} = 1, \quad \lim_{n \rightarrow +\infty} n^2(A_{a,n}A_{-a,n} - 1) = \frac{a^2}{6},$$

and $\lim_{n \rightarrow +\infty} n^2 \log A_{a,n} = a^2/12$. Next,

$$B_{-a,n} = n \int_{-1/n}^{1/n} te^{-at}(1 - n|t|)dt = -\frac{2n}{a^2} \sinh(a/n) + \frac{4n^2}{a^3}(\cosh(a/n) - 1),$$

and

$$\lim_{n \rightarrow +\infty} n^2 B_{-a,n} = -\frac{a}{6}.$$

As to $C_{-a,n}$ we have

$$\begin{aligned} C_{-a,n} &= n \int_{-1/n}^{1/n} t^2(1 - n|t|)e^{-at}dt \\ &= \frac{2}{a^2} \cosh(a/n) - \frac{8n}{a^3} \sinh(a/n) + \frac{12n^2}{a^4}(\cosh(a/n) - 1), \end{aligned}$$

and so

$$\lim_{n \rightarrow +\infty} n^2 C_{-a,n} = \frac{1}{6}.$$

Finally, since the functions K_n have compact supports $[-1/n, 1/n]$, we easily see, as in the previous example, that (ii) of Theorem 5.4 is satisfied. Thus, all the assumptions used in the previous theory are satisfied with $\alpha = 2$. The corresponding asymptotic formula is given by

$$\lim_{n \rightarrow +\infty} n^2((T_n f)(x) - f(x)) = \frac{a^2}{6} f(x) - \frac{a}{4} f'(x) + \frac{1}{12} f''(x).$$

Remark 7.1. The kernel of Example (4) is generated by the spline function of second order, defined by

$$\beta_2(x) = (1 - |x|)\chi_{[-1,1]}(x) \quad (x \in \mathbb{R}).$$

The functions K_n are given by $K_n(x) = n\beta_2(nx)$, for $x \in \mathbb{R}$. The spline functions of order k are given by the formula (see e.g. [11, 24])

$$\beta_k(x) = \frac{1}{(k-1)!} \sum_{j=0}^k (-1)^j \binom{k}{j} \left(\frac{k}{2} + x - j\right)_+^{k-1},$$

where for any real number r , r_+ denotes its positive part. The theory may be applied also to any kernel generated by the spline β_k .

Remark 7.2. Note that the kernels of any of the previous examples satisfy the assumptions of Theorem 5.4, in particular the inequality

$$\frac{1}{a} |\log A_{a,n}| \leq \sqrt{C_{-a,n}},$$

for every $a > 0$ and sufficiently large $n \in \mathbb{N}$. First, note that since in any of the above examples $A_{a,n} \geq 1$, we have $\log A_{a,n} \geq 0$. For the modified Gauss–Weierstrass kernel, one has easily

$$\sqrt{C_{-a,n}} > \frac{a}{2n} > \frac{1}{a} \log A_{a,n}.$$

For the modified Picard kernel, one can employ the calculations of the limits in Example (2): since $nC_{-a,n} \rightarrow 2$ as $n \rightarrow +\infty$, and

$$\lim_{n \rightarrow +\infty} \frac{\sqrt{n}}{a} \log A_{a,n} = 0,$$

for sufficiently large values of n we obtain the assertion. Similar arguments can be used for the remaining examples.

Remark 7.3. Further examples can be obtained using Phillips type operators and Post-Widder type operators which act on functions defined over the positive real axis, (see [16, 18, 19] in which some modified version are introduced).

Remark 7.4. Our approach may be applied also in case of general exponential functions of the form a^{-x} for $a > 1$ as studied in [17] for Baskakov-type operators that act on functions defined over the positive real axis.

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Carlo Bardaro and Ilaria Mantellini
Department of Mathematics and Computer Sciences
University of Perugia
Via Vanvitelli, 1
06123 Perugia
Italy
e-mail: ilaria.mantellini@unipg.it

Carlo Bardaro
e-mail: carlo.bardaro@unipg.it

Gumrah Uysal
Division of Technology of Information Security, Department of Computer Technology
Karabuk University
Karabuk
Turkey
e-mail: fgumrahuyisal@gmail.com

Basar Yilmaz
Department of Mathematics, Faculty of Science and Arts
Kirikkale University
Yahsihan Kirikkale
Turkey
e-mail: basaryilmaz77@yahoo.com

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