De Branges Type Lemma and Approximation in Weighted Spaces

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Abstract. The purpose of this paper is to give some generalizations of de Branges Lemma for weighted spaces to obtain different approximation theorems in weighted spaces for algebras, vector subspaces or convex cones. We recall that the (original) de Branges Lemma (Proc Am Math Soc 10(5):822–824, 1959) was demonstrated for continuous scalar function on a compact space while, the weighted spaces are classes of continuous scalar functions on a locally compact space (e.g. the space of function with compact support, the space of bounded functions, the space of functions vanishing at infinity, the space of functions rapidly decreasing at infinity).

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1. Introduction

Using two fundamental tools in functional analysis: Hahn–Banach and Krein–Milman theorems, in 1959, Louis de Branges [2] give a nice proof of Stone–Weierstrass theorem on algebras of real continuous functions on a compact Hausdorff space.

In this paper, we present, in a natural way, the duality theory for general weighted spaces, i.e. a class of scalar continuous functions on a locally compact space (Lemma 2.1, Theorem 2.1).

Also we extend de Branges Lemma in this new setting: Lemma 3.2 for linear subspaces and Lemma 5.1 for convex cones of a weighted space.

We mention that a characterization of the dual of a weighted space was obtained by Summers in [10], in the particular case $V \leq C^+(X)$, i.e. for any $v \in V$ there is $w \in C^+(X)$ such that $v \leq w$ and a version of de

Branges Lemma for weighted spaces was obtained by Prolla in [5], in the same particular case.

Beside these Lemmata, Theorems 3.1 and 5.1 play a crucial role in the proof of various approximations results: Theorems 4.1–4.5 as well as Corollaries 4.1–4.3

From Corollary 4.3, which is a new result, we deduce a very simple proof of Portenier's theorem of characterization of closed (ordered) ideals in weighted spaces [3] (Theorem 4.5 of our paper).

The last section of this paper is devoted to the theory of approximation in convex cones of weighted spaces (for the most part new) with possible applications in Potential Theory or Processes governed by potentials as weights. Corollary 5.3 and the examples can give an idea of the approximation presented here.

Finally, we mention that this paper is an extension of the work [1], since Theorems 4.1, 4.3 and 5.2 are generalizations of Theorems 3, 4 and 5 from [1].

Throughout this section, X will be a locally compact Hausdorff space and $\mathbb{K}$ be the set of real or complex numbers. Also we shall denote by $C(X, \mathbb{K})$, or simply $C(X)$, the space of all continuous functions on X with values in $\mathbb{K}$.

Definition 1.1. A family V of upper semicontinuous, non-negative functions on X such that for any $v_1, v_2 \in V$ and any $\lambda \in \mathbb{R}$, $\lambda > 0$ there exists $w \in V$ such that

$$v_i(x) \leq \lambda \cdot w(x), \quad \forall x \in X, \quad i = 1, 2$$

will be called a Nachbin family on X . Any element of V will be called a weight.

We shall denote by $CV_0(X, \mathbb{K})$ or by $CV_0(X)$ the weighted space attached to Nachbin family V , i.e. the set of all continuous functions f on X such that the function $f \cdot v$ vanishes at infinity; so the space of all functions $f \in C(X)$ having the property that for any $v \in V$ and any $\varepsilon > 0$, the set

$$[v \cdot |f| \geq \varepsilon] = \{x \in X; v(x) \cdot |f(x)| \geq \varepsilon\}$$

is a compact subset of the starting topological space X .

Any weight $v \in V$ generates a seminorm $p_v : CV_0(X) \rightarrow \mathbb{R}_+$ defined by

$$p_v(f) = \sup\{v(x) \cdot |f(x)|; x \in X\}, \quad \forall f \in CV_0(X).$$

The locally convex topology defined by this family of seminorms is denoted by ω_V and it will be called the weighted topology on $CV_0(X)$.

The family of seminorms $(p_v)_{v \in V}$ is upper directed and the family $(B_v)_{v \in V}$ of subsets of $CV_0(X)$ is a base of neighborhoods of the origin, where

$$B_v = \{f \in CV_0(X); p_v(f) \leq 1\}.$$

As for uniqueness of the limit in the locally convex space $(CV_0(X), \omega_V)$, one can see that this space is Hausdorff iff the interior of the set

$$\bigcap_{v \in V} [v = 0] = \bigcap_{v \in V} \{x \in X; v(x) = 0\},$$

in the topological space X is empty.

- Remark 1.1.* a) If for any point $x \in X$ there exists $v_x \in V$ such that $v_x(x) > 0$, then the weighted space $CV_0(X)$ is a locally convex Hausdorff space.
- b) If for any point $x \in X$ there are $v_x \in V$ and $r \in \mathbb{R}$, $r > 0$ such that the set

$$[v_x > r] = \{y \in X; v_x(y) > r\}$$

is a neighborhood of x in X , then the locally convex spaces $(CV_0(X), \omega_V)$ is a complete space. Further, we mention some particular weighted spaces:

- a) If $V = \{1\}$ then $CV_0(X) = C_0(X)$ —the space of continuous functions vanishing at infinity and the weighted topology ω_V coincide with the uniform convergence topology.
- b) If $V = C_0^+(X)$ then $CV_0(X) = C_b(X)$ —the space of continuous bounded functions on X and the weighted topology ω_V coincide with the strict topology β .
- c) Let $X = \mathbb{R}^n$, and let $\mathcal{P}_n$ be the set of all polynomials defined on $\mathbb{R}^n$ with values in $\mathbb{K}$.

If $V = \{|p|; \forall p \in \mathcal{P}_n\}$, then $CV_0(\mathbb{R}^n)$ coincide with the space of functions rapidly decreasing at infinity.

2. Duality for Weighted Spaces

In this part, X will be a Hausdorff locally compact space, $\mathcal{K}(X)$ will be the space of all continuous real or complex functions on X with compact support.

Obviously $\mathcal{K}(X) \subset CV_0(X)$ and it is well known that the space $\mathcal{K}(X)$ is dense in $CV_0(X)$, with respect to the weighted topology ω_V , i.e. $\overline{\mathcal{K}(X)} = CV_0(X)$.

Hence, any continuous functional on the vector space $\mathcal{K}(X)$ endowed with the trace of ω_V on it (denoted also by ω_V) may be uniquely extended to a continuous linear functional on the locally convex space $(CV_0(X), \omega_V)$ and, therefore, the locally convex spaces $(\mathcal{K}(X), \omega_V)$ and $(CV_0(X), \omega_V)$ have the same dual.

Further, we represent any element θ of this dual under the form:

$$\theta(f) = \int f d\mu, \forall f \in CV_0(X),$$

where μ is the corresponding scalar measure defined on the set $\mathcal{B}(X)$ of all Borel subsets of the locally compact space X .

To this purpose, for any compact subset K of X , let us denote by $\mathcal{K}(X, K)$ the set of all continuous scalar functions on X which vanishes outside K . Obviously, we have

$$\mathcal{K}(X) = \bigcup_K \mathcal{K}(X, K).$$

On the vector space $\mathcal{K}(X)$, there are three remarkable topologies: the topology of uniform convergence denoted by τ_u , the weighted topology ω_V and the

inductive topology, denoted by τ_{ind} , i.e. the finest locally convex topology on $\mathcal{K}(X)$ making continuous the injection maps:

$$i_K : \mathcal{K}(X, K) \rightarrow \mathcal{K}(X),$$

for all compact subsets K of X , where each subspace $\mathcal{K}(X, K)$ is endowed with the uniform topology. What is important here is the fact that giving an arbitrary locally convex space E and a linear map

$$T : (\mathcal{K}(X), \tau_{\text{ind}}) \rightarrow E$$

then T is continuous iff the restriction of T to any $\mathcal{K}(X, K)$ is continuous.

It is no difficult to show that we have

$$\tau_u \subset \tau_{\text{ind}}, \omega_V \subset \tau_{\text{ind}}$$

and, therefore,

$$(\mathcal{K}(X), \tau_u)^* \subset (\mathcal{K}(X), \tau_{\text{ind}})^* \text{ and } (\mathcal{K}(X), \omega_V)^* \subset (\mathcal{K}(X), \tau_{\text{ind}})^*.$$

Any element $\theta \in (\mathcal{K}(X), \tau_{\text{ind}})^*$ is usually called a Radon integral on X . This means that θ is a scalar continuous map whose restriction to any $\mathcal{K}(X, K)$ is continuous with respect to the uniform topology, i.e. for each compact subset K of X there exists a number $a_K > 0$ such that

$$|\theta(f)| \leq a_K \cdot \|f\|_K = a_K \cdot \sup\{|f(x)|; x \in K\}, \forall f \in \mathcal{K}(X, K).$$

The linear, scalar map θ defined on $\mathcal{K}(X)$ belongs to $(\mathcal{K}(X), \tau_u)^*$ iff there exists a number $a > 0$ such that

$$|\theta(f)| \leq a \cdot \|f\|_u = a \cdot \sup\{|f(x)|; \forall x \in X\}, \forall f \in \mathcal{K}(X).$$

We recall that for any Radon integral θ on X there exists a smallest positive Radon integral, denoted by $|\theta|$, such that

$$|\theta(f)| \leq |\theta|(|f|), \forall f \in \mathcal{K}(X).$$

In fact, for any $f \in \mathcal{K}^+(X)$, we have

$$|\theta|(f) = \sup\{|\theta(g)|; g \in \mathcal{K}(X), |g| \leq f\}.$$

Also, any real, positive (on $\mathcal{K}(X)$) functional is a Radon integral on X and for any real, Radon integral θ on $\mathcal{K}(X)$, there exist two positive Radon integral θ^+ and θ^- such that

$$\theta = \theta^+ - \theta^-, \theta^+ = \frac{1}{2}(|\theta| + \theta), \theta^- = \frac{1}{2}(|\theta| - \theta).$$

As usually, a countable additive positive map μ defined on $\mathcal{B}(X)$ is called a Radon measure on X if for any Borel set $A \in \mathcal{B}(X)$, we have

$$\mu(A) = \sup\{\mu(K); K \text{ compact}, K \subset A\}$$

and $\mu(K) < \infty$ for any compact subset K of X .

A countable additive scalar map on $\mathcal{B}(X)$ is called a Radon measure on X if the positive measure $|\mu|$ is a Radon measure.

If μ is a positive Radon measure on X , then for any lower semicontinuous positive function φ on X , we have

$$\int \varphi d\mu = \sup \left\{ \int f d\mu; f \in \mathcal{K}(X), 0 \leq f \leq \varphi \right\}$$

and for any upper directed family $(\varphi_i)_i$ of positive lower semicontinuous functions on X , we have

$$\int \sup_i \varphi_i d\mu = \sup_i \int \varphi_i d\mu.$$

Obviously any function $f \in \mathcal{K}(X)$ is integrable with respect to μ and for any compact $K \subset X$ and any $f \in \mathcal{K}(X, K)$, we have

$$\left| \int f d\mu \right| = \left| \int_K f d\mu \right| \leq \|f\| \cdot \mu(K),$$

where $\|f\| = \sup\{|f(x)|; \forall x \in X\}$.

Let us denote by $\mathcal{M}(X)$ the set of all Radon measure on X . From the above considerations for any $\mu \in \mathcal{M}_+(X)$, the map

$$f \rightarrow \int f d\mu, \forall f \in \mathcal{K}(X)$$

is an element of the dual $(\mathcal{K}(X), \tau_{\text{ind}})^*$. Conversely, for any $\theta \in (\mathcal{K}(X), \tau_{\text{ind}})^{**}$ ($\theta \geq 0$ on $\mathcal{K}(X)$) there exists and it is uniquely determined a positive Radon measure μ_θ , such that

$$\theta(f) = \int f d\mu_\theta, \forall f \in \mathcal{K}(X).$$

In the above correspondence $\theta \rightarrow \mu_\theta$ between $(\mathcal{K}(X), \tau_{\text{ind}})^{**}$ and $\mathcal{M}_+(X)$, the elements $\theta \in (\mathcal{K}(X), \tau_u)^{**}$ are those positive Radon integrals for which the associated measures μ_θ are finite on X , i.e. $\mu_\theta(X) < \infty$.

If θ is a Radon real integral on X then $|\theta|$, θ^+ , θ^- are positive Radon integrals on X and we extend the above correspondence, associating with θ the real sign measure $\mu_\theta := \mu_{\theta^+} - \mu_{\theta^-}$. We have also $\mu_{|\theta|} = |\mu_\theta| = \mu_{\theta^+} + \mu_{\theta^-}$. A similar extension may be done if $\mathcal{K}(X)$ is the set of all continuous complex functions with compact support in X .

In the sequel, we shall freely use the term Radon measure instead of Radon integral or conversely.

If μ is a Radon measure on X and $f : X \rightarrow \overline{\mathbb{R}}$ is μ -integrable then the map $f\mu$ given by

$$A \in \mathcal{B}(X) \xrightarrow{f\mu} \int_A f d\mu$$

is also a Radon measure on X and we have $|f\mu| = |f||\mu|$. We remark also that if μ is a Radon measure such that $\int \frac{1}{v} d|\mu| < \infty$ for some weight $v \in V$ then $CV_0(X) \subset \mathcal{L}^1(\mu)$.

The following results concerning weighted spaces are known, in the particular case $V \leq C^+(X)$ (see Summers [10]). We present them in a more general frame using different proof ideas.

Lemma 2.1. *Let v be an upper semicontinuous non-negative real function on X , let p_v be the seminorm on $\mathcal{K}(X)$ associated with v , i.e.*

$$p_v(f) := \sup\{|f(x) \cdot v(x)|; x \in X\} = \inf \left\{ \alpha \in \mathbb{R}_+; |f(x)| \leq \alpha \cdot \frac{1}{v(x)}, \forall x \in X \right\},$$

and let B_v be the unit ball of $\mathcal{K}(X)$ associated to the seminorm p_v , i.e.

$$\begin{aligned} B_v &= \{f \in \mathcal{K}(X); p_v(f) \leq 1\} \\ &= \left\{ f \in \mathcal{K}(X); |f(x) \cdot v(x)| \leq 1 \text{ or } |f(x)| \leq \frac{1}{v(x)}, \forall x \in X \right\}. \end{aligned}$$

We have

- a) The dual of the locally convex space $(\mathcal{K}(X), p_v)$ is the linear subspace $\mathcal{M}_v$ of all Radon measures μ on $\mathcal{K}(X)$ for which $\int \frac{1}{v} d|\mu| < \infty$.
- b) The polar set of B_v with respect to the duality $(\mathcal{K}(X), \mathcal{M}_v)$, i.e. the set

$$B_v^0 = \{\mu \in \mathcal{M}_v; \|\mu\|_v \leq 1\} := \{\mu \in \mathcal{M}_v; |\mu(f)| \leq 1, \forall f \in B_v\}$$

is the set of all Radon measures μ on $\mathcal{K}(X)$ with the property $\int \frac{1}{v} d|\mu| \leq 1$. This set is compact if we endow the set of all Radon measures $\mathcal{M}(X)$ on X with the weak topology, i.e. the smallest topology on $\mathcal{M}(X)$ making continuous the linear maps on $\mathcal{M}(X)$:

$$\mu \rightarrow \mu(f), \forall f \in \mathcal{K}(X).$$

Proof. Indeed, if we consider a linear map $\theta : \mathcal{K}(X)$ to $\mathbb{K}$, which is continuous with respect to the seminorm $p_v : \mathcal{K}(X) \rightarrow \mathbb{K}$, then we have

$$\|\theta\|_v = \sup\{|\theta(f)|; f \in \mathcal{K}(X), p_v(f) \leq 1\} < \infty.$$

Since for any compact set K the lower semicontinuous function $\frac{1}{v} : X \rightarrow (0, \infty]$ has a strictly positive infimum α_K on K , we deduce that for any function $\varphi \in \mathcal{K}(X)$ which vanishes outside K , we have

$$p_v(\varphi) = \sup\{|v(x) \cdot \varphi(x)|; x \in K\} \leq \frac{1}{\alpha_K} \cdot |\varphi|, \text{ i.e.}$$

$$p_v\left(\frac{\alpha_K}{\|\varphi\|} \cdot \varphi\right) \leq 1, \left|\theta\left(\frac{\alpha_K}{\|\varphi\|} \cdot \varphi\right)\right| \leq \|\theta\|_v, |\theta(\varphi)| \leq \frac{1}{\alpha_K} \cdot \|\theta\|_v \cdot \|\varphi\|.$$

Hence, θ is a Radon measure on X . From the definition of the positive measure $|\theta|$, we have, for any $f \in \mathcal{K}^+(X)$, $f \leq \frac{1}{v}$:

$$\begin{aligned} |\theta|(f) &= \sup\{|\theta(g)|; g \in \mathcal{K}(X), |g| \leq f\}, \\ |\theta|\left(\frac{1}{v}\right) &= \sup\left\{|\theta|(f); f \in \mathcal{K}(X), f \leq \frac{1}{v}\right\} \\ &= \sup\left\{|\theta(g)|; g \in \mathcal{K}(X), |g| \leq \frac{1}{v}\right\} \\ &= \sup\{|\theta(g)|; g \in \mathcal{K}(X), p_v(g) \leq 1\} = \|\theta\|_v. \end{aligned}$$

□

The compactness of the set B_v^0 :

$$B_v^0 = \left\{ \mu \in \mathcal{M}(X); |\mu|\left(\frac{1}{v}\right) \leq 1 \right\}$$

with respect to the weak topology on $\mathcal{M}(X)$ given by the duality $(f, \mu) \rightarrow \mu(f)$ defined on $\mathcal{K}(X) \times \mathcal{M}(X)$ follows now from Alaoglu's theorem applied to the locally convex space $\mathcal{K}(X)$ endowed with the seminorm p_v .

Theorem 2.1. *Let V be a Nachbin family on X and let $CV_0(X)$ be the weighted space associated with the family V , endowed with weighted topology ω_V . Then the dual $CV_0(X)^*$ of the locally convex space $(CV_0(X), \omega_V)$ is identical with the dual of the space $\mathcal{K}(X)$ endowed with the induced ω_V -topology. More precisely $\theta \in CV_0(X)^*$ iff there exist a Radon measure μ on X and $v \in V$ such that*

$$|\mu| \left(\frac{1}{v} \right) < \infty, \quad CV_0(X) \subset \mathcal{L}^1(|\mu|), \quad \text{and } \theta(f) = \int f d\mu, \quad \forall f \in CV_0(X).$$

In the other words, $CV_0(X)^ = V \cdot \mathcal{M}_b(X) = \{v \cdot \lambda; v \in V, \lambda \in \mathcal{M}_b(X)\}$, where $\mathcal{M}_b(X)$ is the set of all bounded measures on X . More exactly*

$$B_v^0 = \left\{ \mu \in \mathcal{M}(X); |\mu| \left(\frac{1}{v} \right) \leq 1 \right\} = \{v \cdot \lambda; \lambda \in \mathcal{M}_b(X), \|\lambda\| \leq 1\}.$$

Proof. Since $\mathcal{K}(X)$ is dense in the locally convex space $(CV_0(X), \omega_V)$ it follows that any element $\theta \in CV_0(X)^*$ is totally determined by its restriction to $\mathcal{K}(X)$. On the other hand, there exists a weight $v \in V$ and $\alpha \in \mathbb{R}_+$ such that

$$|\theta(f)| \leq \alpha \cdot p_v(f), \quad \forall f \in CV_0(X).$$

Hence, the restriction of θ to $\mathcal{K}(X)$ satisfies the same inequality and therefore, using Lemma 2.1, there exists a Radon measure μ on X such that

$$|\mu| \left(\frac{1}{v} \right) < \infty, \quad \theta(f) = \int f d\mu, \quad \forall f \in \mathcal{K}(X).$$

Since any function $g \in CV_0(X)$ is dominated at infinity by $\frac{1}{v}$ and $\frac{1}{v}$ is strictly positive on X , we deduce that there exists $\beta \in \mathbb{R}_+$ such that $|g| \leq \beta \cdot \frac{1}{v}$, on X and, therefore, $g \in \mathcal{L}^1(|\mu|)$. Moreover, there exists a sequence $(f_n)_n \in \mathcal{K}(X)$ such that

$$p_v(g - f_n) \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}^*, \quad \lim_{n \rightarrow \infty} \theta(f_n) = \theta(g)$$

Hence, we have $|g - f_n| \leq \frac{1}{n} \cdot \frac{1}{v}$, $\forall n \in \mathbb{N}^*$.

Particularly, we have

$$\begin{aligned} \theta \in B_v^0 &\Leftrightarrow \sup \left\{ \theta(f); f \in \mathcal{K}(X), |f| \leq \frac{1}{v} \right\} \\ &= \sup \left\{ \int f d\mu; f \in \mathcal{K}(X), |f| \leq \frac{1}{v} \right\} \leq 1 \Leftrightarrow |\mu| \left(\frac{1}{v} \right) := \int \frac{1}{v} d|\mu| \leq 1, \end{aligned}$$

and so the proof is finished. $\square$

3. Lemma De Branges for Weighted Spaces

In this section, for any Radon measure μ on X , we shall use the notation $\sigma(\mu)$ for the support of μ —the smallest closed subset F of X for which $|\mu|(X \setminus F) = 0$.

Lemma 3.1. *Let L be a linear functional on $CV_0(X)$, ω_V -continuous and let $F \subset X$ be a Borel subset such that $|L|(X \setminus F) = 0$, where $|L|$ is the modulus of the Radon measure L on X .*

- a) *If $(f_i)_i$ is a generalized sequence in $CV_0(X)$ such that $f_i \xrightarrow{\omega_V} g$ on F , i.e. for any $v \in V$, any $\varepsilon > 0$ there exists $i_{\varepsilon,v} \in I$ such that $\sup v(x) |f_i(x) - g(x)| < \varepsilon$, $\forall x \in F$, $\forall i \geq i_{\varepsilon,v}$, then $g \in \mathcal{L}^1(L)$ and $\lim_i L(f_i) = \int g dL$.*
- b) *Let $(f_n)_n$ be a sequence of $CV_0(X)$ which is ω_V -bounded on F , i.e. for any $v \in V$ there exists $\alpha_v \in \mathbb{R}_+$ such that $|v(x) \cdot f_n(x)| \leq \alpha_v$, $\forall x \in F$, $\forall n \in \mathbb{N}$. If the sequence $(f_n)_n$ is pointwisely convergent on F to a function g , then $g \in \mathcal{L}^1(L)$ and $\lim_{n \rightarrow \infty} L(f_n) = \int g dL$.*

Proof. Since $L \in [CV_0(X)]^*$ there exists $v \in V$ such that $|L(f)| \leq p_v(f)$ and, therefore, by Theorem 2.1 L is a Radon measure on X with $|L|(\frac{1}{v}) < \infty$.

- a) Since $f_i \xrightarrow{\omega_V} g$ on F , for any $\varepsilon > 0$ there exists $i_\varepsilon \in I$ such that

$$v(x) \cdot |f_i(x) - g(x)| < \varepsilon, \forall x \in F, \forall i \geq i_\varepsilon.$$

Taking $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}^*$, we may consider an increasing sequence $(i_n)_n$ in I such that

$$v(x) \cdot |f_i(x) - g(x)| \leq \frac{1}{n}, \forall x \in F, \forall i \geq i_n.$$

Particularly, we have

$$|f_{i_n} - g| \leq \frac{1}{n} \cdot \frac{1}{v} \text{ on } F,$$

and, therefore, the sequence $(f_{i_n})_n$ is pointwisely convergent to g on the set $F \cap [v > 0]$. Since $X \setminus (F \cap [v > 0]) = (X \setminus F) \cup [v = 0]$ and $|L|(X \setminus F) = 0$, $|L|([v = 0]) = 0$ we deduce that the function g belongs to $\mathcal{L}^1(|L|)$ and we have

$$\left| L(f_i) - \int g dL \right| = \left| \int (f_i - g) d|L| \right| \leq \int |f_i - g| d|L| \leq \frac{1}{n} \cdot \int \frac{1}{v} d|L|,$$

for any $i \geq i_n$, i.e. $\lim_i L(f_i) = \int g dL$.

The assertion b) has a similar proof using Lebesgue domination theorem.

In the sequel, for any $v \in V$, respectively, any $\mu \in [CV_0(X)]^*$, we shall use the notations:

$$B_v^0 = \left\{ \mu \in [CV_0(X)]^*; |\mu| \left(\frac{1}{v} \right) \leq 1 \right\},$$

respectively, $\sigma(\mu)$ the support of μ .

Also, for any linear subspace $\mathcal{W} \subset CV_0(X)$, we denote by $\mathcal{W}^0$ its polar set, i.e.

$$\mathcal{W}^0 = \{ \mu \in [CV_0(X)]^*; \mu(w) = 0, \forall w \in \mathcal{W} \}$$

and for any convex set $S \subset [CV_0(X)]^*$, we denote by $Ext(S)$ the set of all extreme points of S . $\square$

Theorem 3.1. *If $\mathcal{W} \subset CV_0(X)$ is a linear subspace then the closure of $\mathcal{W}$ in $(CV_0(X), \omega_V)$ is given by*

$$\overline{\mathcal{W}} = \left\{ f \in CV_0(X); f|\sigma(\mu) \in \overline{\mathcal{W}|\sigma(\mu)}, \forall \mu \in \text{Ext}(B_v^0 \cap \mathcal{W}^0), \forall v \in V \right\}.$$

Proof. We show only that for any function $g \in CV_0(X) \setminus \overline{\mathcal{W}}$, there exist $v \in V$ and $\mu \in \text{Ext}(B_v^0 \cap \mathcal{W}^0)$ such that $g|\sigma(\mu) \notin \overline{\mathcal{W}|\sigma(\mu)}$.

Indeed, using Hahn–Banach separation theorem there exists $\lambda \in [CV_0(X)]^*$ such that $\lambda \in \mathcal{W}^0$ and $\lambda(g) \neq 0$. Let $v \in V$ be such that $|\lambda(f)| \leq p_v(f)$, $\forall f \in CV_0(X)$, i.e. $|\lambda|(\frac{1}{v}) \leq 1$. Hence, $\lambda \in B_v^0 \cap \mathcal{W}^0$. Since $B_v^0 \cap \mathcal{W}^0$ is a compact convex subset of $[CV_0(X)]^*$ with respect to the weak topology and $\lambda(g) \neq 0$, it follows from Krein–Milman theorem that there exists $\mu \in \text{Ext}(B_v^0 \cap \mathcal{W}^0)$ such that $\mu(g) \neq 0$. Since $\mu \in \mathcal{W}^0$ we deduce, using Lemma 3.1, that $\int \varphi d\mu = 0$ for any $\varphi \in \overline{\mathcal{W}|\sigma(\mu)}$. Hence, $g|\sigma(\mu) \notin \overline{\mathcal{W}|\sigma(\mu)}$. $\square$

Corollary 3.1. *Let $\mathcal{W} \subset CV_0(X)$ be a linear subspace and let $(P_i)_{i \in I}$ be a partition of X such that for any $v \in V$ and any $\mu \in \text{Ext}(B_v^0 \cap \mathcal{W}^0)$ there exists P_{i_μ} such that $\sigma(\mu) \subset P_{i_\mu}$. Then we have*

$$\overline{\mathcal{W}} = \left\{ f \in CV_0(X); f|P_i \in \overline{\mathcal{W}|P_i}, \forall i \in I \right\}.$$

The following result is a generalization of de Branges lemma. In the particular case $V \leq C^+(X)$, this result was obtained by Prolla [5].

Lemma 3.2. *Let $\mathcal{W} \subset CV_0(X)$ be a linear subspace, $\mu \in \text{Ext}(B_v^0 \cap \mathcal{W}^0)$ for some $v \in V$ and let f be a real valued continuous and bounded function on $\sigma(\mu)$ such that $\mu(f \cdot w) = 0$, $\forall w \in \mathcal{W}$. Then f is constant on $\sigma(\mu)$.*

Proof. Let $n, m \in \mathbb{N}^*$ be sufficiently large such that

$$\frac{1}{m} \cdot f < 1; 0 < \frac{1}{n} \cdot \left(1 - \frac{1}{m} \cdot f\right) < 1 \text{ on } \sigma(\mu).$$

Obviously the function $g = \frac{1}{n} \cdot \left(1 - \frac{1}{m} \cdot f\right)$ has the same properties like f but $0 < g < 1$ on $\sigma(\mu)$.

We denote also by g the positive Borel extension of g on X such that $g = 0$ on $X \setminus \sigma(\mu)$. We consider the Radon measures μ_1, μ_2 on X given by

$$\mu_1 = \frac{g \cdot \mu}{|\mu|(\frac{g}{v})}, \mu_2 = \frac{(1-g) \cdot \mu}{|\mu|(\frac{1-g}{v})}.$$

Using Lemma 2.1 and Theorem 2.1, we have for any Radon measure λ on X :

$$\|\lambda\|_v = \sup \{ |\lambda(g)|; g \in \mathcal{K}(X), p_v(g) \leq 1 \} = |\lambda|\left(\frac{1}{v}\right), \text{ and}$$

$$B_v^0 = \left\{ \lambda \in \mathcal{M}(X); |\lambda|\left(\frac{1}{v}\right) \leq 1 \right\}$$

Particularly, we have $|\mu|(\frac{1}{v}) = \|\mu\|_v = 1$ since $\mu \in \text{Ext}(B_v^0 \cap \mathcal{W}^0)$.

Further, we have

$$\|\mu_1\|_v = |\mu_1|\left(\frac{1}{v}\right) = \frac{1}{|\mu|(\frac{g}{v})} \cdot |g| \cdot |\mu|\left(\frac{1}{v}\right) = \frac{|\mu|(\frac{g}{v})}{|\mu|(\frac{g}{v})} = 1,$$

and similarly $\|\mu_2\|_v = 1$. On the other hand, we have $\mu_1(w) = \mu_2(w) = 0$ for all $w \in \mathcal{W}$ and, therefore, $\mu_1, \mu_2 \in B_v^0 \cap \mathcal{W}^0$. If we denote by $\alpha = |\mu|(\frac{g}{v})$ and $\beta = |\mu|(\frac{1-g}{v})$, we have $\alpha + \beta = 1$ and $\alpha \cdot \mu_1 + \beta \cdot \mu_2 = g \cdot \mu + (1-g) \cdot \mu = \mu$. Since μ is an extreme point of $B_v^0 \cap \mathcal{W}^0$ it follows that $\mu_1 = \mu_2 = \mu$. Hence, $\frac{g}{\alpha} \cdot \mu = \mu$ and so $g = \alpha$ on $\sigma(\mu)$. Therefore, the function $f = m \cdot (1 - n \cdot g)$ is constant on $\sigma(\mu)$ and so the proof is finished. $\square$

4. Some Approximation Theorems in Weighted Spaces

Definition 4.1. Let $\mathcal{A} \subset C(X, \mathbb{C})$ be a nonempty set. A subset $S \subset X$ is called antisymmetric with respect to $\mathcal{A}$ if any $a \in \mathcal{A}$ which is real on S is constant on S .

We denote by $\mathcal{S}$ the family of all subsets of X which are antisymmetric with respect to $\mathcal{A}$. Obviously, $\mathcal{S} \neq \emptyset$ because for any $x \in X$, the set $\{x\} \in \mathcal{S}$.

Remark 4.1. The family $\mathcal{S}$ has the following properties:

- (i) If $S_i \in \mathcal{S}$, $i = 1, 2$, and $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2 \in \mathcal{S}$.
- (ii) The closure $\overline{S}$ of any $S \in \mathcal{S}$ belongs to $\mathcal{S}$.
- (iii) Any element $x \in X$ belongs to a maximal (with respect to the inclusion order relation) element of $\mathcal{S}$ denoted by S_x .
- (iv) For any $x, y \in X$, we have one or other of the relations:

$$S_x = S_y, S_x \cap S_y = \emptyset$$

(v)

$$X = \bigcup_{x \in X} S_x.$$

Theorem 4.1. Let $\mathcal{A}$ be a nonempty subset of $C(X, \mathbb{C})$ such that any element $a \in \mathcal{A}$ is a bounded function on the set $[v > 0]$ for each $v \in V$. If $\mathcal{W} \subset CV_0(X)$ is a linear subspace such that $\mathcal{A} \cdot \mathcal{W} \subset \mathcal{W}$, then the closure of $\mathcal{W}$ in $(CV_0(X), \omega_V)$ is given by

$$\overline{\mathcal{W}} = \left\{ f \in CV_0(X); f|_{S_x} \in \overline{\mathcal{W}|_{S_x}}, \forall x \in X \right\}$$

Proof. First, we show that for any $v \in V$ and any extreme element $\mu \in \text{Ext}(B_v^0 \cap \mathcal{W}^0)$ the set $\sigma(\mu)$ is antisymmetric with respect to $\mathcal{A}$. Indeed, since $1 = \|\mu\|_v = |\mu|(\frac{1}{v})$, we deduce that $|\mu|([v = 0]) = 0$. Since any element $a \in \mathcal{A}$ is bounded on the set $[v > 0]$ then a is bounded on the closure $\overline{[v > 0]}$ of this set. Since $v = 0$ on $X \setminus \overline{[v > 0]}$, we get

$$|\mu| \left(X \setminus \overline{[v > 0]} \right) = 0, \sigma(\mu) \subset \overline{[v > 0]},$$

and, therefore, any function $a \in \mathcal{A}$ is bounded on $\sigma(\mu)$. We have $\mu \in \mathcal{W}^0$ and $\mu(a \cdot w) = 0$ for any $a \in \mathcal{A}$ and any $w \in \mathcal{W}$. Using Lemma 3.2, we deduce that any element $a \in \mathcal{A}$ which is real on $\sigma(\mu)$ is constant on $\sigma(\mu)$. Therefore, $\sigma(\mu) \in \mathcal{S}$ and so there exists $x_\mu \in X$ such that $\sigma(\mu) \subset S_{x_\mu}$. $\square$

Remark 4.2. If $\mathcal{A} \subset C(X, \mathbb{C})$ is a self-adjoint algebra then any antisymmetric subset with respect to $\mathcal{A}$ is a set of constancy for $\mathcal{A}$. Particularly for any $x \in X$, we have

$$S_x = [x]_{\mathcal{A}} = \{y \in X; a(y) = a(x), \forall a \in \mathcal{A}\}.$$

Indeed, any element $a \in \mathcal{A}$ is of the form $a = a' + i \cdot a''$, where a', a'' are real functions on X .

Since $a' = \frac{a+\bar{a}}{2} \in \mathcal{A}$, $a'' = \frac{a-\bar{a}}{2 \cdot i} \in \mathcal{A}$, we deduce that a' and a'' are constant on any antisymmetric set with respect to $\mathcal{A}$ and, therefore, a is constant on any such set.

From previous remark, and Theorem 4.1, it follows:

Theorem 4.2. *Nachbin [4] Let $\mathcal{A}$ be a subalgebra of $C_b(X)$, self-adjoint in the complex case, and let $\mathcal{W} \subset CV_0(X)$ be a linear subspace such that $\mathcal{A} \cdot \mathcal{W} \subset \mathcal{W}$. Then $\mathcal{W}$ is localizable with respect to $\mathcal{A}$, i.e.*

$$\overline{\mathcal{W}} = \{f \in CV_0(X); f|_{[x]_{\mathcal{A}}} \in \overline{\mathcal{W}|_{[x]_{\mathcal{A}}}}, \forall x \in X\}.$$

Corollary 4.1. *Let $\mathcal{A}$ be a subalgebra of $C_b(X)$ containing the constant function 1, separating the points of X , and self-adjoint in the complex case. If $\mathcal{W} \subset CV_0(X)$ is a linear subspace such that $\mathcal{A} \cdot \mathcal{W} \subset \mathcal{W}$ and for any $x \in X$ there exists a $w \in \mathcal{W}$ such that $w(x) \neq 0$, then $\mathcal{W}$ is dense in $CV_0(X)$, i.e.*

$$\overline{\mathcal{W}} = CV_0(X).$$

Proof. Since $\mathcal{A}$ separates the points of X it follows that $[x]_{\mathcal{A}} = \{x\}$, $\forall x \in X$. Let $f \in CV_0(X)$ be arbitrary. If $f(x) = 0$, then obviously $f|_{[x]_{\mathcal{A}}} \in \overline{\mathcal{W}|_{[x]_{\mathcal{A}}}}$. If $f(x) \neq 0$, then, there exists $w \in \mathcal{W}$ such that $w(x) \neq 0$. Further, we have

$$f|_{[x]_{\mathcal{A}}} = f(x) = \lambda \cdot w(x) = \lambda \cdot w|_{[x]_{\mathcal{A}}} \in \overline{\mathcal{W}|_{[x]_{\mathcal{A}}}}, \text{ where } \lambda = \frac{f(x)}{w(x)}.$$

Therefore, we have

$$CV_0(X) = \left\{f \in CV_0(X); f|_{[x]_{\mathcal{A}}} \in \overline{\mathcal{W}|_{[x]_{\mathcal{A}}}}, \forall x \in X\right\}.$$

Now, from Theorem 4.2, we deduce

$$\overline{\mathcal{W}} = CV_0(X).$$

□

Definition 4.2. Let $\mathcal{M} \subset C(X, \mathbb{R})$ and $\mathcal{W} \subset CV_0(X)$ be two nonempty subsets. A subset S of X will be called antialgebraic with respect to the pair $(\mathcal{M}, \mathcal{W})$ if any element $m \in \mathcal{M}$ such that

$$m \cdot w|_S \in \mathcal{W}|_S, \forall w \in \mathcal{W}$$

is constant on S .

If we denote by $\mathcal{T}$ the family of all antialgebraic subsets of X with respect to the pair $(\mathcal{M}, \mathcal{W})$ then for any $x \in X$ the singleton $\{x\}$ belongs to $\mathcal{T}$. The set $\mathcal{T}$ endowed with the inclusion order relation has similar properties as family $\mathcal{S}$ in the Remark 4.1. For any $x \in X$ we denote by T_x the maximal $(\mathcal{M}, \mathcal{W})$ -antialgebraic subset containing x . We have $X = \bigcup_{x \in X} T_x$ and $\{T_x\}_{x \in X}$ is a partition of X .

Remark 4.3. If we have two pairs $(\mathcal{M}_1, \mathcal{W})$, $(\mathcal{M}_2, \mathcal{W})$ as above and we denote by $\mathcal{T}_i$ the family of all antialgebraic subsets of X with respect to $(\mathcal{M}_i, \mathcal{W})$, $i = 1, 2$, then we have

$$\mathcal{M}_1 \subset \mathcal{M}_2 \Rightarrow \mathcal{T}_2 \subset \mathcal{T}_1 \Rightarrow T_{2x} \subset T_{1x}, \forall x \in X,$$

where for any $x \in X$, T_{ix} denotes the maximal element from $\mathcal{T}_i$ containing the point x .

Theorem 4.3. *Let $\mathcal{M}$ be a V -bounded (i.e. any function $m \in \mathcal{M}$ is bounded on the support of any weight $v \in V$) nonempty subset of $C(X, \mathbb{R})$ and let $\mathcal{W}$ be a linear subspace of $CV_0(X)$. If we denote by T_x , the maximal antialgebraic set with respect to the pair $(\mathcal{M}, \mathcal{W})$ such that $x \in T_x$, then we have*

$$\overline{\mathcal{W}} = \{f \in CV_0(X); f|_{T_x} \in \overline{\mathcal{W}|_{T_x}}, \forall x \in X\}.$$

Proof. Applying Corollary 3.1, it will be sufficient to show that for any $v \in V$ and any $\mu \in \text{Ext}(B_v^0 \cap \mathcal{W}^0)$ the set $\sigma(\mu)$ is included in T_{x_0} for some $x_0 \in X$ or equivalently to show that $\sigma(\mu)$ is an antialgebraic set with respect to the pair $(\mathcal{M}, \mathcal{W})$. Let $m \in \mathcal{M}$ be such that $m \cdot w|_{\sigma(\mu)} \in \mathcal{W}|_{\sigma(\mu)}$ for all $w \in \mathcal{W}$. Using Lemma 3.2, we deduce that m is constant on $\sigma(\mu)$. Hence, $\sigma(\mu)$ is an antialgebraic set with respect to the pair $(\mathcal{M}, \mathcal{W})$ and the proof is finished. $\square$

In the particular case $V \leq C^+(X)$, this result was obtained in [8].

Theorem 4.4. *Let $\mathcal{W}$ be a linear subspace of $CV_0(X)$ and let $\mathcal{M}$ be a subset of continuous real bounded functions on X . If for any $x \in X$ we denote by T_x the maximal antialgebraic subset with respect to the pair $(\mathcal{M}, \mathcal{W})$ containing x , then we have*

$$\overline{\mathcal{W}} = \left\{ f \in CV_0(X); f|_{T_x} \in \overline{\mathcal{W}|_{T_x}}, \forall x \in X \right\}.$$

Proof. The assertion follows from Theorem 4.3 since any function of $\mathcal{M}$ is V -bounded. $\square$

Corollary 4.2. *Let $\mathcal{M}$, $\mathcal{W}$ be as in Theorem 4.4. If we suppose in addition that $\mathcal{M} \cdot \mathcal{W} \subset \mathcal{W}$, then we have*

$$\overline{\mathcal{W}} = \left\{ f \in CV_0(X); f|_{[x]_{\mathcal{M}}} \in \overline{\mathcal{W}|_{[x]_{\mathcal{M}}}}, \forall x \in X \right\},$$

where for any $x \in X$ we have denoted by

$$[x]_{\mathcal{M}} = \{y \in X; m(y) = m(x), \forall m \in \mathcal{M}\}.$$

Proof. From the hypothesis $\mathcal{M} \cdot \mathcal{W} \subset \mathcal{W}$, we deduce that $[x]_{\mathcal{M}} = T_x$, where T_x is the maximal $(\mathcal{M}, \mathcal{W})$ -antialgebraic set containing x . The proof is finished applying Theorem 4.4. $\square$

Corollary 4.3. *Let $\mathcal{W}$ be a linear subspace of $CV_0(X)$ and let $\mathcal{M}$ be a subset of continuous real bounded functions on X such that $\mathcal{M} \cdot \mathcal{W} \subset \mathcal{W}$. If we suppose in addition that $\mathcal{M}$ separates the points of X and we denote by*

$$X_0 = \{x \in X; w(x) = 0, \forall w \in \mathcal{W}\},$$

then we have

$$\overline{\mathcal{W}} = \begin{cases} CV_0(X), & \text{if } X_0 = \phi \\ \{f \in CV_0(X); f|X_0 = 0\}, & \text{if } X_0 \neq \phi \end{cases}$$

Proof. Since $\mathcal{M} \cdot \mathcal{W} \subset \mathcal{W}$ and $\mathcal{M}$ separates the points of X it follows that

$$T_x = [x]_{\mathcal{M}} = \{y \in X; m(y) = m(x), \forall m \in \mathcal{M}\} = \{x\}, \forall x \in X.$$

On the other hand, if we denote $\mathcal{W}(X) = \{w(x); w \in \mathcal{W}\}$, then we have

$$\mathcal{W}(x) = \begin{cases} \mathbb{R} \text{ or } \mathbb{C} & \text{if } x \notin X_0 \\ \{0\} & \text{if } x \in X_0 \end{cases}$$

We finish the proof applying Corollary 4.2. $\square$

Let $S \subset X$ be a closed set and let $I_S = \{f \in CV_0(X); f|S = 0\}$. Obviously, I_S is an order ideal of $CV_0(X)$. We remark also that I_S is closed with respect to the weighted topology. Indeed, let $x_0 \in S$ be arbitrary and let $v_0 \in V$ be a weight with the property $v_0(x_0) > 0$. If $g \in \overline{I_S}$, then for any $\varepsilon > 0$ and any $v \in V$, there exists $f \in I_S$ such that

$$p_v(g - f) = \sup \{|g(x) - f(x)| \cdot v(x); \forall x \in X\} < \varepsilon.$$

In the particular case $x = x_0$ and $v = v_0$, it results

$$|g(x_0) - f(x_0)| \cdot v_0(x_0) = |g(x_0)| \cdot v_0(x_0) < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we deduce that $g(x_0) = 0$, so $g \in I_S$. In [3], Lemma 3.8, C.Portenier states that any closed order ideal of $CV_0(X, \mathbb{R})$ has the preceding form. Using Corollary 4.2, we give a very simple proof of Portenier's result.

Theorem 4.5. *Let $I \subset CV_0(X, \mathbb{R})$ be an arbitrary ideal. Then there exists a closed subset $S_I \subset X$ such that*

$$\overline{I} = \{f \in CV_0(X, \mathbb{R}); f|S_I = 0\}.$$

Particularly, if I is closed then $I = \{f \in CV_0(X, \mathbb{R}); f|S_I = 0\}$ for some closed subset $S_I \subset X$.

Proof. Let $\mathcal{M} = C(X, [0, 1])$ and for any $x \in X$, let $[x]_{\mathcal{M}}$ be the subset of constancy for the functions from $\mathcal{M}$, i.e. $[x]_{\mathcal{M}} = \{y \in X; m(y) = m(x), \forall m \in \mathcal{M}\}$. The set I being an order ideal we get $\mathcal{M} \cdot I \subset I$. On the other hand, it is obviously that $\mathcal{M} = C(X, [0, 1])$ separates the points of X . The assertion follows now from Corollary 4.3. $\square$

Remark 4.4. If $\mathfrak{I}$ denotes the set of all closed order ideal of $CV_0(X, \mathbb{R})$ and $\mathfrak{F}$ denotes the set of all closed subsets of X then the map:

$$I \rightarrow S_I = \{x \in X; h(x) = 0, \forall h \in I\}$$

is a bijection between $\mathfrak{I}$ and $\mathfrak{F}$ just a decreasing one:

$$I' \subset I'' \Leftrightarrow S_{I''} \subset S_{I'}.$$

This allows us to generalize some results involving different type of closed subset of X (antisymmetric, interpolating, antialgebraic sets) to the abstract case of closed order ideals in a locally convex lattices.

5. Stone–Weierstrass Theorem for Convex Cones in Weighted Spaces

In this section, we consider a convex cone $\mathcal{C} \subset CV_0(X, \mathbb{R})$ and we denote by $\mathcal{C}^0$ its polar set, i.e.

$$\mathcal{C}^0 = \{\mu \in [CV_0(X, \mathbb{R})]^*; \mu(f) \leq 0, \forall f \in \mathcal{C}\}.$$

For any weight $v \in V$, we denote by $\text{Ext}\{B_v^0 \cap \mathcal{C}^0\}$ the set of all extreme points of the compact convex set $B_v^0 \cap \mathcal{C}^0$ of the dual $[CV_0(X, \mathbb{R})]^*$ of the locally convex space $(CV_0(X, \mathbb{R}), \omega_V)$. We remember that the closure of any convex cone in an arbitrary locally convex space coincides with its bipolar with respect to the natural duality. In our case,

$$\overline{\mathcal{C}} = \{f \in CV_0(X); \mu(f) \leq 0, \forall \mu \in \mathcal{C}^0\}.$$

The following result is a generalization of de Branges Lemma for a convex cone.

Lemma 5.1. *Let $\mathcal{C} \subset CV_0(X, \mathbb{R})$ be a convex cone, $v \in V$ and $\mu \in \text{Ext}\{B_v^0 \cap \mathcal{C}^0\}$. If $\sigma(\mu)$ denotes the support of the Radon measure μ , then any function $\varphi \in C(X, \mathbb{R})$ such that*

- (i) $0 \leq \varphi(x) \leq 1, \forall x \in \sigma(\mu)$,
- (ii) $\varphi \cdot f|_{\sigma(\mu)}, (1 - \varphi) \cdot f|_{\sigma(\mu)} \in \overline{\mathcal{C}|\sigma(\mu)}|, \forall f \in \mathcal{C}$

is a constant function on $\sigma(\mu)$.

Proof. Since $\mu \in \text{Ext}\{B_v^0 \cap \mathcal{C}^0\}$ we deduce that $|\mu|_v = |\mu|(\frac{1}{v}) = 1$. If

$$|\mu|(\varphi) = 0 \text{ or } |\mu|(1 - \varphi) = 0,$$

we have $\varphi = 0$ or $\varphi = 1$ on $\sigma(\mu)$. We suppose now that $|\mu|(\varphi) \neq 0$ and $|\mu|(1 - \varphi) \neq 0$ and we consider the measures μ_1, μ_2 given by

$$\mu_1 = \frac{\varphi \cdot \mu}{|\mu|(\frac{\varphi}{v})}, \mu_2 = \frac{(1 - \varphi) \cdot \mu}{|\mu|(\frac{1 - \varphi}{v})}.$$

Further, we have

$$\|\mu_1\|_v = |\mu_1|\left(\frac{1}{v}\right) = \frac{|\mu|(\varphi \cdot \frac{1}{v})}{|\mu|(\frac{\varphi}{v})} = 1, \|\mu_2\|_v = |\mu_2|\left(\frac{1}{v}\right) = \frac{|\mu|((1 - \varphi) \cdot \frac{1}{v})}{|\mu|(\frac{1 - \varphi}{v})} = 1.$$

Since $\varphi \cdot h|_{\sigma(\mu)} \in \overline{\mathcal{C}|\sigma(\mu)}$ and $(1 - \varphi) \cdot h|_{\sigma(\mu)} \in \overline{\mathcal{C}|\sigma(\mu)}, \forall h \in \mathcal{C}$ and using Lemma 3.1, having in mind that $\mu \in \mathcal{C}^0$, we deduce

$$\mu_1(h) = \frac{\mu(\varphi \cdot h)}{|\mu|(\frac{\varphi}{v})} \leq 0, \mu_2(h) = \frac{\mu[(1 - \varphi) \cdot h]}{|\mu|(\frac{1 - \varphi}{v})} \leq 0.$$

Hence, $\mu_1, \mu_2 \in \text{Ext}\{B_v^0 \cap \mathcal{C}^0\}$. On the other hand, since

$$|\mu|\left(\frac{\varphi}{v}\right) \cdot \mu_1 + |\mu|\left(\frac{1 - \varphi}{v}\right) \cdot \mu_2 = \mu \text{ and } |\mu|\left(\frac{\varphi}{v}\right) + |\mu|\left(\frac{1 - \varphi}{v}\right)$$

$$= 1 \text{ we get } \mu_1 = \mu_2 = \mu, \text{ i.e.}$$

$$\frac{\varphi}{|\mu|(\frac{\varphi}{v})} = 1 \text{ on } \sigma(\mu).$$

□

Theorem 5.1. *A function $f \in CV_0(X)$ belongs to the closure $\overline{\mathcal{C}}$ of the convex cone $\mathcal{C}$ in the locally convex space $(CV_0(X, \mathbb{R}), \omega_V)$ if and only if for any weight $v \in V$ and any $\mu \in \text{Ext}\{B_v^0 \cap \mathcal{C}^0\}$, we have*

$$\mu(f) \leq 0.$$

Proof. We show only that if $f \in CV_0(X, \mathbb{R}) \setminus \overline{\mathcal{C}}$, there exist $v \in V$ and $\mu \in \text{Ext}\{B_v^0 \cap \mathcal{C}^0\}$ such that $\mu(f) > 0$. Indeed, if $f \notin \overline{\mathcal{C}} = \mathcal{C}^{00}$ there exists $\lambda \in [CV_0(X, \mathbb{R})]^*$, $\lambda \in \mathcal{C}^0$ such that $\lambda(f) > 0$. Let us consider $v \in V$ such that $|\lambda|(\frac{1}{v}) = 1 = \|\lambda\|_v$, $\lambda \in B_v^0$. Since the following map:

$$\theta : B_v^0 \cap \mathcal{C}^0 \rightarrow \mathbb{R}, \theta(f) = \int f d\lambda$$

is a continuous affine function if we endow the set $B_v^0 \cap \mathcal{C}^0$ with the trace of the weak topology on $[CV_0(X, \mathbb{R})]^*$ and the maximum of this map is realized on a point $\mu \in \text{Ext}\{B_v^0 \cap \mathcal{C}^0\}$, we deduce that $\mu(f) \geq \lambda(f) > 0$. $\square$

The following statement is a procedure to describe the closure of a convex cone in some circumstances.

Corollary 5.1. *Let $\mathcal{C} \subset CV_0(X, \mathbb{R})$ be a convex cone and let $(P_\alpha)_{\alpha \in I}$ be a partition of X such that for any $v \in V$, and any $\mu \in \text{Ext}\{B_v^0 \cap \mathcal{C}^0\}$, there exists $\alpha \in I$ such that the support of $\mu, \sigma(\mu) \subset P_\alpha$. Then we have*

$$\overline{\mathcal{C}} = \left\{ f \in CV_0(X, \mathbb{R}); f|_{P_\alpha} \in \overline{\mathcal{C}|_{P_\alpha}}, \forall \alpha \in I \right\}.$$

Further, we state such kind of such circumstances. A nonempty subset $\mathcal{M}$ of $C(X, [0, 1])$ is said to be a set with complement if for any $\varphi \in \mathcal{M}$, we have $1 - \varphi \in \mathcal{M}$.

The following definition is analogous with Definition 4.6. of [9]

Definition 5.1. A subset $S \subset X$ is called antisymmetric with respect to the pair $(\mathcal{M}, \mathcal{C})$ if any function $\varphi \in \mathcal{M}$ with the properties:

$$\varphi \cdot f|_S \in \overline{\mathcal{C}|_S}, (1 - \varphi) \cdot f|_S \in \overline{\mathcal{C}|_S}, \forall f \in \mathcal{C}$$

is a constant function on S .

Further, we denote by $\mathcal{B}$ the family of all subsets of X antisymmetric with respect to the pair $(\mathcal{M}, \mathcal{C})$. The following assertions are almost obvious.

- (i) $\{x\} \in \mathcal{B}, \forall x \in X$
- (ii) $B_1, B_2 \in \mathcal{B}, B_1 \cap B_2 \neq \emptyset \Rightarrow B_1 \cup B_2 \in \mathcal{B}$
- (iii) For any upper directed family $(B_\alpha)_{\alpha \in I}$ from $\mathcal{B}$ we have $\bigcup_{\alpha \in I} B_\alpha \in \mathcal{B}$.

For any $x \in X$, we denote $B_x = \bigcup \{B; B \in \mathcal{B}, x \in B\}$. We have

$$B_x = \overline{B_x} \in \mathcal{B}, B_x \cap B_y = \emptyset \text{ if } B_x \neq B_y.$$

The family $(B_x)_{x \in X}$ is a partition of X and for any $B \in \mathcal{B}$, there exists $x \in X$ such that $B \subset B_x$.

Remark 5.1. With the above notations, we have for any $v \in V$ and any $\mu \in \text{Ext}\{B_v^0 \cap \mathcal{C}^0\}$ the set $\sigma(\mu)$ belongs to $\mathcal{B}$. Indeed, if $\varphi \in \mathcal{M}$ is such that $\varphi \cdot f|_{\sigma(\mu)} \in \overline{\mathcal{C}|\sigma(\mu)}$, $(1 - \varphi) \cdot f|_{\sigma(\mu)} \in \overline{\mathcal{C}|\sigma(\mu)}$, $\forall f \in \mathcal{C}$, we deduce from Lemma 5.1, that φ is a constant function on $\sigma(\mu)$.

Theorem 5.2. *Let X be a locally compact Hausdorff space, V be a Nachbin family of weights on X , $\mathcal{C} \subset CV_0(X, \mathbb{R})$ be a convex cone and $\mathcal{M} \subset C(X, [0, 1])$ be a nonempty subset with complement. Then*

$$\overline{\mathcal{C}} = \left\{ f \in CV_0(X, \mathbb{R}); f|_{B_x} \in \overline{\mathcal{C}|\overline{B_x}}, \forall x \in X \right\},$$

where $(B_x)_{x \in X}$ is the family of all maximal subsets of X antisymmetric with respect to the pair $(\mathcal{M}, \mathcal{C})$.

Proof. The assertion follows from the above remark and from the Corollary 5.1. First, this result was obtained in the case $V \leq C^+(X)$ in [7]. See also [6] for compact spaces. $\square$

Corollary 5.2. *Let $X, V, \mathcal{C}, \mathcal{M}$ as in Theorem 5.2 such that $\mathcal{M} \cdot \mathcal{C} \subset \mathcal{C}$. Then we have*

$$\overline{\mathcal{C}} = \left\{ f \in CV_0(X, \mathbb{R}); f|_{[x]_{\mathcal{M}}} \in \overline{\mathcal{C}|\overline{[x]_{\mathcal{M}}}}, \forall x \in X \right\},$$

where, for any $x \in X, [x]_{\mathcal{M}} = \{y \in X; m(y) = m(x), \forall m \in \mathcal{M}\}$.

Proof. Since $\mathcal{M}$ is a set with complement and $\mathcal{M} \cdot \mathcal{C} \subset \mathcal{C}$ we deduce that all functions from $\mathcal{M}$ are constant on any antisymmetric set with respect to the pair $(\mathcal{M}, \mathcal{C})$ and for any $x \in X$ the set $[x]_{\mathcal{M}}$ is antisymmetric set with respect to the pair $(\mathcal{M}, \mathcal{C})$. Hence, the set $[x]_{\mathcal{M}} = B_x, \forall x \in X$. $\square$

The assertion follows now from Theorem 5.2.

Corollary 5.3. *Let $X, V, \mathcal{C}, \mathcal{M}$ as in Corollary 5.2. Moreover, we suppose that $\mathcal{M}$ separates the points of X , i.e. for any $x, y \in X, x \neq y$ there exists $m \in \mathcal{M}$ such that $m(x) \neq m(y)$. Let us denote*

$$\begin{aligned} X_- &= \{x \in X; f(x) \leq 0, \forall f \in \mathcal{C}\} \\ X_+ &= \{x \in X; f(x) \geq 0, \forall f \in \mathcal{C}\}. \end{aligned}$$

Then we have

$$\overline{\mathcal{C}} = \{f \in CV_0(X, \mathbb{R}); f \geq 0 \text{ on } X_+, f \leq 0 \text{ on } X_-\}.$$

Example 5.1. Let $X = \mathbb{R}$ and let $\varphi : \mathbb{R} \rightarrow (0, 1)$ be the strictly continuous homeomorphism given by $\varphi(x) = \frac{2}{\pi} \cdot \arctan(e^x)$. On the space $C_b(\mathbb{R})$ of all real bounded and continuous functions on $\mathbb{R}$, we consider the strict topology β given by the family of seminorms:

$$f \rightarrow p_g(f) = \sup_{x \in \mathbb{R}} g(x) \cdot |f(x)|, \forall f \in C_b(\mathbb{R}),$$

where g runs the set $C_0^+(\mathbb{R})$ of all positive, continuous functions on $\mathbb{R}$ vanishing at infinity. In fact $\beta = \omega_V$, where the Nachbin family of weights V is just $C_0^+(\mathbb{R})$. We know that

$$CV_0(\mathbb{R}) = C_b(\mathbb{R}).$$

Let us consider the convex cone $\mathcal{C}$ in $C_b^+(\mathbb{R})$ given by

$$\mathcal{C} = \left\{ P(\varphi, 1 - \varphi); P(x, y) = \sum_{i,j=1}^n a_{ij} \cdot x^i \cdot y^j, a_{ij} \geq 0, n \in \mathbb{N} \right\}.$$

Obviously, $\mathcal{C}$ separates the points of X since $\varphi \in \mathcal{C}$. Using the notations from Corollary 5.3, we have $X_0 = \phi$, $X_+ = \mathbb{R}$ and, therefore, $\overline{\mathcal{C}} = C_b^+(\mathbb{R})$, i.e. for any $f \in C_b^+(\mathbb{R})$ and any $g \in C_0^+(\mathbb{R})$, there exists a sequence $(P_k)_k$ of polynomials $P_k(x, y) = \sum_{i,j=1}^{n_k} a_{ij}^k \cdot x^i \cdot y^j$, $a_{ij}^k \geq 0$ such that the sequence $(g \cdot P_k(\varphi, 1 - \varphi))_k$ converges uniformly to $g \cdot f$ on $\mathbb{R}$.

Example 5.2. Let $X = (0, 1)$ and let $V = C_0^+(X)$ be the set of all positive, continuous functions v on X vanishing at infinity, i.e. $\lim_{x \rightarrow 0} v(x) = 0 = \lim_{x \rightarrow 1} v(x)$. On the space $C_b(X)$ of all real continuous functions on X we consider the strict topology β given by the family of seminorms:

$$f \rightarrow p_v(f) = \sup_{x \in X} v(x) \cdot |f(x)|, \forall f \in C_b(X).$$

In fact, $\beta = \omega_V$ and as it is known we have $CV_0(X) = C_b(X)$. Let us consider the convex cone $\mathcal{C}$ in $C_b^+(X)$ of all functions of the form:

$$x \in X \rightarrow c(x) = \sum_{i,j=1}^n a_{ij} \cdot x^i \cdot (1-x)^j,$$

where n runs the set $\mathbb{N}^*$, $a_{i,j} \in \mathbb{R}$, $a_{i,j} \geq 0$. The required conditions of Corollary 5.3 are satisfied if we take $\mathcal{M} = \{x, 1-x\}$. Hence, $\overline{\mathcal{C}} = C_b^+(X)$, i.e. any continuous, positive and bounded function on X may be approximated with functions of the form $\sum_{i,j=1}^n a_{ij} \cdot x^i \cdot (1-x)^j$, $a_{i,j} \geq 0$.

Example 5.3. On the space $X = (0, 2)$, we consider $V = C_0^+(X)$, $C_b(X)$, p_v with $v \in V$, as in the previous example and we consider also the convex cone $\mathcal{C}$ of all functions of the form:

$$x \in X \rightarrow c(x) = \sum_{i,j,k=1}^n a_{ijk} \cdot x^i \cdot (1-x)^{2j-1} \cdot (2-x)^k, n \in \mathbb{N}^*, a_{i,j,k} \geq 0(1)$$

We take $\mathcal{M} = \{\frac{1}{2} \cdot x, 1 - \frac{1}{2} \cdot x\}$ and we have, using the notations of Corollary 5.3, $X_+ = (0, 1]$, $X_- = [1, 2)$ and, therefore, any continuous, real, bounded function on $(0, 2)$ such that $f(x) \geq 0$, $\forall x \in (0, 1)$, $f(1) = 0$, $f(x) \leq 0$, $\forall x \in (1, 2)$ may be approximated with respect to the strict topology on the open interval $(0, 2)$ by functions of the form (1).

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The following example was suggested by the reviewer:

If in the previous example we replace the convex cone $\mathcal{C}$ in C_1 —the convex cone of all polynomial functions $\varphi : (0, 2) \rightarrow \mathbb{R}$ of the form $\varphi(x) = \sum_{i,k} a_{i,k} \cdot x^i \cdot (1-x) \cdot (2-x)^k$, $a_{i,k} \geq 0$, then any continuous, real, bounded function $f : (0, 2) \rightarrow \mathbb{R}$, with the properties:
 $f(0) = f(1) = f(2) = 0$, $f \geq 0$ on $(0, 1)$ and $f \leq 0$ on $(1, 2)$

may be approximated with respect to the strict topology on $C_b((0,1))$ by functions from C_1 .

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