Mediterr. J. Math. (2021) 18:157
https://doi.org/10.1007/s00009-021-01798-2
1660-5446/21/040001-19
published online June 9, 2021

Mediterranean Journal of Mathematics

Some Results on the Approximation of Solutions of Variational Inequalities for Multivalued Maps on Banach Spaces

Luigi Muglia and Giuseppe Marino

Abstract. Multivalued $*$-nonexpansive mappings are studied in Banach spaces. The demiclosedness principle is established. Here we focus on the problem of solving a variational inequality which is defined on the set of fixed points of a multivalued $*$-nonexpansive mapping. For this purpose, we introduce two algorithms approximating the unique solution of the variational inequality.

Mathematics Subject Classification. 47J20, 47J25, 49J40.

Keywords. Strongly accretive operators, variational inequality problem, iterative method, $*$-nonexpansive multivalued mapping.

1. Introduction

The notion of $*$-nonexpansive maps seems to be interesting because the $*$-nonexpansivity holds when given two sets that are images of two different points of the domain; it is possible to choose for each set (at least) a closest point to the corresponding point of the domain so that the distance between these two does not exceed the distance between the starting points. Therefore, it is an idea that immediately calls back the usual nonexpansivity of the single-valued case.

More precisely, let X be a Banach space and let C be a subset of X. Let $K(C)$ be the family of compact subsets of C.

Definition 1.1. [10] A mapping $W : C \to K(C)$ is said to be $*$-nonexpansive if for all $x, y \in C$ and $x^W \in Wx$ such that $\|x - x^W\| = d(x, Wx)$, there exists $y^W \in Wy$ with $\|y - y^W\| = d(y, Wy)$ such that

$$\|x^W - y^W\| \leq \|x - y\|$$

Recall that a point $x \in C$ is said to be a fixed point for a multivalued mapping W if $x \in Wx$.

The concept of $*$-nonexpansive multivalued maps was introduced by Husain and Latif [10] in 1988; it is a generalisation of the known notion of nonexpansiveness for single-valued maps. In general, $*$-nonexpansive multivalued maps may neither be continuous (Example 1.1 in [9]) nor nonexpansive with respect to the definition obtained by means of Hausdorff metric(see also [26]).

However, $*$-nonexpansivity and multivalued nonexpansivity are not so far. In Theorem 3 of [15], it is proved that a multivalued map $W : C \to K(C)$ is $*$-nonexpansive if and only if the metric projection

$$P_W(x) := \left\{ u_x \in Wx : \|x - u_x\| = \inf_{y \in Wx} \|x - y\| \right\}$$

is nonexpansive.

Existence results of fixed points for multivalued mappings are, in general, subtle and sometimes, surprising. For instance, the multivalued improvement of the classical Banach contraction principle, proved by Nadler in 1969 [18], guarantees the existence of a fixed point for a multivalued contraction, but not its uniqueness. Again, unlike to the single-valued case, the set of fixed points of a multivalued nonexpansive mapping $W : C \to K(C)$ on a strictly convex Banach space is not, in general, a convex set, see [12] Section 3, and the same holds for $*$-nonexpansive mappings too.

Xu, 1991 [26], has proved two existence results of fixed points for $*$-nonexpansive on strictly convex Banach spaces; Lopez-Acedo and Xu in [15] (1995) have obtained existence result in the setting of Banach space satisfying Opial condition.

Other surprising results, compared to the single-valued case, can be found in the literature about the approximation of fixed points of multivalued mappings. We refer to a well-known counterexample due to Pietramala, proved in [20] (1991): she proved that Browder approximation Theorem 1 in [2] cannot be extended to the genuine multivalued case even on a finite dimensional space $\mathbb{R}^2$.

The problem that we are concerned within this paper is the following: given a reflexive Banach space X and a closed subset $C \subset X$, to find $x^* \in C$ such that

$$\langle Ax^*, j(y - x^*)\rangle \geq 0, \qquad \forall y \in C \subset D(A), \tag{1.1}$$

where

- $j(x) \in J(x)$ and $J : X \to X^*$ is the normalized duality mapping defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = x^*(x) = \|x\|^2, \|x^*\| = \|x\|\}. \tag{1.2}$$

- $A : D(A) \subset X \to X$ is a $\eta-$strongly accretive operator, i.e. it satisfies

$$\langle Ax - Ay, j(x - y)\rangle \geq \eta\|x - y\|^2.$$

The solution of (VIP)(1.1) is a singleton; indeed, given x^* and $\bar{x}$ two different solutions, one immediately notes that

$$\langle Ax^*, j(\bar{x} - x^*)\rangle \geq 0 \text{ and } \langle A\bar{x}, j(x^* - \bar{x})\rangle \geq 0$$

hold jointly. Therefore, adding the inequalities

$$-\eta\|\bar{x} - x^*\|^2 \geq -\langle A\bar{x} - Ax^*, j(\bar{x} - x^*)\rangle \geq 0,$$

i.e. $\bar{x} = x^*$.

In a Hilbert space H, (VIP)(1.1) is equivalent to a variational inequality problem on the set of fixed points $Fix(W)$ of a suitable nonexpansive mapping W; for instance, the metric projection on the subset C. In the setting of a general Banach space, since there exist closed and convex sets that are not fixed point sets of a nonexpansive mapping $W : X \to X$ (see page 25 in [6]), this is no longer true.

In this note, we will work on a feasible set C that is the fixed point set of a multivalued $*$-nonexpansive mapping, i.e. given a strongly accretive operator $A : X \to X$ and a multivalued $*$-nonexpansive mapping W with fixed points, we focus on some approximation algorithms of the unique solution of the variational inequality

$$\langle Ax^*, j(y - x^*)\rangle \geq 0, \qquad \forall y \in Fix(W). \tag{1.3}$$

This problem encloses, as a particular case, viscosity problems

$$\langle (I - f)x^*, j(y - x^*)\rangle \geq 0, \qquad \forall y \in Fix(W), \tag{1.4}$$

when $A = I - f$ and, if $A = I - u$, the problem

$$\langle x^* - u, j(y - x^*)\rangle \geq 0, \qquad \forall y \in Fix(W), \tag{1.5}$$

that is equivalent to the minimum problem $\min_{x \in Fix(W)} \|x - u\|^2$.

These problems are widely studied as for the single-valued as for the multivalued case; for details one should refer to [5,17,19,25,29].

The novelty of our work can be immediately recognised: the use of $*$-nonexpansive mapping is no longer developed with respect to multivalued nonexpansive although they can be of interest in view of Example 1.1 in [9] and Example 1-2 in [26] respectively.

In our approach, with respect to multivalued nonexpansive case, we do not use Banach limit.

Remark 1.2. We want to emphasise here that this last approach is not always fully correct. Indeed, taking into account [31], we note that in many papers, see for instance [8,11,24,32], Banach limits are used to define a function ϕ by

$$\phi(x) := \mathrm{LIM}_{n\to+\infty}\|x_n - x\|^2, \quad x \in X,$$

where $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X (which is generated by an iterative method). It is easily verified that ϕ is continuous, convex and coercive (i.e. $\phi(x) \to \infty$, as $\|x\| \to +\infty$). Hence, reflexivity of X ensures that ϕ attains its minimum on a closed convex set C. Let $p \in C$ be a minimiser of ϕ over C. If C is a nonexpansive retract of X, this minimum is a global minimum on X. The goal is to prove that p is a fixed point of W. Using compactness arguments, they proved that, given $(w_n)_{n\in\mathbb{N}} \subset Wp$, there exists a subsequence strongly convergent to $w \in Tp$ (wrongfully indicated by the same sequence $(w_n)_{n\in\mathbb{N}}$). Therefore, using the formula that defines the iteration, they proved that

$$\phi(w) = \mathrm{LIM}_{n\to\infty}\|x_n - w\|^2 \leq \ldots \leq \mathrm{LIM}_{n\to\infty}\|x_n - p\|^2 = \phi(p) = \min_X \phi \tag{1.6}$$

and then drew the conclusion that $w = p$ and thus $p \in Wp$.

Unfortunately, the above argument holds for a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$ only, and so (1.6) holds for a subsequence (x_{n_k}) only; that is, the correct statement of (1.6) should be

$$\mathrm{LIM}_{k\to\infty}\|x_{n_k} - w\|^2 \leq \ldots \leq \mathrm{LIM}_{k\to\infty}\|x_{n_k} - p\|^2. \tag{1.7}$$

Consequently, the conclusion $w = p$ cannot be drawn from (1.7). Notice that Banach limits are sensitive to subsequences, as the following simple example shows: consider the real sequence $a_n = 1 + (-1)^n$; then we have

$$\mathrm{LIM}_{n\to\infty} a_n = 1,\ \mathrm{LIM}_{n\to\infty} a_{2n+1} = 0,\ \mathrm{LIM}_{n\to\infty} a_{2n} = 2.$$

Therefore, the claim $w = p$ in the above proof is not convincing.

The paper is organised as follows: in the next section, we introduce some definitions and tools which are used in our proofs. In Sect. 3, we prove our results and raise some open problems.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a Banach space. Denote by $K(X)$ the family of compact subset of X.

In (1.2), we have quickly introduced the normalised duality mapping; indeed it is the special case of the following.

A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *gauge* if:

1. $\varphi(0) = 0$;
2. φ is continuous and strictly increasing;
3. $\varphi(t) \to +\infty$, as $t \to +\infty$.

Associated with a gauge φ is the duality map

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \varphi(\|x\|) = \|x^*\|\}.$$

Choosing $\varphi(t) = t^{p-1}$, for some $p \in (1, +\infty)$, the duality map is referred to as the generalised duality map of order p; for $p = 2$, we get $J(x)$. It is well known that the Asplund's result [1] proved that J_φ is the sub-differential of the convex functional $\Phi(\|\cdot\|)$ defined as

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s.$$

Since the relationship

$$J(x)\varphi(\|x\|) = \|x\|^2 J_\varphi(x) \tag{2.1}$$

[1](see, for instance, [4, 30])

holds, it is easy to notice that the (VIP)(1.1) is equivalent to

$$\langle Ax^*, j_\varphi(y - x^*)\rangle \geq 0, \qquad \forall y \in C.$$

Following Browder [1], recall that a Banach space X has a weakly sequentially continuous duality map J_φ for some gauge φ if $J_\varphi x_n \to J_\varphi x$ in the weak* topology of X^* whenever $x_n \to x$ in the weak topology of X. The following result is useful in fixed point theory and geometry of Banach spaces [14].

Lemma 2.1. *Let X be a Banach space with a weakly sequentially continuous duality map J_φ for some gauge φ. Assume (x_n) is a sequence in X weakly converging to x^*. Then*

$$\limsup_{n\to\infty} \Phi(\|x_n - x\|) = \limsup_{n\to\infty} \Phi(\|x_n - x^*\|) + \Phi(\|x - x^*\|) \tag{2.2}$$

for all $x \in X$. In particular, X satisfies Opial's condition, i.e.

$$\limsup_{n\to\infty} \|x_n - x^*\| < \limsup_{n\to\infty} \|x_n - x\|, \qquad \forall x \in X \tag{2.3}$$

(but not vice versa [7,28]).

Definition 2.2. [3] An operator $A : X \to X$ is said to be $\lambda-$strict pseudocontractive ($\lambda \in (0, 1)$) if for every $x, y \in X$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \leq \|x - y\|^2 - \lambda\|(I - A)x - (I - A)y\|^2.$$

Proposition 2.3. [3] *Let X be a smooth Banach space, $A : X \to X$ be an operator.*

(i) If A is $\lambda-$strict pseudocontractive then A is $L-$Lipschitzian, where $L = 1 + \lambda^{-1}$.

(ii) If A is $\eta-$strongly accretive and $\lambda-$ strict pseudocontractive with $\eta+\lambda > 1$ then $(I - \tau A)$ is a $(1 - \tau\rho)$-contraction, for all $\tau \in (0, 1)$, where $\rho := \left(1 - \sqrt{\dfrac{1-\eta}{\lambda}}\right)$.

Remark 2.4. Each linear operator A defined by $Ax = kx$, $k > 1$, is not a strict pseudocontractive mapping therefore vice versa of statement (i) does not hold.

In [28] it is proved that if A is a η-strongly monotone and $L-$Lipschitzian operator, then $(I - \tau A)$ is a contraction if $\tau \leq \dfrac{2\eta}{L^2}$ in the setting of a Hilbert space.

A similar result is proved on q-uniformly smooth Banach spaces (see [16]).

Next Lemma, proved in [27], will be a useful tool for our proof.

Lemma 2.5. *Assume $(b_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative numbers for which,*

$$b_{n+1} \leq (1 - a_n)b_n + \delta_n, \quad n \geq 0,$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence in $(0, 1)$ and $(\delta_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that,

1. $\sum_{n=1}^{\infty} a_n = \infty$;
2. $\limsup_{n\to\infty} \frac{\delta_n}{a_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} b_n = 0$.

3. Results

Let $W : X \to K(X)$ be a $*$-nonexpansive multivalued mapping with nonempty $Fix(W)$.

In view of Definition 1.1, for a given $x \in X$ and $p \in Fix(W)$, for any $x^W \in Wx$ such that $\|x - x^W\| = d(x, Wx)$, i.e $x^W \in P_W x$, there exists p^W such that $\|p - p^W\| = d(p, Wp) = 0$, i.e. $p = p^W \in P_W p$, and

$$\|x^W - p\| \le \|x - p\|. \tag{3.1}$$

Let us start with two easy examples showing that the convergence of a classical viscosity method ([17,29]) is not guaranteed without any attention on the choice of $x^W \in Wx$.

Counter-Example 3.1. *Let* $W : \mathbb{R} \to K(\mathbb{R})$ *be defined as* $Wx = \{x, x + 1\}$.

Note that W *is* $*$*-nonexpansive mapping,* $Fix(W) = \mathbb{R}$ *and* $Wx \ne \{x\}$ *if* $x \in Fix(W)$.

Let $f(x) = \dfrac{x}{2}$ *and let us consider the implicit iteration:*

$$x_n = \tau_n f(x_n) + (1 - \tau_n)x_n^W, \tag{3.2}$$

where $x_n^W \in Wx_n$.

Since $Wx = \{x, x + 1\}$, *we choose* $W(x_n) \ni x_n^W = x_n + 1$ *and our iteration becomes:*

$$x_n = \frac{2(1 - \tau_n)}{\tau_n}. \tag{3.3}$$

It is clear that $(x_n)_{n\in\mathbb{N}}$ *does not converge for any null sequence* $(\tau_n)_{n\in\mathbb{N}}$ *(i.e.* $\tau_n \to 0$ *as* $n \to +\infty$*).*

Lemma 3.2. *Let* $W : X \to K(X)$ *and let* P_W *be defined as*

$$P_W x := \{u_x \in Wx : \|x - u_x\| = d(x, Wx)\},$$

i.e. the projection of x *on the set* Wx*. Then the following hold:*

(i) *If* P_W *is the identity mapping then* W *is* $*$*-nonexpansive.*
(ii) W *is* $*$*-nonexpansive if and only if* P_W *is nonexpansive.*
(iii) *If* $Fix(W) \ne \emptyset$ *then* $P_W|_{Fix(W)}$ *is single-valued, i.e.* $P_W x = \{x\}$ *for each* $x \in Fix(W)$ *and* $Fix(W) = Fix(P_W)$.

Proof. (i) follows by definitions and (ii) is proved in Theorem 3 in [15].

To prove (iii), if $\tilde{x} \in P_W(\tilde{x})$ then $\tilde{x} \in \{u_{\tilde{x}} \in W\tilde{x} : \|\tilde{x} - u_{\tilde{x}}\| = d(\tilde{x}, W\tilde{x})\}$; hence, $d(\tilde{x}, W\tilde{x}) = 0$. This implies that $\tilde{x} \in Fix(W)$, and

$$Fix(P_W) \subset Fix(W)$$

is proved. On the other hand, if $x \in Fix(W)$, $d(x, Wx) = 0$ then $P_W x = \{u_x : \|u_x - x\| = 0\}$, i.e. $x = u_x$.

Therefore, $P_W(x) = \{x\}$, i.e P_W is single-valued on the set of fixed points of W and $x \in Fix(P_W)$ that conclude our proof. □

The next counter-example shows that the convergence of a classical viscosity method is not certain even under the strong condition that the metric projection P_W is single-valued on $Fix(W)$.

Counter-Example 3.3. *Let $W : [0,1] \to K([0,1])$ defined as:*

$$Wx = \begin{cases} \left[x, x + \frac{1}{2}\right], & 0 \le x < \frac{1}{2} \\ \left[x - \frac{1}{2}, x\right], & \frac{1}{2} \le x \le 1. \end{cases}$$

Since P_W is the identity mapping, W is $*$-nonexpansive by Lemma 3.2 (i).

Let us consider iteration

$$x_n = \tau_n f(x_{n-1}) + (1 - \tau_n) x_n^W,$$

where $f : [0,1] \to [0,1]$ is a contraction such that $f(x) < x$, $x > 0$ (so the unique fixed point is $x = 0$) and the following choice for x_n^W is done:

$$x_n^W = \begin{cases} x_n + \frac{1}{2}, & 0 \le x_n < \frac{1}{2} \\ x_n - \frac{1}{2}, & \frac{1}{2} \le x_n \le 1. \end{cases}$$

By induction we prove that if $x_0 \le \frac{1}{2}$ then $x_n \le \frac{1}{2}$ for all $n \in \mathbb{N}$.

Let us suppose, by contradiction, that $x_n \le \frac{1}{2}$ and $x_{n+1} > \frac{1}{2}$.

Thus, $x_{n+1}^W = x_{n+1} - \frac{1}{2}$ and

$$x_{n+1} = \tau_{n+1} f(x_n) + (1 - \tau_{n+1}) x_{n+1} - \frac{(1 - \tau_{n+1})}{2}$$
$$\Rightarrow x_{n+1} = f(x_n) - \frac{1}{2} \frac{(1 - \tau_{n+1})}{\tau_{n+1}} \le f(x_n) \le x_n \le \frac{1}{2}$$

by inductive hypothesis. This is a contradiction; therefore, $x_0 \le \frac{1}{2}$ implies $x_n \le \frac{1}{2}$ for all $n \in \mathbb{N}$. [2]

Let $x_0 \in \left[0, \frac{1}{2}\right]$; then the entire sequence lies in the same interval. This is not possible because it can be written as

$$x_n = f(x_{n-1}) + \frac{(1 - \tau_n)}{2\tau_n}.$$

[2]Note that same result can be obtained for $f(x) := u_0 \le \frac{1}{2}$.

Since f is positive $x_n > \dfrac{(1-\tau_n)}{2\tau_n}$ therefore our sequence $(x_n)_{n\in\mathbb{N}}$ can not be found in $\left[0, \dfrac{1}{2}\right]$ for any null sequence $(\tau_n)_{n\in\mathbb{N}}$. This contradiction shows that by such a choice of x_n^W, the algorithm does not work.

3.1. Iterative Approach

To prove our convergence results, the following demiclosedness type-principle is necessary; for multivalued nonexpansive mapping demiclosedness principle is well known by Theorem 3.1- [13]; for multivalued $*$-nonexpansive mapping demiclosedness principle given in the next Lemma 3.4 seems to be new.

Lemma 3.4. *Let X be a reflexive space satisfying Opial condition* (2.3)*.*

Let $W : X \to K(X)$ be a $*$-nonexpansive multivalued mapping with fixed points.

Let $(y_n)_{n\in\mathbb{N}}$ be a bounded sequence such that

$$d(y_n, Wy_n) \to 0, \ \text{as } n \to \infty.$$

Then the weak cluster points of $(y_n)_{n\in\mathbb{N}}$ belong to $Fix(W)$, (i.e. $\omega_w(y_n) \subset Fix(W)$).

Proof. Since X is reflexive, let $(y_{n_k})_{k\in\mathbb{N}} \subset (y_n)_{n\in\mathbb{N}}$ weak convergent to z.

Since Wz is compact, it is closed and there exists $(z_{n_k})_{k\in\mathbb{N}} \subset Wz$ such that

$$\|y_{n_k} - z_{n_k}\| = d(y_{n_k}, Wz).$$

Still because of the compactness of Wz, there exists a subsequence $(z_{n_{k_j}})_j \subset (z_{n_k})_k \subset Wz$ strongly convergent to $\tilde{z} \in Wz$.

By definition of $*-$nonexpansivity, for any $j \in \mathbb{N}$ and $y_{n_{k_j}}$ there exists $u_{y(j)} \in Wy_{n_{k_j}}$ with $\|y_{n_{k_j}} - u_{y(j)}\| = d(y_{n_{k_j}}, Wy_{n_{k_j}})$ and $u_{z(j)} \in Wz$ with $\|u_{z(j)} - z\| = d(z, Wz)$ such that

$$\|u_{y(j)} - u_{z(j)}\| \le \|y_{n_{k_j}} - z\|.$$

We now prove that $z = \tilde{z}$, so will be $z \in Wz$. If not, since $d(y_n, Wy_n) \to 0$ and using the Opial's inequality

$$\begin{aligned}
\limsup_{j\to\infty} \|y_{n_{k_j}} - \tilde{z}\| &\le \limsup_{j\to\infty} [\|y_{n_{k_j}} - z_{n_{k_j}}\| + \|z_{n_{k_j}} - \tilde{z}\|] \\
&= \limsup_{j\to\infty} d(y_{n_{k_j}}, Tz) \le \limsup_{j\to\infty} \|y_{n_{k_j}} - u_{z(j)}\| \\
&\le \limsup_{j\to\infty} [\|y_{n_{k_j}} - u_{y(j)}\| + \|u_{y(j)} - u_{z(j)}\|] \\
&= \limsup_{j\to\infty} [d(y_{n_{k_j}}, Ty_{n_{k_j}}) + \|u_{y(j)} - u_{z(j)}\|] \\
&\le \limsup_{j\to\infty} \|y_{n_{k_j}} - z\| \\
&< \limsup_{j\to\infty} \|y_{n_{k_j}} - \tilde{z}\|,
\end{aligned}$$

and this is absurd. Therefore $z \in Wz$, i.e. $z \in Fix(W)$. □

Our first result concerns the existence of a unique solution of our (VIP) on the set of fixed point of a $*$-nonexpansive mapping.

Proposition 3.5. *Let X be a reflexive Banach space with duality mapping J_φ that is weakly sequentially continuous, for some gauge φ.*

Let $W : X \to K(X)$ a $*$-nonexpansive multivalued mapping such that $Fix(W)$ is nonempty.

Let $A : X \to X$ an η-strongly accretive and $k-$strict pseudocontractive such that $\eta + k > 1$.

Then

$$\langle Ax^*, j(y - x^*)\rangle \geq 0, \qquad \forall y \in Fix(W) \tag{3.4}$$

has a unique solution.

Proof. Uniqueness of the solution is already noted by means of the strong accretivity of A. Let us prove the existence.

Since W is $*$-nonexpansive, P_W is nonexpansive by Lemma 3.2 (ii). Let $(\alpha_n)_{n\in\mathbb{N}} \subset (0, 1)$ be a sequence such that $\alpha_n \to 0$ as $n \to +\infty$ and let $\mu \in (0, 1)$. For any $n \in \mathbb{N}$, consider the multivalued mapping

$$\Gamma_n := \alpha_n(I - \mu A) + (1 - \alpha_n)P_W.$$

It is easy to verify that each Γ_n is a contraction. Indeed, if $x, y \in X$, $w = (I - \mu_n A)x \in X$, $v = (I - \mu_n A)y \in X$ we obtain that

$$\begin{aligned} H(\Gamma_n x, \Gamma_n y) &\leq \alpha_n\|w - v\| + (1 - \alpha_n)H(P_W x, P_W y) \\ &\leq \alpha_n\|(I-\mu A)x-(I-\mu A)y\|+(1-\alpha_n)H(P_W x, P_W y) \\ \text{(by Proposition 2.3(ii))} &\leq \alpha_n(1 - \mu\rho)\|x - y\| + (1 - \alpha_n)\|x - y\| \\ &= (1 - \alpha_n\mu\rho)\|x - y\|. \end{aligned}$$

Then by Nadler fixed point principle, Γ_n has fixed point and

$$x_n = \alpha_n(I - \mu_n A)x_n + (1 - \alpha_n)x_n^P$$

is well defined for an opportune $x_n^P \in P_W x_n$. Let $p \in Fix(W)$; then $p = Fix(P_W)$ by Lemma 3.2 (iii) and $P_W p = \{p\}$ i.e. P_W is single-valued on $Fix(W)$. Thus

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n(I - \mu A)x_n + (1 - \alpha_n)x_n^P - p\| \\ &\leq \alpha_n\|(I - \mu A)x_n - (I - \mu A)p\| + \alpha_n\mu\|Ap\| + (1 - \alpha_n)\|x_n^P - p\| \\ \text{(by (3.1))} &\leq \alpha_n(1 - \mu\rho)\|x_n - p\| + \alpha_n\mu\|Ap\| + (1 - \alpha_n)\|x_n - p\| \\ &= (1 - \alpha_n\mu\rho)\|x_n - p\| + \alpha_n\mu\rho\frac{\|Ap\|}{\rho}; \end{aligned}$$

therefore,

$$\|x_n - p\| \leq \frac{\|Ap\|}{\rho},$$

i.e. our sequence is bounded.

Moreover, for each $w \in Fix(W)$,

$$\begin{aligned} \|x_n - w\|^2 &= \langle x_n - w, j(x_n - w)\rangle \\ &\leq \alpha_n\langle x_n - \mu Ax_n - w, j(x_n - w)\rangle + (1 - \alpha_n)\|x_n - w\|^2 \end{aligned}$$

$$= \|x_n - w\|^2 - \alpha_n \mu \langle Ax_n, j(x_n - w)\rangle; \tag{3.5}$$

hence,

$$\langle Ax_n, j(x_n - w)\rangle \le 0, \qquad \forall w \in Fix(T). \tag{3.6}$$

On the other hand, since A is η-strongly accretive, it follows from (3.6) that

$$\begin{aligned} 0 &\ge \langle Ax_n - Aw, J(x_n - w)\rangle + \langle Aw, J(x_n - w)\rangle \\ &\ge \eta\|x_n - w\|^2 + \langle Aw, j(x_n - w)\rangle. \end{aligned}$$

This implies that

$$\|x_n - w\|^2 \le -\frac{1}{\eta}\langle Aw, j(x_n - w)\rangle. \tag{3.7}$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded and $(\alpha_n)_{n\in\mathbb{N}}$ is a null sequence it holds

$$\|x_n - x_n^P\| = \alpha_n\|(I - \mu A)x_n - x_n^P\| \to 0, \text{ as } n \to \infty;$$

therefore, $d(x_n, P_W x_n) \to 0$ as $n \to \infty$ and, as a rule, $d(x_n, W x_n) \to 0$ as $n \to \infty$. By Lemma 3.4, the weak limit of $(x_n)_{n\in\mathbb{N}}$ are fixed points for W.

Recalling (2.1) we can write (3.7) as

$$\varphi(\|w - x_n\|)\|x_n - w\|^2 \le \frac{1}{\eta}\|w - x_n\|^2\langle Aw, j_\varphi(w - x_n)\rangle) \tag{3.8}$$

and so

$$\varphi(\|w - x_n\|) \le \frac{1}{\eta}\langle Aw, j_\varphi(w - x_n)\rangle).$$

Let $w \in \omega_w(x_n)$; there exists $x_{n_k} \rightharpoonup w$ and $w \in Fix(T)$. Since the duality map J_φ is weakly sequentially continuous

$$\varphi(\|x_{n_k} - w\|) \le \frac{1}{\eta}\langle Aw, j_\varphi(w - x_{n_k})\rangle \to 0,$$

as $k \to \infty$; hence, $x_{n_k} \to w$, by properties of φ. Rewriting (3.6) with respect to J_φ we get

$$0 \ge \langle Ax_{n_k}, j_\varphi(x_{n_k} - p)\rangle \to \langle Aw, j_\varphi(w - p)\rangle, \qquad \forall p \in Fix(T).$$

Then w is a solution of (VIP) and, by the uniqueness of the solution, $\omega_w(x_n) = \omega_s(x_n) = \{w\}$ and the thesis follows. □

Next we can define our first iteration.

Let $A : D(A) \subset X \to X$ be a strongly accretive operator and strict pseudocontractive.

Let $W : X \to 2^X$ be a multivalued $*$-nonexpansive mapping; let x_0 and $x_0^W \in Wx_0$ such that $\|x_0 - x_0^W\| = d(x_0, Wx_0)$, i.e. $x_0^W \in P_W x_0$. Let

$$x_1 = \lambda_0(x_0 - \mu_0 A x_0) + (1 - \lambda_0)x_0^W.$$

Using definition of $*-$nonexpansivity, there exists $x_1^W \in Wx_1$ such that $\|x_1 - x_1^W\| = d(x_1, Wx_1)$, i.e. $x_1^W = P_W x_1$ and

$$\|x_1^W - x_0^W\| \le \|x_1 - x_0\|.$$

In a same manner, let

$$x_2 = \lambda_1(x_1 - \mu_1 A x_1) + (1 - \lambda_0)x_1^W,$$

and choose $x_2^W \in P_W x_2$ and

$$\|x_2^W - x_1^W\| \le \|x_2 - x_1\|.$$

Iterating this process we get a sequence

$$x_{n+1} = \lambda_n(x_n - \mu_n A x_n) + (1 - \lambda_n)x_n^W \tag{3.9}$$

such that

$$\|x_{n+1}^W - x_n^W\| \le \|x_{n+1} - x_n\|. \tag{3.10}$$

Theorem 3.6. *Let X be a reflexive Banach space with duality mapping J_φ that is weakly sequentially continuous .*

Let $W : X \to K(X)$ a $*$-nonexpansive multivalued mapping such that $Fix(W)$ is nonempty.

Let $A : X \to X$ an η-strongly accretive and $k-$strict pseudocontractive such that $\eta + k > 1$.

Let $(\mu_n)_{n\in\mathbb{N}} \subset (0,1)$ and $(\lambda_n)_{n\in\mathbb{N}} \subset [0,a] \subset [0,1)$ such that

- $\lambda_n\mu_n \to 0$, *as* $n \to \infty$ *and* $\sum_{n\in\mathbb{N}} \lambda_n\mu_n = \infty$.
- $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n\mu_n} = 0.$
- $\lim_{n\to\infty} \frac{|\mu_n - \mu_{n-1}|}{\mu_n} = 0.$

Then, for any choice x_0 as a starting point, the explicit process

$$x_{n+1} = \lambda_n(I - \mu_n A)x_n + (1 - \lambda_n)x_n^W, \tag{3.11}$$

defined choosing x_n^W in such a way that (3.10) is satisfied, strongly converges, as $n \to \infty$, to the unique solution of (VIP)

$$\langle Ax^*, j(y - x^*)\rangle \ge 0, \qquad \forall y \in Fix(W). \tag{3.12}$$

Proof. Defining $B_n := (I - \mu_n A)$, our iteration can be described by

$$x_{n+1} = \lambda_n B_n x_n + (1 - \lambda_n)x_n^W.$$

By hypotheses, every B_n is a contraction using Proposition 2.3 (ii).

Let $p \in Fix(W)$ be a given fixed point of W; then by (3.1),

$$\begin{aligned}\|x_{n+1} - p\| &\le \lambda_n\|B_n x_n - p\| + (1 - \lambda_n)\|x_n^W - p\| \\ &\le \lambda_n\|B_n x_n - B_n p\| + \lambda_n\|B_n p - p\| + (1 - \lambda_n)\|x_n - p\| \\ &\le \lambda_n(1 - \mu_n\rho)\|x_n - p\| + (1 - \lambda_n)\|x_n - p\| + \lambda_n\mu_n\|Ap\| \\ &= (1 - \lambda_n\mu_n\rho)\,\|x_n - p\| + \lambda_n\mu_n\rho\frac{\|Ap\|}{\rho} \\ &\le \max\left\{\|x_n - p\|, \frac{\|Ap\|}{\rho}\right\} \le \ldots \le \max\left\{\|x_1 - p\|, \frac{\|Ap\|}{\rho}\right\},\end{aligned}$$

and the boundedness of our sequence immediately holds.

Recalling that X is reflexive and since it satisfies Opial condition because it has a weakly sequentially continuous duality mapping J_φ (Lemma 2.1), our next step is: *to show that $\omega_w(x_n) \subset Fix(W)$.*

The claim will follow by Lemma 3.4 and by the asymptotic regularity of $(x_n)_{n\in\mathbb{N}}$. Computing:

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\lambda_n B_n x_n + (1-\lambda_n)x_n^W - \lambda_{n-1}B_{n-1}x_{n-1} - (1-\lambda_{n-1})x_{n-1}^W\|\\ &\le \lambda_n\|B_n x_n - B_{n-1}x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|B_{n-1}x_{n-1} - x_{n-1}^W\|\\ &\quad +(1-\lambda_n)\|x_n^W - x_{n-1}^W\|\\ &\le \lambda_n\|B_n x_n - B_n x_{n-1}\| + \lambda_n\|B_n x_{n-1} - B_{n-1}x_{n-1}\|\\ &\quad +|\lambda_n - \lambda_{n-1}|\|B_{n-1}x_{n-1} - x_{n-1}^W\|\\ &\quad +(1-\lambda_n)\|x_n - x_{n-1}\|\\ &\le \lambda_n(1-\mu_n\rho)\|x_n - x_{n-1}\| + \lambda_n\|B_n x_{n-1} - B_{n-1}x_{n-1}\|\\ &\quad +|\lambda_n - \lambda_{n-1}|\|B_{n-1}x_{n-1} - x_{n-1}^W\| + (1-\lambda_n)\|x_n - x_{n-1}\|\\ &\le (1-\lambda_n\mu_n\rho)\|x_n - x_{n-1}\| + \lambda_n|\mu_n - \mu_{n-1}|\|Ax_{n-1}\|\\ &\quad +|\lambda_n - \lambda_{n-1}|\|B_{n-1}x_{n-1} - x_{n-1}^W\|.\end{aligned}$$

The boundedness of $(x_n)_{n\in\mathbb{N}}$ guarantees that there exists a constant M such that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le (1-\lambda_n\mu_n\rho)\|x_n - x_{n-1}\| + M\left[\lambda_n|\mu_n - \mu_{n-1}| + |\lambda_n - \lambda_{n-1}|\right]\\ &= (1-a_n)\|x_n - x_{n-1}\| + M\delta_n,\end{aligned}$$

where

$$a_n := \lambda_n\mu_n\rho; \qquad \delta_n = \left[\lambda_n|\mu_n - \mu_{n-1}| + |\lambda_n - \lambda_{n-1}|\right];$$

hence, asymptotic regularity for $(x_n)_{n\in\mathbb{N}}$ follows by Lemma 2.5. Moreover,

$$\begin{aligned}\|x_n - x_n^W\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - x_n^W\|\\ &\le \|x_n - x_{n+1}\| + \lambda_n\|B_n x_n - x_n^W\|;\end{aligned}$$

thus,

$$(1-\lambda_n)\|x_n - x_n^W\| \le \|x_n - x_{n+1}\| + \lambda_n\mu_n\|Ax_n^W\|.$$

Since $(\lambda_n\mu_n)_{n\in\mathbb{N}}$ is a null sequence and by asymptotic regularity, $\|x_n - x_n^W\| \to 0$.

Observing that

$$d(x_n, Wx_n) \le \|x_n - x_n^W\| \to 0,$$

as $n\to\infty$, by Lemma 3.4, the weak limits of $(x_n)_{n\in\mathbb{N}}$ are fixed points for W.

To prove the strong convergence, let $w \in Fix(W)$ the unique solution for (1.1). Such a (unique) solution exists by Proposition 3.5.

Since J_φ is the sub-differential of Φ, we have

$$\begin{aligned}\Phi(\|x_{n+1}-w\|) &= \Phi(\|\lambda_n(B_n x_n - w) + (1-\lambda_n)(x_n^W - w)\|)\\ &= \Phi(\|\lambda_n(B_n x_n - B_n w) + \lambda_n(B_n w - w) + (1-\lambda_n)(x_n^W - w)\|)\\ &\le \Phi(\|\lambda_n(B_n x_n - B_n w) + (1-\lambda_n)x_n^W - w)\|)\\ &\quad -\lambda_n\mu_n\langle Aw, j_\varphi(x_{n+1}-w)\rangle\\ &\le \lambda_n(1-\mu_n\rho)\Phi(\|x_n - w\|) + (1-\lambda_n)\Phi(\|x_n^W - w\|)\\ &\quad -\lambda_n\mu_n\langle Aw, j_\varphi(x_{n+1}-w)\rangle\end{aligned}$$

$$\leq [1 - \lambda_n \mu_n \rho]\, \Phi(\|x_n - w\|) - \lambda_n \mu_n \langle Aw, j_\varphi(x_{n+1} - w)\rangle. \tag{3.13}$$

Then

$$\Phi(\|x_{n+1} - w\|) \leq (1 - a_n)\, \Phi(\|x_n - w\|) + a_n \frac{\langle -Aw, j_\varphi(x_{n+1} - w)\rangle}{,} \rho,$$

where $a_n = \lambda_n \mu_n \rho$ satisfies Lemma 2.5 (1). To apply Lemma 2.5, note that there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ for which

$$\limsup_{n\to\infty} \langle -Aw, j_\varphi(x_{n+1} - w)\rangle = \lim_{k\to\infty} \langle -Aw, j_\varphi(x_{n_k} - w)\rangle.$$

Since $(x_n)_{n\in\mathbb{N}}$ is bounded and X is reflexive, there exists a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}} \subset (x_{n_k})_{k\in\mathbb{N}}$ weak convergence to p; moreover, $p \in Fix(W)$. By the weak sequential continuity of the duality map we, therefore, conclude that

$$\limsup_{n\to\infty} \langle Aw, j_\varphi(w - x_{n+1})\rangle = \lim_{j\to\infty} \langle Aw, j_\varphi(w - x_{n_{k_j}})\rangle = \langle Aw, j_\varphi(w - p)\rangle \leq 0$$

since w is the unique solution of (1.1). Lemma 2.5 gives that $\|x_n - w\| \to 0$, as $n \to \infty$, being $\Phi(\|x_n - w\|) \to 0$, and the thesis follows. □

By means of Theorem 3.6, we obtain viscosity iteration and Halpern approach to minimisation problem

Corollary 3.7. *Let X be a reflexive Banach space with duality mapping that is weakly sequentially continuous, J_φ.*

Let $W : X \to K(X)$ a $*$-nonexpansive multivalued mapping such that $Fix(W)$ is nonempty. Let $f : X \to X$ a η-contraction.

Let $(\lambda_n)_{n\in\mathbb{N}} \subset [0, a] \subset [0, 1)$ such that

$$\lim_{n\to+\infty} \lambda_n = 0, \sum_{n\in\mathbb{N}} \lambda_n = \infty \text{ and } \lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} = 0.$$

Then, for any choice x_0 as a starting point, the explicit process

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n) x_n^W$$

strongly converges, as $n \to \infty$, to the unique solution of (VIP)

$$\langle (I - f)x^*, j(y - x^*)\rangle \geq 0, \qquad \forall y \in Fix(W). \tag{3.14}$$

Corollary 3.8. *Let X be a reflexive Banach space with duality mapping that is weakly sequentially continuous, J_φ.*

Let $W : X \to K(X)$ a $*$-nonexpansive multivalued mapping such that $Fix(W)$ is nonempty. Let $(\lambda_n)_{n\in\mathbb{N}} \subset [0, a] \subset [0, 1)$ such that

$$\lim_{n\to+\infty} \lambda_n = 0, \sum_{n\in\mathbb{N}} \lambda_n = \infty \text{ and } \lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} = 0.$$

Then, for any choice x_0 as a starting point, the explicit process

$$x_{n+1} = \lambda_n u + (1 - \lambda_n) x_n^W$$

strongly converges, as $n \to \infty$, to the unique solution of minimisation problem

$$\min_{x \in Fix(W)} \|x - u\|. \tag{3.15}$$

Example. Let us apply Corollary 3.8 to W defined as in counter-example 3.3. Let $W : [0,1] \to K([0,1])$ defined as

$$Wx = \begin{cases} \left[x, x + \frac{1}{2}\right], & 0 \le x < \frac{1}{2} \\ \\ \left[x - \frac{1}{2}, x\right], & \frac{1}{2} \le x \le 1. \end{cases}$$

We have already noted that $P_W = I$ and $Fix(W) = \mathbb{R}$. For any $(\lambda_n)_{n \in \mathbb{N}}$ satisfying the assumption of Corollary 3.8, following construction (3.11) our iteration becomes

$$x_{n+1} = \lambda_n u + (1 - \lambda_n) x_n.$$

Then a simple realisation of Lemma 2.5 applied to the unknown sequence $(x_n - u)$ gives the convergence of $(x_n)_{n \in \mathbb{N}}$ to u that solves the minimization problem (3.15).

By a suitable modification of the idea that defines (3.11), we can define another iteration; the proof of the following is based on the idea of Theorem 3.6.

Theorem 3.9. *Let X, W and A as in Theorem 3.6. Let $(\mu_n)_{n \in \mathbb{N}} \subset (0,1)$ and $(\lambda_n)_{n \in \mathbb{N}} \subset [0,a] \subset [0,1)$ such that*

- *$\mu_n \to 0$, as $n \to \infty$ and $\sum_{n \in \mathbb{N}} \mu_n = \infty$.*
- $\lim_{n \to \infty} \frac{|\lambda_n - \lambda_{n-1}|}{\mu_n} = 0.$
- $\lim_{n \to \infty} \frac{|\mu_n - \mu_{n-1}|}{\mu_n} = 0.$

Then choosing x_n^W as in the previous theorem, i.e. in such a way that (3.10) is satisfied, the explicit iteration

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) x_n^W - (1 - \lambda_n) \mu_n A x_n^W \tag{3.16}$$

strongly converges, as $n \to \infty$, to the unique solution of (VIP)

$$\langle Ax^*, j(y - x^*) \rangle \ge 0, \qquad \forall y \in Fix(W). \tag{3.17}$$

Proof. Again define $B_n := (I - \mu_n A)$ in such a way that our iteration becomes:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) B_n x_n^W.$$

Let $p \in Fix(W)$ be a given fixed point of W. Then

$$\begin{aligned} \|x_{n+1} - p\| &\le \lambda_n \|x_n - p\| + (1 - \lambda_n) \|B_n x_n^W - B_n p\| + (1 - \lambda_n) \mu_n \|Ap\| \\ &\le \lambda_n \|x_n - p\| + (1 - \lambda_n)(1 - \mu_n \rho) \|x_n^W - p\| + (1 - \lambda_n) \mu_n \|Ap\| \end{aligned}$$

$$\begin{aligned} \text{by (3.1)} &\leq \lambda_n \|x_n - p\| + (1-\lambda_n)(1-\mu_n\rho)\|x_n - p\| + (1-\lambda_n)\mu_n\|Ap\| \\ &= (1-(1-\lambda_n)\mu_n\rho)\,\|x_n - p\| + (1-\lambda_n)\mu_n\rho\frac{\|Ap\|}{\rho} \\ &\leq \max\left\{\|x_n - p\|, \frac{\|Ap\|}{\rho}\right\} \leq \ldots \leq \max\left\{\|x_1 - p\|, \frac{\|Ap\|}{\rho}\right\}, \end{aligned}$$

and the boundedness of our sequence immediately holds. To show that $\omega_w(x_n) \subset Fix(W)$ asymptotic regularity is needed; therefore, let us compute

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|x_{n-1} - B_{n-1}x^W_{n-1}\| + \\ &\quad (1-\lambda_n)\|B_n x^W_n - B_{n-1}x^W_{n-1}\| \\ &\leq \lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|x_{n-1} - B_{n-1}x^W_{n-1}\| + \\ &\quad (1-\lambda_n)\|B_n x^T_n - B_n x^W_{n-1}\| + \\ &\quad (1-\lambda_n)\|B_n x^W_{n-1} - B_{n-1}x^W_{n-1}\| \\ &\leq \lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|x_{n-1} - B_{n-1}x^W_{n-1}\| + \\ &\quad (1-\lambda_n)(1-\mu_n\rho)\|x^W_n - x^W_{n-1}\| + \\ &\quad (1-\lambda_n)|\mu_n - \mu_{n-1}|\|Ax^W_{n-1}\| \\ \text{by (3.10)} &\leq \lambda_n\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|x_{n-1} - B_{n-1}x^W_{n-1}\| + \\ &\quad (1-\lambda_n)(1-\mu_n\rho)\|x_n - x_{n-1}\| + \\ &\quad (1-\lambda_n)|\mu_n - \mu_{n-1}|\|Ax^W_{n-1}\|. \end{aligned}$$

The boundedness of $(x_n)_{n\in\mathbb{N}}$ guarantees that there exists a constant M such that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq [\lambda_n + (1-\lambda_n)(1-\mu_n\rho)]\|x_n - x_{n-1} \\ &\quad + M\left[|\lambda_n - \lambda_{n-1}| + |\mu_n - \mu_{n-1}|\right] \\ &= [1-(1-\lambda_n)\mu_n\rho)]\|x_n - x_{n-1}\| \\ &\quad + M\left[|\lambda_n - \lambda_{n-1}| + |\mu_n - \mu_{n-1}|\right] \\ &= (1-a_n)\|x_n - x_{n-1}\| + M\delta_n, \end{aligned}$$

where $a_n := (1-\lambda_n)\mu_n\rho$ and $\delta_n = [|\lambda_n - \lambda_{n-1}| + |\mu_n - \mu_{n-1}|]$.
Applying Lemma 2.5, we obtain asymptotic regularity for $(x_n)_{n\in\mathbb{N}}$. Moreover,

$$\begin{aligned} \|x_n - x^W_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x^W_n\| \\ &\leq \|x_n - x_{n+1}\| + \|\lambda_n(x_n - x^W_n) + (1-\lambda_n)\mu_n Ax^W_n\| \\ &\leq \|x_n - x_{n+1}\| + \lambda_n\|x_n - x^W_n\| + (1-\lambda_n)\mu_n\|Ax^W_n\|; \end{aligned}$$

thus,

$$(1-\lambda_n)\|x_n - x^W_n\| \leq \|x_n - x_{n+1}\| + (1-\lambda_n)\mu_n\|Ax^W_n\|.$$

Since $(\mu_n)_{n\in\mathbb{N}}$ is a null sequence and by asymptotic regularity, $\|x_n - x^W_n\| \to 0$. Observing that

$$d(x_n, Wx_n) \leq \|x_n - x^W_n\| \to 0,$$

as $n \to \infty$, by Lemma 3.4, the weak limits of $(x_n)_{n\in\mathbb{N}}$ are fixed points for W.

To conclude, let $w \in Fix(W)$ the unique solution for (1.1) that there exists by Proposition 3.5. Thus,

$$\begin{aligned}\Phi(\|x_{n+1}-w\|) &= \Phi(\|\lambda_n(x_n-w)+(1-\lambda_n)(B_nx_n^W-w)\|)\\ &= \Phi(\|\lambda_n(x_n-w)+(1-\lambda_n)(B_nx_n^W-B_nw)\\ &\quad -(1-\lambda_n)\mu_nAw)\|)\\ &\le \Phi(\|\lambda_n(x_n-w)+(1-\lambda_n)(B_nx_n^W-B_nw)\|)\\ &\quad -(1-\lambda_n)\mu_n\langle Aw, j_\varphi(x_{n+1}-w)\rangle\\ &\le \lambda_n\Phi(\|x_n-w\|)+(1-\lambda_n)(1-\mu_n\rho)\Phi(\|x_n^W-w\|)\\ &\quad -(1-\lambda_n)\mu_n\langle Aw, j_\varphi(x_{n+1}-w)\rangle\\ &\le [1-(1-\lambda_n)\mu_n\rho]\,\Phi(\|x_n-w\|)\\ &\quad -(1-\lambda_n)\mu_n\langle Aw, j_\varphi(x_{n+1}-w)\rangle,\end{aligned}$$

i.e.

$$\Phi(\|x_{n+1}-w\|) \le (1-a_n)\,\Phi(\|x_n-w\|)+a_n\frac{\langle -Aw, j_\varphi(x_{n+1}-w)\rangle}{\rho},$$

where $a_n=(1-\lambda_n)\mu_n\rho$. To apply Lemma 2.5, following proof of Theorem 3.6, we get that

$$\limsup_{n\to\infty}\langle Aw, j_\varphi(w-x_{n+1})\rangle = \lim_{j\to\infty}\langle Aw, j_\varphi(w-x_{n_{k_j}})\rangle = \langle Aw, j_\varphi(w-p)\rangle \le 0$$

since w is the unique solution of (1.1). Lemma 2.5 gives that $\|x_n-w\|\to 0$, as $n\to\infty$, being $\Phi(\|x_n-w\|)\to 0$, and the thesis follows. □

Example. Take $X=l^p$, $p>1$, that is reflexive and it has a weakly sequentially continuous duality mapping J_φ with gauge $\varphi(t)=t^{p-1}$.

Take $Wx := \{x, 2x\}$, that is $*$-nonexpansive by Lemma 3.2(i).

Take $A = I-u$, u be fixed.

Consider the iterative process (3.16) with $\lambda_n=\frac{1}{2}$ and $\mu_n=\frac{1}{\sqrt{n}}$. Then all the hypotheses of Theorem 3.9 are satisfied and the iteration process (3.16) becomes

$$x_{n+1}-u=\left(1-\frac{1}{2\sqrt{n}}\right)(x_n-u).$$

So Lemma 2.5 still yields that $x_n\to u$, solution of the variational inequality (1.5).

3.2. Open Questions

In what follows, we include some open problems that we think that they may be of interest:

- Does the conclusion of our Theorems hold under weaker conditions on underlying Banach spaces?
- Is it possible to replace the strict pseudocontractivity of A with Lipschitzianity as in the setting of Hilbert spaces?

3.3. Conclusions

We studied multivalued $*$-nonexpansive mappings in Banach spaces. The demiclosedness principle is established in reflexive Banach spaces satisfying Opial's condition (Lemma 3.4). Thus, the demiclosedness principle holds also in reflexive Banach spaces with duality mapping that is weakly sequentially continuous since these satisfy Opial's condition.

We proved the existence of a unique solution of our Variational Inequality Problem (VIP) on the set of fixed point of a $*$-nonexpansive mapping (Proposition 3.5) when X is a reflexive Banach space with duality mapping that is weakly sequentially continuous J_φ for some gauge φ. We constructed two iterative schemes (3.11) and (3.16) that converge to the solution of the (VIP) in reflexive Banach spaces with duality mapping that is weakly sequentially continuous (Theorems 3.6 and Theorem 3.9, respectively).

Some examples and counter-examples are given.

Some open questions whose answer could be interesting are pointed out.

Author contributions Each author equally contributed to this paper, read and approved the final manuscript.

Funding Open access funding provided by Universitá della Calabria within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare that they have no competing interests.

References

[1] Browder, F.E.: Fixed point theorems for nonlinear semicontractive mappings in Banach spaces. Arch. Ration. Mech. Anal. **21**(4), 259–269 (1966)

[2] Browder, F.E.: Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. Arch. Ration. Mech. Anal. **24**(1), 82–90 (1967)

[3] Ceng, L.C., Ansari, Q.H., Yao, J.C.: Mann-type steepest-descent and modified hybrid steepest-descent methods for variational inequalities in Banach spaces. Numer. Funct. Anal. Optim. **29**(9–10), 987–1033 (2008)

[4] Cioranescu, I.: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, vol. 62. Springer Science & Business Media (2012)

[5] Cui, Y., Fei, Z.Z., Hudzik, H.: Viscosity approximation methods for multivalued mappings in Banach spaces. Numer. Funct. Anal. Optim. **33**(11), 1288–1303 (2012)

[6] Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Marcel Dekker INC (1984)

[7] Gossez, J.P., Lami, Dozo E.: Some geometric properties related to the fixed point theory for nonexpansive mappings. Pac. J. Math. **40**(3), 565–573 (1972)

[8] He, H., Liu, S., Chen, R.: Convergence results of multi-valued nonexpansive mappings in Banach spaces. J. Inequal. Appl. **2014**(1), 1–12 (2014)

[9] Hussain, N., Khan, A.R.: Applications of the best approximation operator to $*$-nonexpansive maps in Hilbert spaces. Numer. Funct. Anal. Optim. **24**(3–4), 327–338 (2003)

[10] Husain, T., Latif, A.: Fixed points of multivalued nonexpansive maps. Math. Japonica **33**, 385–391 (1988)

[11] Jung, J.S.: Strong convergence theorems for multivalued nonexpansive nonself-mappings in Banach spaces. Nonlinear Anal. TMA **66**(11), 2345–2354 (2007)

[12] Ko, H.M.: Fixed point theorems for point-to-set mappings and the set of fixed points. Pac. J. Math. **42**(2), 369–379 (1972)

[13] Lami, Dozo E.: Multivalued nonexpansive mappings and Opial condition. Proc. Am. Math. Soc. **38**(2), 286–292 (1973)

[14] Lim, T.C., Xu, H.K.: Fixed point theorems of asymptotically nonexpansive mappings. Nonlinear Anal. TMA **22**(11), 1345–1355 (1994)

[15] López, Acedo G., Xu, H.K.: Remarks on multivalued nonexpansive mappings. Soochow J. Math. **21**(1), 107–115 (1995)

[16] Marino, G., Muglia, L., Yao, Y.: The uniform asymptotical regularity of families of mappings and solutions of variational inequality problems. J. Nonlinear Convex Anal. **15**(3), 477–492 (2014)

[17] Moudafi, A.: Viscosity approximation methods for fixed-points problems. J. Math. Anal. Appl. **241**(1), 46–55 (2000)

[18] Nadler, S.: Multi-valued contraction mappings. Pac. J. Math. **30**(2), 475–488 (1969)

[19] Panyanak B., Suantai S.: Viscosity approximation methods for multivalued nonexpansive mappings in geodesic spaces. Fixed Point Theory. Appl. **2015**(1), 1–14 (2015)

[20] Pietramala, P.: Convergence of approximating fixed points sets for multivalued nonexpansive mappings. Comment. Math. Univ. Carol. **32**(4), 697–701 (1991)

[21] Reich, S.: Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **67**(2), 274–276 (1979)

[22] Reich, S.: Strong convergence theorems for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl. **75**(1), 287–292 (1980)

[23] Sahu, D.R.: Strong convergence theorems for nonexpansive type and non-self multi-valued mappings. Nonlinear Anal. TMA **37**(3), 401–407 (1999)

[24] Shahzad, N., Zegeye, H.: Strong convergence results for nonself multimaps in Banach spaces. Proc. Am. Math. Soc. **136**(2), 539–548 (2007)

[25] Wu, X., Zhao, L.: Viscosity approximation methods for multivalued monexpansive mappings. Mediterr. J. Math. **13**(5), 2645–2657 (2016)

[26] Xu, H.K.: On weakly nonexpansive and ∗-nonexpansive multivalued mappings. Math. Jpn. **36**(3), 441–445 (1991)

[27] Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**(1), 240–256 (2002)

[28] Xu, H.K., Kim, T.H.: Convergence of hybrid steepest-descent methods for variational inequalities. J. Optim. Theory Appl. **119**(1), 185–201 (2003)

[29] Xu, H.K.: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**(1), 279–291 (2004)

[30] Xu, H.K., Kim, T.H., Yin, X.: Weak continuity of the normalized duality map. J. Nonlinear Convex Anal. **15**(3), 595–604 (2014)

[31] Xu, H.K., Alghamdi, M.A., Shahzad, N.: Remarks on iterative methods for multivalued nonexpansive mappings. J. Nonlinear Conv. Anal. **18**(1), 161–171 (2017)

[32] Zegeye, H., Shahzad, N.: Viscosity approximation methods for nonexpansive multimaps in Banach spaces. Acta Mathematica Sinica Engl Ser **26**(6), 1165–1176 (2010)

Luigi Muglia and Giuseppe Marino
Dipartimento di Matematica
Universitá della Calabria
via P. Bucci
87036 Arcavacata di Rende CS
Italy
e-mail: muglia@mat.unical.it

Giuseppe Marino
e-mail: giuseppe.marino@unical.it

Received: May 5, 2020.
Revised: November 4, 2020.
Accepted: May 28, 2021.