On the Existence of an Extremal Function in the Delsarte Extremal Problem

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Abstract. This paper is concerned with a Delsarte-type extremal problem. Denote by $\mathcal{P}(G)$ the set of positive definite continuous functions on a locally compact abelian group G . We consider the function class, which was originally introduced by Gorbachev,

$$\mathcal{G}(W, Q)_G = \{f \in \mathcal{P}(G) \cap L^1(G) : \\ f(0) = 1, \text{ supp } f_+ \subseteq W, \text{ supp } \hat{f} \subseteq Q\}$$

where $W \subseteq G$ is closed and of finite Haar measure and $Q \subseteq \hat{G}$ is compact. We also consider the related Delsarte-type problem of finding the extremal quantity

$$\mathcal{D}(W, Q)_G = \sup \left\{ \int_G f(g) d\lambda_G(g) : f \in \mathcal{G}(W, Q)_G \right\}.$$

The main objective of the current paper is to prove the existence of an extremal function for the Delsarte-type extremal problem $\mathcal{D}(W, Q)_G$. The existence of the extremal function has recently been established by Berdysheva and Révész in the most immediate case where $G = \mathbb{R}^d$. So, the novelty here is that we consider the problem in the general setting of locally compact abelian groups. In this way, our result provides a far reaching generalization of the former work of Berdysheva and Révész.

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1. Introduction

The Fourier analytic formulation of the so-called Delsarte extremal problem on $\mathbb{R}^d$ incorporates the calculation of the numerical quantity

$$\sup \hat{f}(0) = \sup \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(x) dx,$$

provided that

- (i) $f \in L^1(\mathbb{R}^d)$, f is continuous and bounded on $\mathbb{R}^d$,
- (ii) $f(0) = 1$,
- (iii) $f(x) \leq 0$ for $\|x\| \geq 2$ and
- (iv) $\widehat{f}(y) \geq 0$. The last property (iv) can be interpreted as f being positive definite; see the precise definition of positive definiteness in the forthcoming section.

The Delsarte extremal problem has generated broad interest because of its intimate connections to different problems from various branches of mathematics. First of all, the linear programming bound of Delsarte is useful in coding and design theory as well. Second, let us mention that relying on Delsarte's problem, upper bounds can be derived for the sphere packing density of $\mathbb{R}^d$ [2, 8, 17, 23, 24]. Moreover, Gorbachev and Tikhonov [10] worked out a further concrete application of the Delsarte problem for the so-called Wiener problem.

A few of years ago, Viazovska [22] solved the sphere packing problem in dimension 8, combining the Delsarte extremal problem with modular form techniques. Subsequently, in the paper [4], Cohn et al. resolved the problem also in dimension 24.

Besides solving the Delsarte problem, further challenging and closely related questions come into picture. As for recent investigations in this direction, we refer to the seminal paper of Berdysheva and Révész [1]. They have pointed out the independence of the extremal constant from the underlying function class. Furthermore, they showed the existence of an extremal function in band-limited cases. The main objective of the current paper is to prove an analogous result for general LCA groups. Actually we discuss two proofs, the key difference being the use in the second one of a suggestion for which we thank our referee.

2. The Result

Before moving on, we need some more preliminaries. In the first part of the section, we summarize the necessary background from the field of abstract harmonic analysis. Let G be a locally compact abelian group (LCA group for short). The *dual group* of G is denoted by $\widehat{G}$, by which we mean the set of continuous homomorphisms of G into the complex unit circle $\mathbb{T}$, the multiplication being the pointwise multiplication of functions. For a compact set $K \subseteq G$ and an open set $U \subseteq \mathbb{T}$, consider the set

$$P(K, U) := \{\chi \in \widehat{G} : \chi(K) \subseteq U\}.$$

Then the *compact open topology* on $\widehat{G}$ contains the sets $P(K, U)$ as a subbasis. By this topology, $\widehat{G}$ acquires an LCA group structure. The *Pontryagin-van Kampen Duality Theorem* asserts that G is isomorphic to $\widehat{\widehat{G}}$, both as groups and as topological spaces. In this case δ stands for the corresponding natural

isomorphism, that is,

$$\delta_g(\chi) := \chi(g), \quad \chi \in \widehat{G}$$

and $\delta : G \rightarrow \widehat{\widehat{G}}, g \mapsto \delta_g$ which is usually called the *Pontryagin map*.

Recall that a *continuous* function $f \in C(G)$ is called *positive definite* (denoted by $f \gg 0$) if the inequality

$$\sum_{j=1}^n \sum_{k=1}^n c_j \overline{c_k} f(g_j - g_k) \geq 0 \quad (1)$$

holds for all choices of $n \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $g_j \in G$ for $j = 1, \dots, n$. Throughout the paper, the set of continuous positive definite functions defined on G will be denoted by $\mathcal{P}(G)$. If λ_G is a (fixed, conveniently normalized) Haar measure on G , then the condition (1) for continuous f is equivalent to

$$\int_G \int_G f(g-s) \varphi(g) \overline{\varphi(s)} d\lambda_G(g) d\lambda_G(s) \geq 0$$

for every $\varphi \in L^1(G)$ (see, for instance [6, Proposition 13.4.4.]). The next properties will be quite useful in the sequel. We have [12, 32.4.]

Lemma 1. *Let G be an LCA group and denote by $\star$ the convolution.*

- (1) *If f is a positive definite function on G , then*
 - (a) $|f(g)| \leq f(0) = \|f\|_\infty$ for all $g \in G$;
 - (b) $\int_G f d\lambda_G \geq 0$.
- (2) *If $\varphi \in L^2(G)$ and $\tilde{\varphi}$ is defined as $\tilde{\varphi}(g) := \overline{\varphi(-g)}$ ($g \in G$), then the convolution square $\varphi \star \tilde{\varphi}$ is a continuous positive definite function.*

For any $f \in L^1(G)$, its *Fourier transform* $\widehat{f}$ is defined on $\widehat{G}$ as

$$\widehat{f}(\chi) = \int_G f(g) \overline{\chi(g)} d\lambda_G(g), \quad \chi \in \widehat{G}.$$

The *Inversion Theorem* (cf. [20, Sect. 1.5]) asserts that if f belongs to $[\mathcal{P}(G) \cap L^1(G)]$, the subspace generated by $\mathcal{P}(G) \cap L^1(G)$, then $\widehat{f} \in L^1(\widehat{G})$ and the

Haar measure $\lambda_{\widehat{G}}$ on $\widehat{G}$ can be normalized so that $f(g) = \widehat{\widehat{f}}(\delta_{g^{-1}})$. We shall use this Haar measure (the so-called Plancherel measure) on the dual group $\widehat{G}$. For $k \in L^1(\widehat{G})$, introducing the *conjugate Fourier transform* $\mathcal{F}^*$ as

$$\mathcal{F}^*(k)(g) := \int_{\widehat{G}} k(\chi) \delta_g(\chi) d\lambda_{\widehat{G}}(\chi), \quad g \in G,$$

the Inversion Theorem can be rephrased as $f = \mathcal{F}^*(\widehat{f})$ and is satisfied for every $f \in [\mathcal{P}(G) \cap L^1(G)]$.

Another important tool in our study is the *Plancherel Theorem* which asserts that the Fourier transform $\mathcal{F} : [\mathcal{P}(G) \cap L^1(G)] \rightarrow L^2(\widehat{G})$ can be extended to a unitary equivalence $U : L^2(G) \rightarrow L^2(\widehat{G})$. This unitary operator is called the *Plancherel transform*. We abuse notation and do not distinguish the usual Fourier transform and the latter extension.

Denote, as usual, $x_+ := \max(x, 0)$ and $x_- := \max(-x, 0)$ for any $x \in \mathbb{R}$, with similar notation for functions as well. In this paper, we consider the function class

$$\mathcal{G}(W, Q)_G = \{f \in \mathcal{P}(G) \cap L^1(G) : f(0) = 1, \operatorname{supp} f_+ \subseteq W, \operatorname{supp} \widehat{f} \subseteq Q\}, \quad (2)$$

where $W \subseteq G$ is closed and of finite Haar measure and $Q \subseteq \widehat{G}$ is compact. It was originally introduced by Gorbachev [8] in connection with the Delsarte-type problem of finding the extremal quantity

$$\mathcal{D}(W, Q)_G = \sup \left\{ \int_G f(g) d\lambda_G(g) : f \in \mathcal{G}(W, Q)_G \right\} \quad (3)$$

in the most immediate case where $G = \mathbb{R}^d$,

$$W = \mathbb{B} = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$$

and $Q = r\mathbb{B}$ with some real number $r > 0$.

In the very recent publication [1], Berdysheva and Révész analyzed in detail the aforementioned Delsarte-type extremal quantity. When $G = \mathbb{R}^d$, they collect and work up extensive information, which were in part either folklore or just available in different unpublished sources, to clarify the existence of extremal functions $f \in \mathcal{G}(W, Q)_{\mathbb{R}^d}$ in certain band-limited cases, that is, when W is closed and of finite Lebesgue measure and Q is compact. Since the problem of existence of the extremal function makes sense also in case of general LCA groups, our objective is to obtain a completely analogous counterpart of the aforementioned result in the general setting of LCA groups. More precisely, we intend to prove the following.

Theorem 2. *Let G be any LCA group. If $W \subseteq G$ is closed with positive, finite Haar measure and $Q \subseteq \widehat{G}$ is compact, then there exists an extremal function $f \in \mathcal{G}(W, Q)_G$ satisfying $\int_G f d\lambda_G = \mathcal{D}(W, Q)_G$.*

Note that the existence of an extremal function might be helpful in calculating or estimating the extremal constant itself. That explains the effort undertaken, for instance in [1, 3, 4, 7–10, 14, 15], to prove the existence of extremal functions. As we will see, the argument of [1] cannot be directly copied here. Indeed, in [1], the authors use estimation of modulus of smoothness, Bessel functions and their decrease estimates, and σ -compactness of the underlying group $\mathbb{R}^d$; however, for our general groups, all these are no longer available.

3. Preliminary Lemmata

Recall that a Banach space X is called *weakly compactly generated* (WCG for short) if it has a weakly compact subset whose linear span is dense in X . Fundamental examples of such spaces are separable normed spaces and reflexive Banach spaces.

An important property what we shall apply in our argument is that the unit ball of the dual space of a WCG space is weak-* sequentially compact (see [5], p. 148).

For a general LCA group G , it might be difficult to characterize when $L^1(G)$ turns to be a WCG space; however, a sufficient condition for that is the σ -compactness of G . This sufficiency can be seen by composing two well-known results. First, note that the space $L^1(X, \mu)$ is WCG when the occurring measure μ is σ -finite on X (see [19], p. 36). Second, the Haar measure on the LCA group G is σ -finite exactly when G is σ -compact. Moreover, in that case, we have the duality $(L^1(G))^* = L^\infty(G)$ of Banach spaces (see [13, Theorem 20.20.] and cf. [21], p. 11) because a σ -finite measure is decomposable.

The proof of Theorem 2 rests heavily on a technical lemma.

Lemma 3. *Let $W \subseteq G$ be closed and of finite Haar measure and let $Q \subseteq \widehat{G}$ be compact. Then the function class $\mathcal{G}(W, Q)_G \subseteq C(G)$ is relatively compact in the compact convergence topology.*

In the setting $G = \mathbb{R}^d$, the above lemma has been a part of the proof of [1, 3.5. Proposition], and its proof is based on the Arzelá-Ascoli Theorem and on the estimation of the modulus of continuity. We are unable to carry out this argument in the general case of LCA groups. Thus, we will prove the LCA group counterpart in a slightly different way, involving some basic notions and properties from the theory of topological vector spaces, which are given in the forthcoming paragraphs.

Let $A \subseteq X$ be any subset of a locally convex (Hausdorff) topological vector space X . Then, A is called *totally bounded*, whenever for every neighbourhood V of the origin, there is a finite subset $S \subseteq A$ such that A is contained in $S + V$. Obviously, this requirement can be equivalently assumed only for neighbourhoods belonging to a given neighbourhood base of X .

A topological vector space X is called *complete* if every Cauchy net has a limit in X . Further for any locally convex topological space X , there is a unique (up to a linear homeomorphism) pair $(\tilde{X}, j)$ of a complete space $\tilde{X}$ and a linear homeomorphic embedding $j : X \rightarrow \tilde{X}$ such that $j(X)$ is dense in $\tilde{X}$.

If G is an LCA group, then the relative compactness can be verified by using the following result from the theory of locally convex spaces (see, for instance [16, Theorem 3.5.1.]).

Lemma 4. *For every subset E of a locally convex topological vector space X , the following are equivalent.*

- (1) E is totally bounded.
- (2) E is relatively compact in the completion of X .
- (3) Every sequence of E has a cluster point in the completion of X .

Proof of Lemma 3. According to Lemma 4, we are going to show that $\mathcal{G}(W, Q)_G$ is totally bounded. First note that in the space $C(G)$ (equipped with the compact convergence topology) for any compact set $K \subseteq G$ and any $\varepsilon > 0$, the $U(f; K, \varepsilon)$ -neighbourhood of the function $f \in C(G)$ is defined as

$$U(f; K, \varepsilon) = \{h \in C(G) : \|h - f\|_{C(K)} < \varepsilon\}.$$

This forms the defining neighbourhood base for compact convergence on $C(G)$.

So, our aim is to show that for any $\varepsilon > 0$ and any compact set $K \subseteq G$, there exists a finite set $\{f_1, \dots, f_n\} \subseteq \mathcal{G}(W, Q)_G$ such that

$$\mathcal{G}(W, Q)_G \subseteq \bigcup_{j=1}^n U(f_j; K, \varepsilon).$$

As by assumption, $Q \subseteq \widehat{G}$ is compact in the compact convergence topology, and Q is totally bounded as well. It means that there exists a finite set $\{\chi_1, \dots, \chi_n\} \subseteq Q$ such that for every $\gamma \in Q$, we get that $\|\gamma - \chi_j\|_{C(K)} < \varepsilon$ for some $j \in \{1, \dots, n\}$. Via the disjointization procedure,

$$Q_1 := U(\chi_1; K, \varepsilon) \cap Q,$$

$$Q_2 := U(\chi_2; K, \varepsilon) \cap (Q \setminus Q_1),$$

$$\vdots$$

$$Q_n := U(\chi_n; K, \varepsilon) \cap (Q \setminus (Q_1 \cup \dots \cup Q_{n-1})),$$

we obtain a partition $\{Q_1, \dots, Q_n\}$ of Q such that every Q_j ($j = 1, \dots, n$) is a Borel sets with compact closure, and for any $\gamma \in Q_j$ we have $\|\gamma - \chi_j\|_{C(K)} < \varepsilon$.

Next, choose an element $f \in \mathcal{G}(W, Q)_G$ and define

$$F(g) := \sum_{j=1}^n \chi_j(g) \int_{Q_j} \widehat{f}(\chi) d\lambda_{\widehat{G}}(\chi) \equiv \sum_{j=1}^n c_j(f) \chi_j(g)$$

with $c_j(f) := \int_{Q_j} \widehat{f}(\chi) d\lambda_{\widehat{G}}(\chi)$, where $c_j(f) \geq 0$ because of the positive definiteness of f , and

$$\sum_{j=1}^n c_j(f) = \int_{\widehat{G}} \widehat{f}(\chi) d\lambda_{\widehat{G}}(\chi) = f(0) = 1.$$

Note that $f \in \mathcal{G}(W, Q)_G \subseteq L^1(G)$ and $\text{supp } \widehat{f} \subseteq Q$ implies $\widehat{f} \in L^p(\widehat{G})$ ($1 \leq p \leq \infty$). By the Inversion Theorem, we get for $g \in K$ in view of $\widehat{f} \geq 0$ that

$$|f(g) - F(g)| = \left| \sum_{j=1}^n \int_{Q_j} (\chi(g) - \chi_j(g)) \widehat{f}(\chi) d\lambda_{\widehat{G}}(\chi) \right| \leq \varepsilon \cdot f(0) = \varepsilon$$

and so $\|f - F\|_{C(K)} \leq \varepsilon$. Let $m > n/\varepsilon$ be an integer, and define $d_j(f) := [m \cdot c_j(f)]/m$. Then we have

$$\left\| \sum_{j=1}^n c_j(f) \chi_j - \sum_{j=1}^n d_j(f) \chi_j \right\|_{C(K)} < \sum_{j=1}^n \frac{1}{m} = \frac{n}{m} < \varepsilon.$$

It follows that

$$\left\| f - \sum_{j=1}^n d_j(f) \chi_j \right\|_{C(K)} \leq \|f - F\|_{C(K)} + \left\| F - \sum_{j=1}^n d_j(f) \chi_j \right\|_{C(K)} < 2\varepsilon.$$

For any choice of the function $f \in \mathcal{G}(W, Q)_G$, one has $d_j(f) \in \{0, 1/m, \dots, 1\}$, whence the set

$$\left\{ \sum_{j=1}^m r_j \chi_j : r_j \in \{0, 1/m, \dots, 1\} \right\} \quad (4)$$

forms a finite 2ε -net for $\mathcal{G}(W, Q)_G$ on K with respect to the compact convergence topology. This holds for any base neighbourhood of the form $U(\mathbf{0}; K, 2\varepsilon)$. Therefore, we found a finite net (4) such that the respective translates of $U(\mathbf{0}; K, 2\varepsilon)$ cover $\mathcal{G}(W, Q)_G$, so the function set $\mathcal{G}(W, Q)_G$ is totally bounded. $\square$

4. First Proof of the Theorem

In the sequel, we make crucial use of the following selection lemma.

Lemma 5. *Suppose that G is σ -compact. Let (f_n) be a sequence in $\mathcal{G}(W, Q)_G$. Then there exists a subsequence of (f_n) which converges to a function $f \in \mathcal{G}(W, Q)_G$ uniformly on every compact set and also in a weak- $*$ sense. Moreover, we have the inequality*

$$\int_G f d\lambda_G \geq \limsup_{n \rightarrow \infty} \int_G f_n d\lambda_G. \quad (5)$$

Proof of Lemma 5. In the first part of the proof, we use the arguments given in [1]. Let (f_n) be a sequence in $\mathcal{G}(W, Q)_G$. Using Lemmata 3, 4 and the completeness of $C(G)$ with respect to the compact convergence topology, we conclude that there exists a subsequence of (f_n) which tends to some $f \in C(G)$ uniformly on every compact set, and thus also in the pointwise sense. Without loss of generality, we may and do assume that (f_n) itself converges to f .

Next, we intend to show that $f \in \mathcal{G}(W, Q)_G$. Since the pointwise limit of positive definite functions is likewise positive definite, it follows that $f \gg 0$ holds. As $W \subseteq G$ is closed, we clearly have $\text{supp } f_+ \subseteq \overline{W} = W$, $f(0) = 1$ and $|f| \leq 1$.

Now, we are concerned with verifying that f belongs to $L^1(G)$. Writing $f = f_+ - f_-$ and, in a similar fashion, $f_n = (f_n)_+ - (f_n)_-$ one has $(f_n)_\pm \rightarrow f_\pm$ in the pointwise sense. An application of Fatou's lemma gives us that

$$\int_G f_- d\lambda_G \leq \liminf_{n \rightarrow \infty} \int_G (f_n)_- d\lambda_G. \quad (6)$$

For the positive parts, note that $(f_n)_+$ and f_+ are all supported in W , and $|(f_n)_+| \leq (f_n)_+(0) = 1$, all the functions f_n belonging to $\mathcal{G}(W, Q)_G$. That is, $(f_n)_+ \leq \mathbb{1}_W$, which is integrable because W has finite Haar measure. Therefore, the Lebesgue Dominated Convergence Theorem yields

$$\int_G f_+ d\lambda_G = \lim_{n \rightarrow \infty} \int_G (f_n)_+ d\lambda_G. \quad (7)$$

Note that then

$$\begin{aligned} \int_G |f| d\lambda_G &= \int_G f_+ d\lambda_G + \int_G f_- d\lambda_G \leq \lim_{n \rightarrow \infty} \int_G (f_n)_+ d\lambda_G + \\ &\quad + \liminf_{n \rightarrow \infty} \int_G (f_n)_- d\lambda_G \leq 2 \lim_{n \rightarrow \infty} \int_G (f_n)_+ d\lambda_G \leq 2\lambda_G(W), \end{aligned}$$

because

$$\int_G (f_n)_- d\lambda_G = \int_G ((f_n)_+ - f_n) d\lambda_G \leq \int_G (f_n)_+ d\lambda_G$$

for each n , for $\int_G f_n d\lambda_G \geq 0$ in view of $f_n \gg 0$. Therefore, we have also proved $f \in L^1(G)$, that is, also $f \in C(G) \cap L^1(G) \cap L^\infty(G)$, whence it belongs to $L^2(G)$. In particular, $\widehat{f}$ does exist, is continuous and belongs to $L^2(\widehat{G})$.

Note that by subtracting (6) from (7), we immediately get (5), too:

$$\begin{aligned} \int_G f d\lambda_G &\geq \lim_{n \rightarrow \infty} \int_G (f_n)_+ d\lambda_G - \liminf_{n \rightarrow \infty} \int_G (f_n)_- d\lambda_G \\ &\geq \limsup_{n \rightarrow \infty} \int_G ((f_n)_+ - (f_n)_-) = \limsup_{n \rightarrow \infty} \int_G f_n d\lambda_G. \end{aligned}$$

It remains to show that $\text{supp } \widehat{f} \subseteq Q$. Here, we need to argue in a different way than [1] does. Clearly, the linear functional

$$\psi_\rho(\varphi) := \int_G \varphi \bar{\rho} d\lambda_G, \quad \text{for } \varphi \in L^1(G)$$

belongs to the unit ball in the dual space of $L^1(G)$ for $\rho = f_n$ or $\rho = f$.

Using that G is σ -compact, we have that the space $L^1(G)$ is WCG; moreover, $(L^1(G))^* = L^\infty(G)$. Thus, there is a subsequence of (f_n) (supposed to be itself (f_n) again) which converges to some $f_0 \in L^\infty(G)$ in the weak-* sense. It is not difficult to verify that f_0 must coincide with the locally uniform limit function f , that is, we have

$$\int_G f_n \varphi d\lambda_G \longrightarrow \int_G f \varphi d\lambda_G, \quad \text{for } \varphi \in L^1(G). \quad (8)$$

Take any $\gamma \in \widehat{G} \setminus Q$, and a small symmetric neighbourhood B of the unit element $\mathbf{1}$ of $\widehat{G}$ with compact closure satisfying $\gamma B B \cap Q = \emptyset$. Define the functions $\theta_\gamma(\chi) := (\mathbf{1}_B \star \mathbf{1}_B)(\chi \gamma^{-1})$ and $\theta(\chi) := \theta_{\mathbf{1}}(\chi)$. Note that θ is compactly supported and $\theta(\mathbf{1}) = (\mathbf{1}_B \star \mathbf{1}_B)(\mathbf{1}) = \lambda_{\widehat{G}}(B)$.

Since B is symmetric, we get $\mathcal{F}^*(\theta) = \left| \widehat{\mathbf{1}_B} \right|^2$, by elementary properties of the L^2 -Fourier transform [20, Sect. 1.6.]. This immediately yields that $\mathcal{F}^*(\theta) \in L^1(G)$. Indeed, one has

$$\|\mathcal{F}^*(\theta)\|_1 = \left\| \widehat{\mathbf{1}_B} \right\|_2^2 = \|\mathbf{1}_B\|_2^2 = \lambda_{\widehat{G}}(B) < +\infty,$$

because B has compact closure and the Haar measure is locally finite. Thus, we also have

$$h(g) := \mathcal{F}^*(\theta_\gamma)(g) = \gamma(g) \cdot \mathcal{F}^*(\mathbf{1}_B \star \mathbf{1}_B)(g)$$

$$= \gamma(g) \cdot |\mathcal{F}^*(\mathbf{1}_B)(g)|^2 = \gamma(g) \cdot \left| \widehat{\mathbf{1}_B}(\delta_g) \right|^2,$$

where the last equality follows from the symmetry of B . We see that $h \in L^1(G)$. So we can take $k := f \star h \in L^1(G)$ for which we clearly have $\mathcal{F}(k) = \widehat{f}\theta_\gamma$.

Hence, we conclude from (8) via the Plancherel Theorem that

$$\begin{aligned} k(s) &= \lim_{n \rightarrow \infty} \int_G f_n(g) h(s-g) d\lambda_G(g) \\ &= \lim_{n \rightarrow \infty} \int_{\widehat{G}} \widehat{f_n}(\chi) \delta_s(\chi) \theta_\gamma(\chi) d\lambda_{\widehat{G}}(\chi) = 0 \end{aligned}$$

in view of $\text{supp } \widehat{f_n} \subseteq Q$ and $\{\theta_\gamma \neq 0\} \cap Q = \gamma B B \cap Q = \emptyset$. Therefore, $k(s) = 0$ holds for all $s \in G$. Taking Fourier transform gives $\widehat{f}\theta_\gamma \equiv 0$, in particular,

$$0 = \widehat{f}(\gamma) \theta_\gamma(\gamma) = \widehat{f}(\gamma) \theta(\mathbf{1}) = \widehat{f}(\gamma) \lambda_{\widehat{G}}(B).$$

Thus, at any point γ outside the set Q , the function $\widehat{f}$ vanishes. It follows that $\text{supp } \widehat{f} \subseteq Q$ and so $f \in \mathcal{G}(W, Q)_G$ as wanted. $\square$

From the definition it is clear that the restriction of a positive definite function to a subgroup remains positive definite on the subgroup as well. For the following fact, the reader can consult with [12, 32.43. (a)].

Lemma 6. *Let $H \leq G$ be a closed subgroup of G . If the function $f : H \rightarrow \mathbb{C}$ is continuous and positive definite, then so is its trivial extension $\widetilde{f} : G \rightarrow \mathbb{C}$ defined by*

$$\widetilde{f}(g) = \begin{cases} f(g) & \text{if } g \in H; \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Now, we are in a position to prove Theorem 2. Our strategy is the following. First we prove the theorem in the case where the underlying group is σ -compact, and then we reduce the general case after some technical preparation to the σ -compact one.

Proof of Theorem 2. At first, we suppose that G is σ -compact. Using the definition of sup, there is an extremal sequence $(f_n) \subseteq \mathcal{G}(W, Q)_G$ such that

$$\int_G f_n d\lambda_G > \mathcal{D}(W, Q)_G - \frac{1}{n}, \quad n \in \mathbb{N}^+. \quad (10)$$

According to Lemma 5, there exists a limit function $f \in \mathcal{G}(W, Q)_G$ such that a subsequence of (f_n) converges to f . Without loss of generality, we may and do suppose that (f_n) converges to f . We show that f is the extremal function in $\mathcal{G}(W, Q)_G$, that is, $\int_G f d\lambda_G = \mathcal{D}(W, Q)_G$. By using the definition of the extremal constant, inequality (5) and definition (10), we get

$$\mathcal{D}(W, Q)_G \geq \int_G f d\lambda_G \geq \limsup_{n \rightarrow \infty} \int_G f_n d\lambda \geq \mathcal{D}(W, Q)_G$$

and thus we have equality everywhere in the last displayed chain of inequalities. This completes the proof when G is σ -compact.

Assume now that G is not σ -compact. Let G_0 denote the open, σ -compact subgroup which is generated by W , that is,

$$V := W - W, \quad G_0 := \bigcup_{n \in \mathbb{N}} nV.$$

Then, G_0 is an LCA group and a Haar measure on G_0 is given by $\lambda_{G_0} := \lambda_G|_{G_0}$. Define the sets Q^*, Q_0 as

$$\begin{aligned} Q^* &:= \left\{ \gamma \in \widehat{G_0} : \text{all the extensions of } \gamma \text{ to } G \text{ lie in } Q \right\} \\ Q_0 &:= \left\{ \gamma \in \widehat{G_0} : \exists \chi \in Q \text{ such that } \chi|_{G_0} = \gamma \right\}. \end{aligned}$$

Claim 1. *The set $Q^* \subseteq \widehat{G_0}$ is compact.*

Clearly, we have $Q^* \subseteq Q_0$. The set Q_0 is the image of the compact set Q under the restriction map

$$\Phi : \widehat{G} \rightarrow \widehat{G_0}, \quad \chi \mapsto \chi|_{G_0}.$$

Since G_0 is open, according to Lemma [11, 24.5.] Φ is continuous. So Q^* is compact if and only if it is a closed subset of the compact set Q_0 . We can write the complement of Q^* as

$$\begin{aligned} (Q^*)^c &= \{ \xi \in \widehat{G_0} : \exists \chi \in \widehat{G}, \chi|_{G_0} = \xi, \chi \notin Q \} \\ &= \bigcup_{\chi \in \widehat{G} \setminus Q} \{ \chi|_{G_0} \} = \Phi \left(\widehat{G} \setminus Q \right), \end{aligned}$$

where the latter set is open because Φ is an open mapping, again by [11, Lemma 24.5.].

Similarly to (2) and (3), we consider the function class $\mathcal{G}(W, Q^*)_{G_0}$ and the extremal quantity $\mathcal{D}(W, Q^*)_{G_0}$.

Claim 2. *We have $h^0 \in \mathcal{G}(W, Q^*)_{G_0}$ if and only if for its extension h , we have $h \in \mathcal{G}(W, Q)_G$.*

First, assume that $h \in \mathcal{G}(W, Q)_G$. Since further properties of $h^0 := h|_{G_0}$ are inherited to that of h , we intend to show that $\text{supp } \widehat{h^0} \subseteq Q^*$. Choose a $\gamma \in \widehat{G_0}$ for which $\widehat{h^0}(\gamma) \neq 0$. Let $\chi \in \widehat{G}$ be any extension of γ such that $\chi|_{G_0} = \gamma$. As $h \in L^1(G)$ with $\text{supp } h \subseteq G_0$, there holds the computation

$$\widehat{h^0}(\gamma) = \int_{G_0} h^0(g) \overline{\gamma(g)} d\lambda_{G_0}(g) = \int_G h(g) \overline{\chi(g)} d\lambda_G(g) = \widehat{h}(\chi), \quad (11)$$

whence $\widehat{h}(\chi) \neq 0$ holds. It follows that every extension $\chi \in \widehat{G}$ of γ lies in Q , in other words $\gamma \in Q^*$. Thus, $\text{supp } \widehat{h^0} \subseteq Q^*$, as wanted.

To see the converse, again from the computation (11) we see that $\widehat{h}(\chi) \neq 0$ implies $\widehat{h^0}(\gamma) \neq 0$ whenever γ is the restriction of χ . By assumption and the definition of the set Q^* , the character χ lies in Q . So $\text{supp } \widehat{h} \subseteq Q$, and thus $h \in \mathcal{G}(W, Q)_G$.

Claim 3. *We have $\mathcal{D}(W, Q)_G = \mathcal{D}(W, Q^*)_{G_0}$.*

The inequality $\mathcal{D}(W, Q)_G \leq \mathcal{D}(W, Q^*)_{G_0}$ is in fact easy to verify. Indeed, by the $\mathcal{D}(W, Q)_G$ -extremality of the sequence (f_n) and $f_n^0 := f_n|_{G_0} \in \mathcal{G}(W, Q^*)_{G_0}$ for every $n \in \mathbb{N}$, we get

$$\mathcal{D}(W, Q^*)_{G_0} \geq \int_{G_0} f_n^0 d\lambda_{G_0} = \int_G f_n d\lambda_G > \mathcal{D}(W, Q)_G - \frac{1}{n}$$

for every $n \in \mathbb{N}^+$, as wanted.

To see the converse, consider an extremal sequence $(f_n^0) \subseteq \mathcal{G}(W, Q^*)_{G_0}$ on G_0 and extend it in the trivial way to a sequence $(\widetilde{f_n})$ on G . Then in virtue of (11), $\widetilde{f_n}(\chi) \neq 0$ implies $\widetilde{f_n^0}(\gamma) \neq 0$ where $\gamma = \chi|_{G_0}$. The latter condition gives us that $\gamma \in Q^*$, so it follows directly that $\text{supp } \widetilde{f_n} \subseteq Q$. Hence, $\widetilde{f_n} \in \mathcal{G}(W, Q)_G$ and thus

$$\mathcal{D}(W, Q)_G \geq \int_G \widetilde{f_n} d\lambda_G = \int_{G_0} f_n^0 d\lambda_{G_0} > \mathcal{D}(W, Q^*)_{G_0} - \frac{1}{n}$$

for every integer $n \geq 1$ which implies $\mathcal{D}(W, Q)_G \geq \mathcal{D}(W, Q^*)_{G_0}$.

Now, we can finish the proof of Theorem 2 quite easily. To do so, consider a $\mathcal{D}(W, Q^*)_{G_0}$ -extremal sequence (f_n^0) on G_0 . Then according to Lemma 5, there is a subsequence of (f_n^0) which tends to a function $f_0 \in \mathcal{G}(W, Q^*)_{G_0}$, and we have

$$\int_{G_0} f_0 d\lambda_{G_0} = \mathcal{D}(W, Q^*)_{G_0} = \mathcal{D}(W, Q)_G.$$

For the trivial extension $\widetilde{f}$ of f_0 , one has

$$\int_G \widetilde{f} d\lambda_G = \int_{G_0} f_0 d\lambda_{G_0} = \mathcal{D}(W, Q)_G.$$

Since $\widetilde{f} \in \mathcal{G}(W, Q)_G$, the last displayed equality shows that the function $\widetilde{f}$ is $\mathcal{D}(W, Q)_G$ -extremal.

Remark. It is apparent from the construction presented in the last part of the proof of Theorem 2 that the extremal function can be chosen to be supported in the open σ -compact subgroup of G generated by W .

5. Second Proof of the Theorem

Our original argument was the above proof with the somewhat heavy use of $L^1(G)^* = L^\infty(G)$ for WCG spaces, allowing to conclude for G σ -compact and then a somewhat involved argument to transfer the result from the σ -compact case to the general one.

A simpler, more direct proof was, however, hinted by our anonymous referee, whose suggestion we gratefully acknowledge here. The key point is that instead of using weak-* convergence in the $L^\infty(G)$ sense of f_n in (8), it is available to use weak $L^2(G)$ convergence, too. Although this innocent-looking modification seems to provide only an equivalent version of the original argument, actually it gives a way to an essential simplification, too, because (e.g. referring to the Eberline–Smulian Theorem, James' Theorem etc., see in [18,

2.8.6 and 2.8.9]) the modification allows us to get rid of the σ -compactness condition in Lemma 5. (In fact, for obtaining the assertion of Lemma 5 regarding weak- L^2 convergence instead of weak- $^* L^\infty$ convergence, in our case also a direct calculation works, avoiding references to deeper functional analysis results.) Thus we became enabled to deduce Theorem 2 shortly, without the heavy work for the transference of the result from the σ -compact case to the general one.

At present, it seems that the original proof still has a little extra yield, which may justify its presentation in spite of the available shorter argument. Namely, the concluding Remark of the section is seen from this heavier work on transference, but is not immediate from the L^2 version. Therefore, we have decided to keep the above proof and describe the more direct L^2 version separately here.

Lemma 7. *Let (f_n) be a sequence in $\mathcal{G}(W, Q)_G$. Then there exists a subsequence of (f_n) which converges to a function $f \in \mathcal{G}(W, Q)_G$ uniformly on every compact set and also weakly in the L^2 sense. Moreover, we have the inequality*

$$\int_G f d\lambda_G \geq \limsup_{n \rightarrow \infty} \int_G f_n d\lambda_G. \quad (12)$$

Proof of Lemma 7. The first part of the proof is the same as the proof of Lemma 5: we show that $f \in C(G) \cap L^1(G) \cap L^\infty(G)$, whence it belongs to $L^2(G)$. In particular, $\hat{f}$ does exist, is continuous and belongs to $L^2(\hat{G})$.

Observe that (12) is exactly the formula (5) from Lemma 5. Although in the latter σ -compactness of G was assumed, actually the proof of this formula did not use σ -compactness (which was only used later for L^∞ weak- * convergence). Therefore, this formula can be proved by repeating the respective argument in the proof of Lemma 5.

Furthermore, once our sequence f_n converges to f locally uniformly, it also converges weakly to the same limit function in $L^2(G)$. Indeed, consider the linear functionals defined by our f_n and f on $L^2(G)$: we need to see that it holds that

$$\int_G \varphi f_n d\lambda_G \rightarrow \int_G \varphi f d\lambda_G \quad (\varphi \in L^2(G)). \quad (13)$$

Let $\varepsilon > 0$ be arbitrarily chosen, and take a compact subset $K \subset G$ such that $\|\varphi\|_{L^2(G)}^2 < \int_K |\varphi|^2 d\lambda_G + \varepsilon$, i.e. $\int_{G \setminus K} |\varphi|^2 d\lambda_G < \varepsilon$. Further, take a sufficiently large index $n_0 := n_0(K)$ such that for $n \geq n_0$ we have $\|f - f_n\|_{L^\infty(K)} < \varepsilon$. Then for $n \geq n_0$, we are led to

$$\begin{aligned} \left| \int_G \varphi(f_n - f) d\lambda_G \right| &\leq \left| \int_K \varphi(f_n - f) d\lambda_G \right| + \left| \int_{G \setminus K} \varphi(f_n - f) d\lambda_G \right| \\ &\leq \sqrt{\varepsilon^2 \|\varphi\|_{L^2(G)}^2} + \sqrt{2\varepsilon}, \end{aligned}$$

using also the Hölder inequality in the first and the uniform norm estimate $\|f_n - f\|_\infty \leq \|f_n\|_\infty + \|f\|_\infty = 2$ in the second term.

It remains to show that $\text{supp } \hat{f} \subseteq Q$. As above, we take any $\gamma \in \hat{G} \setminus Q$, and a small symmetric neighbourhood B of the unit element 1 of $\hat{G}$ with

compact closure, satisfying $\gamma BB \cap Q = \emptyset$. Also, we consider the functions $\theta_\gamma(\chi) := (\mathbb{K}_B \star \mathbb{K}_B)(\chi\gamma^{-1})$ and $\theta(\chi) := \theta_1(\chi)$, which are compactly supported, continuous functions with integrable inverse Fourier transform, giving rise to the construction of $h := \mathcal{F}^*(\theta_\gamma)$ with $h \in L^1(G)$. So we can take $k := f \star h \in L^1(G)$ for which we clearly have $\mathcal{F}(k) = \widehat{f}\theta_\gamma$ [20, Theorem 1.2.4].

It is also easy to see that $h \in L^2(G)$, for it is the inverse Fourier transform of the continuous, compactly supported—whence $L^2(\widehat{G})$ —function θ_γ . Hence we can apply (13), followed by an application of the Plancherel Theorem [20, 1.6.2] to infer

$$\begin{aligned} k(s) &= \lim_{n \rightarrow \infty} \int_G f_n(g) h(s-g) d\lambda_G(g) \\ &= \lim_{n \rightarrow \infty} \int_{\widehat{G}} \widehat{f_n}(\chi) \delta_s(\chi) \theta_\gamma(\chi) d\lambda_{\widehat{G}}(\chi) = 0 \end{aligned}$$

in view of $\text{supp } \widehat{f_n} \subseteq Q$ and $\{\theta_\gamma \neq 0\} \cap Q = \gamma BB \cap Q = \emptyset$. Therefore, $k(s) = 0$ holds for all $s \in G$. Via Fourier transform, we get $\widehat{f}\theta_\gamma = \widehat{k} \equiv 0$, as in the proof of Lemma 5, and we deduce mutatis mutandis $\text{supp } \widehat{f} \subseteq Q$ and $f \in \mathcal{G}(W, Q)_G$ as wanted. $\square$

Second proof of Theorem 2. Using the definition of sup, there is an extremal sequence $(f_n) \subseteq \mathcal{G}(W, Q)_G$ such that

$$\int_G f_n d\lambda_G > \mathcal{D}(W, Q)_G - \frac{1}{n}, \quad n \in \mathbb{N}^+. \quad (14)$$

According to Lemma 7, there exists a limit function $f \in \mathcal{G}(W, Q)_G$ such that a subsequence of (f_n) converges to f uniformly on every compact set and also weakly in the L^2 sense. Without loss of generality, we may and do suppose that (f_n) converges to f . We show that f is the extremal function in $\mathcal{G}(W, Q)_G$, that is, $\int_G f d\lambda_G = \mathcal{D}(W, Q)_G$. By using the definition of the extremal constant, inequality (12) and definition (14), we get

$$\mathcal{D}(W, Q)_G \geq \int_G f d\lambda_G \geq \limsup_{n \rightarrow \infty} \int_G f_n d\lambda \geq \mathcal{D}(W, Q)_G$$

and thus we have equality everywhere in the last displayed chain of inequalities. This completes the proof. $\square$

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References

- [1] Berdysheva, E., Révész, S.Z.G.Y.: Delsarte's Extremal Problem and Packing on Locally Compact Abelian Groups, [arXiv:1904.03614](https://arxiv.org/abs/1904.03614)
- [2] Cohn, H.: New upper bounds on sphere packings. II, *Geom. Topol.* **6** 329–353 (2002)
- [3] Cohn, H., Gonçalves, F.: An optimal uncertainty principle in twelve dimensions via modular forms. *Invent. Math.* **217**, 799–831 (2019)
- [4] Cohn, H., Kumar, A., Miller, S.D., Radchenko, D., Viazovska, M.: The sphere packing problem in dimension 24. *Ann. Math.* **185**, 1017–1033 (2017)
- [5] Diestel, J.: *Geometry of Banach Spaces, Lecture notes in mathematics*, 485. Springer, Berlin, Heidelberg (1975)
- [6] Dixmier, J.: *C*-algebras*. North-Holland Publishing Company, Amsterdam, New York-Oxford (1977)
- [7] Gonçalves, F., Oliveira e Silva, D., Steinerberger, S.: Hermite polynomials, linear flows on the torus, and an uncertainty principle for roots. *J. Math. Anal. Appl.* **451** 678–711 (2017)
- [8] Gorbachev, D.V.: Extremal problems for entire functions of exponential spherical type, connected with the Levenshtein bound on the sphere packing density in $\mathbb{R}^n$ (Russian), *Izvestiya of the Tula State University. Ser. Math. Mech. Inform.* **6**, 71–78 (2000)
- [9] Gorbachev, D.V., Ivanov, V.I.: Turán's and Fejér's extremal problems for Jacobi transform. *Anal. Math.* **44**, 419–432 (2018)
- [10] Gorbachev, D.V., Tikhonov, S.Y.: Wiener's problem for positive definite functions. *Math. Z.* **289**, 859–874 (2018)
- [11] Hewitt, E., Ross, K.A.: *Abstract harmonic analysis, I, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*, vol. 115. Springer, Berlin, Göttingen, Heidelberg (1963)

- [12] Hewitt, E., Ross, K.A.: Abstract harmonic analysis, II, Die Grundlehren der mathematischen Wissenschaften, vol. 152. Springer, Berlin, Heidelberg, New York, Budapest (1970)
- [13] Hewitt, E., Stromberg, K.: Real and Abstract Analysis, A modern treatment of the theory of functions of real variable. Springer, New York, Heidelberg (1965)
- [14] Ivanov, V.I., Ivanov, A.V.: *Turán problems for periodic positive definite functions*. Ann. Univ. Sci. Budapest. Sect. Comp. **33** 219–237 (2010)
- [15] Ivanov, V.I., Rudomazina, YuD: On the Turán problem for periodic functions with nonnegative Fourier coefficients and small support. Math. Notes **77**, 870–875 (2005)
- [16] Jarchov, H. : Locally convex spaces. Vieweg+Teubner (1981)
- [17] Levenshtein, V.I.: Bounds for packings in n -dimensional Euclidean space. Dokl. Akad. Nauk SSSR **245**, 1299–1303 (1979)
- [18] Megginson, R.E.: An introduction to banach space theory, graduate texts in mathematics, vol. 183. Springer, New York (1998)
- [19] Phelps, R.R.: Monotone Operators, Convex Functions and Differentiability, second edn, Lecture Notes in Mathematics, 1364. Springer (1993)
- [20] Rudin, W.: Fourier analysis on groups. Intersci. Tracts Pure Appl. Math. 12. New York, London (1962)
- [21] Székelyhidi, L.: Discrete spectral synthesis. Springer, Berlin (2006)
- [22] Viazovska, M.: The sphere packing problem in dimension. Ann. Math. **185**, 991–1015 (2017)
- [23] Yudin, V.A.: Packings of balls in Euclidean space, and extremal problems for trigonometric polynomials (Russian). Diskret. Mat. **1**, 155–158 (1989)
- [24] Yudin, V.A.: translation in Discrete Math. Appl. **1**, 69–72 (1991)

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