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Atomic Operators in Vector Lattices

Ralph Chill and Marat Pliev

Abstract. In this paper, we introduce a new class of operators on vector lattices. We say that a linear or nonlinear operator T from a vector lattice E to a vector lattice F is atomic if there exists a Boolean homomorphism Φ from the Boolean algebra $\mathfrak{B}(E)$ of all order projections on E to $\mathfrak{B}(F)$ such that $T\pi = \Phi(\pi)T$ for every order projection $\pi \in \mathfrak{B}(E)$. We show that the set of all atomic operators defined on a vector lattice E with the principal projection property and taking values in a Dedekind complete vector lattice F is a band in the vector lattice of all regular orthogonally additive operators from E to F. We give the formula for the order projection onto this band, and we obtain an analytic representation for atomic operators between spaces of measurable functions. Finally, we consider the procedure of the extension of an atomic map from a lateral ideal to the whole space.

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1. Introduction and Preliminaries

Local operators and, more generally, atomic operators in classical function spaces find numerous applications in control theory, the theory of dynamical systems and the theory of partial differential equations (see [6,19,23]). The concept of a local operator was in the context of vector lattices first introduced in [22]. It is an abstract form of the well-known property of a nonlinear superposition operator and can be stated in the following form: the value of the image function on a certain set depends only on the values of the preimage function on the same set. In this article, we analyse the notion of an atomic operator in the framework of the theory of vector lattices and

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orthogonally additive operators. Today, the theory of orthogonally additive operators in vector lattices is an active area in functional analysis; see for instance [1,2,7,8,10,11,13,14,16–18,25]. Abstract results of this theory can be applied to the theory of nonlinear integral operators [12,21], and there are connections with problems of convex geometry [24].

Let us introduce some basic facts concerning vector lattices and orthogonally additive operators. We assume that the reader is acquainted with the theory of vector lattices and Boolean algebras. For the standard information, we refer to [3,4,9]. All vector lattices below are assumed to be Archimedean.

Let E be a vector lattice. A net $(x_\alpha)_{\alpha\in\Lambda}$ in E *order converges* to an element $x \in E$ (notation $x_\alpha \xrightarrow{(\mathrm{o})} x$) if there exists a net $(u_\alpha)_{\alpha\in\Lambda}$ in E_+ such that $u_\alpha \downarrow 0$ and $|x_\alpha - x| \le u_\alpha$ for all $\alpha \in \Lambda$ satisfying $\alpha \ge \alpha_0$ for some $\alpha_0 \in \Lambda$. Two elements x, y of the vector lattice E are *disjoint* (notation $x\bot y$), if $|x| \wedge |y| = 0$. The sum $x + y$ of two disjoint elements x and y is denoted by $x \sqcup y$. The equality $x = \bigsqcup_{i=1}^n x_i$ means that $x = \sum_{i=1}^n x_i$ and $x_i\bot x_j$ if $i \ne j$. An element y of E is called a *fragment* of an element $x \in E$, provided $y\bot(x - y)$. The notation $y \sqsubseteq x$ means that y is a fragment of x. If E is a vector lattice and $x \in E$ then we denote by $\mathcal{F}_x$ the set of all fragments of x. A positive, linear projection $\pi : E \to E$ is said to be an *order projection* if $0 \le \pi \le Id$, where Id is the identity operator on E. The set of all order projections on E is denoted by $\mathfrak{B}(E)$. The set $\mathfrak{B}(E)$ is ordered by $\pi \le \rho :\Leftrightarrow \pi \circ \rho = \pi$, and it is a Boolean algebra with respect to the Boolean operations:

$$\begin{aligned} \pi \wedge \rho :=& \pi \circ \rho; \\ \pi \vee \rho :=& \pi + \rho - \pi \circ \rho; \\ \overline{\pi} =& Id - \pi. \end{aligned}$$

An element x of a vector lattice E is called a *projection element* if the band generated by x is a projection band, and then we denote by π_x the order projection onto the band generated by x. A vector lattice E is said to have the *principal projection property* if every element of E is a projection element. For example, every σ-Dedekind complete vector lattice has the principal projection property.

A (possibly nonlinear) operator $T : E \to F$ from a vector lattice E into a real vector space is called *orthogonally additive* if $T(x + y) = T(x) + T(y)$ for every disjoint elements x, $y \in E$. It is clear that if T is orthogonally additive, then $T(0) = 0$. The set of all orthogonally additive operators from E into F, denoted by $\mathcal{OA}(E, F)$, is a real vector space for the natural linear operations.

An operator $T : E \to F$ between two vector lattices E and F is said to be

- *positive*, if $Tx \ge 0$ for all $x \in E$;
- *order bounded*, if T maps order bounded sets in E to order bounded sets in F;
- *laterally-to-order bounded*, if for every $x \in E$ the set $T(\mathcal{F}_x)$ is order bounded in F.

An orthogonally additive operator $T : E \to F$ is

- *regular*, if $T = T_1 - T_2$ for two positive, orthogonally additive operators T_1 and T_2 from E to F.

An orthogonally additive, order bounded operator $T : E \to F$ is called an *abstract Urysohn operator*. This class of operators was introduced and studied in 1990 by Mazón and Segura de León [12]. We notice that the order boundedness is a restrictive condition for orthogonally additive operator. Indeed, every operator $T : \mathbb{R} \to \mathbb{R}$ satisfying $T(0) = 0$ is orthogonally additive, but not every operator of this form is order bounded. Consider, for instance, the positive function T defined by

$$T(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The notion of a laterally-to-order bounded operator was introduced in [15]. It is obviously weaker than the notion of order bounded operator. An orthogonally additive, laterally-to-order bounded operator $T : E \to F$ is also called a *Popov operator*.

We denote by $\mathcal{OA}_+(E, F)$ the set of all positive, orthogonally additive operators from E to F (so that $\mathcal{OA}(E, F)$ becomes an ordered vector space with this cone), by $\mathcal{OA}_r(E, F) := \mathcal{OA}_+(E, F) - \mathcal{OA}_+(E, F)$ the regular, orthogonally additive operators, and by $\mathcal{P}(E, F)$ the laterally-to-order bounded, orthogonally additive operators from E to F. Also $\mathcal{OA}_r(E, F)$ and $\mathcal{P}(E, F)$ are ordered vector spaces. In general, $\mathcal{OA}_r(E, F) \neq \mathcal{P}(E, F)$ (see [15]), but for a Dedekind complete vector lattice F we have the following strong properties of $\mathcal{OA}_r(E, F)$ and $\mathcal{P}(E, F)$.

Theorem 1.1 [15, Theorem 3.6]. *Let E and F be vector lattices, and assume that F is Dedekind complete. Then $\mathcal{OA}_r(E, F) = \mathcal{P}(E, F)$, and $\mathcal{OA}_r(E, F)$ is a Dedekind complete vector lattice. Moreover, for every S, $T \in \mathcal{OA}_r(E, F)$ and every $x \in E$,*

(1) $(T \vee S)(x) = \sup\{Ty + Sz : x = y \sqcup z\}$;
(2) $(T \wedge S)(x) = \inf\{Ty + Sz : x = y \sqcup z\}$;
(3) $(T)^+(x) = \sup\{Ty : y \sqsubseteq x\}$;
(4) $(T)^-(x) = -\inf\{Ty : y \sqsubseteq x\}$;
(5) $|Tx| \leq |T|(x)$.

2. Basic Properties of Atomic Operators

In this section, we introduce a new subclass of orthogonally additive operators, namely the class of *atomic operators*, and show that under some assumptions on the vector lattices E and F the set of all atomic operators from E to F subordinate to a Boolean homomorphism $\Phi : \mathfrak{B}(E) \to \mathfrak{B}(F)$ is a band in the vector lattice $\mathcal{OA}_r(E, F)$. We further obtain a formula for the order projection onto this band.

Let us first recall the definition of Boolean homomorphisms. Let $\mathfrak{A}$, $\mathfrak{B}$ be Boolean algebras. A map $\Phi : \mathfrak{A} \to \mathfrak{B}$ is called a *Boolean homomorphism*, if the following conditions hold:

(1) $\Phi(x \vee y) = \Phi(x) \vee \Phi(y)$ for all x, $y \in \mathfrak{A}$.
(2) $\Phi(x \wedge y) = \Phi(x) \wedge \Phi(y)$ for all x, $y \in \mathfrak{A}$.
(3) $\Phi(\overline{x}) = \overline{\Phi(x)}$ for all $x \in \mathfrak{A}$.

It is clear that $\Phi(\mathbf{0}_{\mathfrak{A}}) = \mathbf{0}_{\mathfrak{B}}$ and $\Phi(\mathbf{1}_{\mathfrak{A}}) = \mathbf{1}_{\mathfrak{B}}$. If, moreover, $\Phi(\bigvee_{\lambda\in\Lambda} x_\lambda) = \bigvee_{\lambda\in\Lambda} \Phi(x_\lambda)$ for every family (resp. countable family) $(x_\lambda)_{\lambda\in\Lambda}$ of elements of $\mathfrak{A}$, then Φ is said to be an *order continuous* (resp. a *sequentially order continuous*) Boolean homomorphism.

Let E and F be vector lattices and Φ be a Boolean homomorphism from $\mathfrak{B}(E)$ to $\mathfrak{B}(F)$. A map $T : E \to F$ is said to be an *atomic operator subordinate to* Φ, or briefly *atomic operator*, if $T\pi = \Phi(\pi)T$ for every order projection $\pi \in \mathfrak{B}(E)$. The set of all atomic operators from E to F subordinate to Φ is denoted by $\Phi(E, F)$.

We remark that the class of atomic operators was first introduced in [1]. It is easy to verify that $\Phi(E, F)$ is a vector space. Indeed, let $\lambda \in \mathbb{R}$, T, $S \in \Phi(E, F)$, $x \in E$ and $\pi \in \mathfrak{B}(E)$. Then

$$\begin{aligned} \Phi(\pi)\lambda T(x) &= \lambda\Phi(\pi)T(x) = \lambda T\pi(x) \text{ and} \\ \Phi(\pi)(T + S)(x) &= \Phi(\pi)T(x) + \Phi(\pi)S(x) \\ &= T\pi(x) + S\pi(x) \\ &= (T + S)\pi(x). \end{aligned}$$

Let us consider some examples of atomic operators.

Example 2.1. Recall that an operator $T : E \to E$ on a vector lattice is said to be *band preserving* if $T(D) \subseteq D$ for every band D of E. By [4, Theorem 2.37], if E is a vector lattice with the principal projection property, then a linear operator $T : E \to E$ is band preserving if and only if T commutes with every order projection on E. In other words, if E is a vector lattice with the principal projection property, then a linear operator $T : E \to E$ is band preserving if and only if it is atomic subordinate to the identity homomorphism $\Phi : \mathfrak{B}(E) \to \mathfrak{B}(E)$. In particular, if E has the principal projection property, then every linear *orthomorphism* $T : E \to E$ (a band-preserving, order-bounded operator) is atomic with respect to the identity homomorphism.

Example 2.2. Let $E = l^p(\mathbb{Z})$ with $1 \leq p \leq \infty$. For every subset $A \subseteq \mathbb{Z}$, one can define an order projection π_A which corresponds, in fact, to the multiplication by the characteristic function 1_A. This gives a one-to-one correspondence between the Boolean algebra $\mathcal{P}(\mathbb{Z})$ of all subsets of $\mathbb{Z}$ and the Boolean algebra of order projections on E. With this identification and for fixed $k \in \mathbb{Z}$, if we define the shift Boolean homomorphism $\Phi_k : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$, $A \mapsto \Phi_k(A) = \{i + k : i \in A\}$ and the shift operator $T_k : E \to E$, $f \mapsto T_k f$ with $(T_k f)(i) = f(i - k)$, then T_k is an atomic operator subordinate to Φ_k.

The following is an example of a nonlinear atomic operator.

Example 2.3. Let (B, Ξ, ν) be a σ-finite measure space, $L_0(B, \Xi, \nu) = L_0(\nu)$ the vector space of all (equivalence classes of) measurable real valued functions on B. A function $N : B \times \mathbb{R} \to \mathbb{R}$ is called a $\mathfrak{K}$-*function* if it satisfies the conditions:

(C_0) $N(s, 0) = 0$ for ν-almost all $s \in B$;
(C_1) $N(\cdot, r)$ is measurable for all $r \in \mathbb{R}$;
(C_2) $N(s, \cdot)$ is continuous on $\mathbb{R}$ for ν-almost all $s \in B$.

If the function satisfies only the conditions (C_1) and (C_2), then we call it a *Carathéodory function*. Given a Caratheodory function $N : B \times \mathbb{R} \to \mathbb{R}$, one defines the *superposition operator* $T_N : L_0(\nu) \to L_0(\nu)$ by

$$T_N f := N(\cdot, f(\cdot)) \quad (f \in L_0(\nu)).$$

We note that a superposition operator is in the literature also known as *Nemytskii operator*. The theory of these operators is widely represented in the literature (see [5]).

Lemma 2.4. *If* $N : B \times \mathbb{R} \to \mathbb{R}$ *is a* $\mathfrak{K}$*-function, then the superposition operator* $T_N : L_0(\nu) \to L_0(\nu)$ *is an atomic operator subordinate to the identity homomorphism* $Id : \mathfrak{B}(L_0(\nu)) \to \mathfrak{B}(L_0(\nu))$.

Proof. Let $\Xi_0 = \{D \in \Xi : \nu(D) = 0\}$ be the set of all ν-null sets. Then Ξ_0 is an ideal in the Boolean algebra Ξ. We let $\Xi' := \Xi/\Xi_0$ be the factor algebra. It is well known that the Boolean algebra $\mathfrak{B}(L_0(\nu))$ of all order projections on $L_0(\nu)$ is isomorphic to the Boolean algebra Ξ'. In fact, to every (equivalence class of a) measurable subset $D \in \Xi'$ there corresponds an order projection π_D which is in fact the multiplication by the characteristic function 1_D, and vice versa; see [3, Section 1.6]. Now we show that

$$N(s, r1_D(s)) = N(s, r)1_D(s) \quad \text{for all } D \in \Xi'.$$

First, for every $s \in D$,

$$N(s, r1_D(s)) = N(s, r) = N(s, r)1_D(s).$$

Second, for every $s \in B \backslash D$, by condition (C_0),

$$N(s, r1_D(s)) = N(s, 0) = 0 = N(s, r)1_D(s).$$

Hence, for every $f \in L_0(\nu)$ and $\pi = \pi_D \in \mathfrak{B}(L_0(\nu))$,

$$T\pi f = T(f1_D) = N(\cdot, f1_D(\cdot)) = N(\cdot, f(\cdot))1_D(\cdot) = \pi T f,$$

and the assertion is proved. $\square$

The following lemma shows that on Banach lattices with the principal projection property every atomic operator is regular orthogonally additive.

Lemma 2.5. *Let* E *be a vector lattice with the principal projection property,* F *be a vector lattice,* Φ *be a Boolean homomorphism from* $\mathfrak{B}(E)$ *to* $\mathfrak{B}(F)$ *and* $T \in \Phi(E, F)$*. Then* T *is orthogonally additive, laterally-to-order bounded (that is,* $T \in \mathcal{P}(E, F)$*) and disjointness preserving.*

Proof. Fix x, $y \in E$ with $x \perp y$. Then

$$\begin{aligned}
T(x+y) &= T(\pi_x + \pi_y)(x+y) \\
&= T(\pi_x \vee \pi_y)(x+y) \\
&= \Phi(\pi_x \vee \pi_y)T(x+y) \\
&= (\Phi(\pi_x) \vee \Phi(\pi_y))T(x+y) \\
&= (\Phi(\pi_x) + \Phi(\pi_y))T(x+y) \\
&= \Phi(\pi_x)T(x+y) + \Phi(\pi_y)T(x+y) \\
&= T\pi_x(x+y) + T\pi_y(x+y) \\
&= Tx + Ty.
\end{aligned}$$

Hence, T is orthogonally additive.

We next show that T is disjointness preserving. Let x, $y \in E$ be disjoint elements. Then the order projections π_x, π_y are disjoint elements in the Boolean algebra $\mathfrak{B}(E)$, and hence $\Phi(\pi_x)$ and $\Phi(\pi_y)$ are disjoint elements in the Boolean algebra $\mathfrak{B}(F)$. Since

$$\begin{aligned}
Tx &= T\pi_x x = \Phi(\pi_x)Tx \quad \text{and} \\
Ty &= T\pi_y y = \Phi(\pi_y)Ty,
\end{aligned}$$

then $Tx \perp Ty$.

Now fix $x \in E$ and assume that $y \in \mathcal{F}_x$. Then by definition $y \perp (x-y)$, and therefore, since T is disjointness preserving, $Ty \perp T(x-y)$. Since T is orthogonally additive, this implies $T(\mathcal{F}_x) \subseteq \mathcal{F}_{Tx}$, and hence $|Ty| \leq |Tx|$. Hence, T is a laterally-to-order bounded. □

It is worth to notice that without any assumption on the vector lattices E and F the space $\mathcal{P}(E,F)$ is not a vector lattice and we say nothing about the order structure of the space of laterally-to-order bounded orthogonally additive operators. Nevertheless, the next lemma shows that $\Phi(E,F)$ is a vector lattice if E has the principal projection property.

Lemma 2.6. *Let E be a vector lattice with the principal projection property, F be a vector lattice, and Φ be a Boolean homomorphism from $\mathfrak{B}(E)$ to $\mathfrak{B}(F)$. Then $\Phi(E,F)$ is a vector lattice.*

Proof. Let $T \in \Phi(E,F)$. It suffices to show that $|T| = T \vee (-T)$ exists. Define the operator $R : E \to F$ by

$$Rx := |Tx|, \quad x \in E.$$

For every $x \in E$,

$$Tx \leq |Tx| = Rx \quad \text{and} \quad (-Tx) \leq |Tx| = Rx.$$

Thus, $T \leq R$ and $(-T) \leq R$. Assume that G is an orthogonally additive operator from E to F such that $T \leq G$ and $(-T) \leq G$. Then for every $x \in E$, $Tx \leq Gx$ and $(-T)x \leq Gx$, and therefore, $Tx \vee (-Tx) = |Tx| = Rx \leq Gx$. Hence, $R = T \vee (-T)$.

We show that R is orthogonally additive. Indeed, let x, $y \in E$ with $x \perp y$. By Lemma 2.5, T is disjointness preserving. Then

$$R(x+y) = |T(x+y)| = |Tx+Ty| = |Tx| + |Ty| = Rx + Ry.$$

Finally, we show that R is an atomic operator. Let $\pi \in \mathfrak{B}(E)$ and $x \in E$. Then

$$R\pi(x) = |T\pi(x)| = |\Phi(\pi)Tx| = \Phi(\pi)|Tx| = \Phi(\pi)Rx,$$

and the proof is complete. □

The following theorem is the first main result of this section.

Theorem 2.7. *Let E be a vector lattice with the principal projection property, F be a Dedekind complete vector lattice and Φ be a Boolean homomorphism from $\mathfrak{B}(E)$ to $\mathfrak{B}(F)$. Then $\Phi(E,F)$ is a band in the vector lattice $\mathcal{OA}_r(E,F)$ and for any T, $S \in \Phi(E,F)$, $x \in E$ the following relations hold:*

(1) $(T \vee S)x = Tx \vee Sx$;
(2) $(T \wedge S)x = Tx \wedge Sx$;
(3) $(T)^+x = (Tx)^+$;
(4) $(T)^-(x) = (Tx)^-$;
(5) $|T|x = |Tx|$.

Proof. By Theorem 1.1, $\mathcal{OA}_r(E,F)$ is a vector lattice, and by Lemmas 2.5 and 2.6and Theorem 1.1, $\Phi(E,F)$ is a linear sublattice of $\mathcal{OA}_r(E,F) = \mathcal{P}(E,F)$. Suppose T, $S \in \Phi(E,F)$ and $x \in E$. By Theorem 1.1,

$$(T \vee S)(x) = \sup\{Ty + Sz : \, x = y \sqcup z\} \geq Tx \vee Sx.$$

We remark that if $x = y \sqcup z$, then $y = \pi_y x = \pi_y y$, $z = \pi_z x = \pi_z z$, $\Phi(\pi_y) \perp \Phi(\pi_z)$, $\Phi(\pi_x) = \Phi(\pi_y) + \Phi(\pi_z)$ and $\Phi(\pi_x)(Tx \vee Sx) = Tx \vee Sx$. Hence,

$$\begin{aligned} Ty + Sz &= T\pi_y y + S\pi_z z \\ &= T\pi_y x + S\pi_z x \\ &= \Phi(\pi_y)Tx + \Phi(\pi_z)Sx \\ &\leq \Phi(\pi_y)(Tx \vee Sx) + \Phi(\pi_z)(Tx \vee Sx) \\ &= \Phi(\pi_x)(Tx \vee Sx) \\ &= Tx \vee Sx. \end{aligned}$$

Passing to the supremum in the left-hand side of the above inequality over all y, $z \in \mathcal{F}_x$ such that $x = y \sqcup z$ yields to

$$(T \vee S)(x) \leq Tx \vee Sx$$

and it follows that $(T \vee S)(x) = Tx \vee Sx$. Now it is easy to deduce formulas for the infimum, module, positive and negative parts of operators.

$$\begin{aligned} (T \wedge S)(x) &= -\left((-T) \vee (-S)(x)\right) = -\left((-Tx) \vee (-Sx)\right) = Tx \wedge Sx; \\ T^+(x) &= (T \vee 0)(x) = Tx \vee 0 = (Tx)^+; \\ T^-(x) &= (-T \vee 0)(x) = -Tx \vee 0 = (Tx)^-; \\ |T|x &= (T \vee (-T))(x) = Tx \vee (-Tx) = |Tx|. \end{aligned}$$

Suppose $S \in \mathcal{OA}_+(E,F)$, $T \in \Phi(E,F)$, and $0 \leq S \leq T$. Then $0 \leq S\pi(x) \leq T\pi(x)$ for any $\pi \in \mathfrak{B}(E)$ and $x \in E$. Since $\Phi(\pi)T = T\pi$ and $\Phi(\pi) \perp \Phi(\pi^{\perp}) = (\Phi(\pi))^{\perp}$, it follows that

$$(\Phi(\pi))^{\perp}T\pi(x) = 0 \Rightarrow (\Phi(\pi)^{\perp})S\pi(x) = (\Phi(\pi))^{\perp}S(x) = 0.$$

Thus, $S\pi(E) \subseteq \Phi(\pi)S(E)$ and, therefore, $\Phi(\pi)S = S\pi$. Hence, $S \in \Phi(E,F)$, and we have shown that $\Phi(E,F)$ is an ideal in $\mathcal{OA}_r(E,F)$.

Finally, we show that $\Phi(E,F)$ is a band in $\mathcal{OA}_r(E,F)$. Assume that $\pi \in \mathfrak{B}(E)$ and $T_\lambda \xrightarrow{(\mathrm{o})} T$, where $(T_\lambda)_{\lambda\in\Lambda} \subseteq \Phi(E,F)$ and $T \in \mathcal{OA}_r(E,F)$. Then we have

$$\begin{aligned}|T\pi - \Phi(\pi)T| &= |T\pi - T_\lambda\pi + T_\lambda\pi - \Phi(\pi)T| \\ &\leq |T\pi - T_\lambda\pi| + |\Phi(\pi)T - T_\lambda\pi| \\ &= |T\pi - T_\lambda\pi| + |\Phi(\pi)T - \Phi(\pi)T_\lambda|.\end{aligned}$$

Since the net $(|T\pi - T_\lambda\pi| + |\Phi(\pi)T - \Phi(\pi)T_\lambda|)$ order converges to 0 it follows that $T\pi = \Phi(\pi)T$ for any $\pi \in \mathfrak{B}(E)$. □

Let E be a vector lattice with the principal projection property and F be a Dedekind complete vector lattice. Then $\mathcal{OA}_r(E,F)$ is Dedekind complete by Theorem 1.1, and therefore, by a theorem of F. Riesz, every band is a projection band. By Theorem 2.7, every positive orthogonally additive operator $T : E \to F$ thus has a unique decomposition $T = T_1 + T_2$ with $0 \leq T_1 \in \Phi(E,F)$ and $T_2 \in \Phi(E,F)^{\perp}$. The next theorem, which is the second main result of this section, gives a description of the band projection onto $\Phi(E,F)$.

By $\mathfrak{D}_0(E)$, or $\mathfrak{D}_0$ for short, we denote the set of all finite partitions of the identity operator Id, that is,

$$\mathfrak{D}_0 = \left\{(\pi_i) : \ \pi_k \wedge \pi_j = 0, k \neq j; \sum_{i=1}^{n} \pi_i = Id; \, n \in \mathbb{N}\right\}.$$

Theorem 2.8. *Let E be a vector lattice with the principal projection property and F be a Dedekind complete vector lattice and $T \in \mathcal{OA}_+(E,F)$. Then the component $T_1 \in \Phi(E,F)$ is given by*

$$\inf\left\{\sum_{i=1}^{n} \Phi(\pi_i)T\pi_i : \, (\pi_i) \in \mathfrak{D}_0\right\}.$$

Proof. For any $T \in \mathcal{OA}_+(E,F)$, set

$$\mathfrak{A}(T) := \left\{\sum_{i=1}^{n} \Phi(\pi_i)T\pi_i : \, (\pi_i) \in \mathfrak{D}_0\right\}.$$

Clearly, $\mathfrak{A}(T)$ is a downward directed set of positive orthogonally additive operators and taking into account the Dedekind completeness of the vector lattice $\mathcal{OA}_r(E,F)$ we deduce that there exists $R(T) := \inf \mathfrak{A}(T)$. We verify the following properties for every $T \in \mathcal{OA}_+(E,F)$:

(1) $0 \leq R(T) \leq T$;
(2) $R : \mathcal{OA}_r(E,F) \to \mathcal{OA}_r(E,F)$ extends to a linear operator;

(3) $R(T) = T \Leftrightarrow T \in \Phi(E, F)$;
(4) $R(R(T)) = R(T)$.

The relation (1) is obvious. To prove (2), we show that R is additive on the positive cone. If T_1, $T_2 \in \mathcal{OA}_+(E, F)$, then for arbitrary (π_i), $(\pi_j) \in \mathfrak{D}_0$, we have

$$\begin{aligned}
&\sum_i \Phi(\pi_i)T_1\pi_i + \sum_j \Phi(\pi_j)T_2\pi_j \\
&\quad \geq \sum_k \Phi(\pi_k)(T_1 + T_2)\pi_k \\
&\quad = \sum_k \Phi(\pi_k)T_1\pi_k + \sum_k \Phi(\pi_k)T_2\pi_k,
\end{aligned}$$

where $(\pi_k) \in \mathfrak{D}_0$ is finer than (π_i) and (π_j). Taking the infimum, we obtain

$$R(T_1) + R(T_2) = R(T_1 + T_2).$$

Let us prove the equivalence in (3). Assume that $0 \leq T \in \Phi(E, F)$. We notice that for every $(\pi_i) \in \mathfrak{D}_0$, we have that $\Phi(\pi_i) = Id$. Thus,

$$\sum_i \Phi(\pi_i)T\pi_i = \sum_i \Phi(\pi_i)^2 T = \sum_i \Phi(\pi_i)T = T.$$

Passing to infimum on the left-hand side of the above equality over all $(\pi_i) \in \mathfrak{D}_0$, we get that $R(T) = T$. On the other hand, assume that $R(T) = T$. We show that $T \in \Phi(E, F)$. Indeed, fix $\pi \in \mathfrak{B}(E)$. Then $T \leq \Phi(\pi)T\pi + \Phi(Id - \pi)T(Id - \pi)$. It follows that $\Phi(\pi)T \leq \Phi(\pi)T\pi$ and, therefore, $\Phi(\pi)T = T\pi$.

It remains to verify the equality (4). Suppose that $W = R(T)$, with $T \in \mathcal{OA}_+(E, F)$. For every $\rho \in \mathfrak{B}(E)$, we may write

$$\begin{aligned}
W\rho &= \inf\left\{\sum_i \Phi(\pi_i)T\pi_i\rho : \ (\pi_i) \in \mathfrak{D}_0\right\} \\
&= \inf\left\{\sum_i \Phi(\pi_i')T\pi_i'\rho : \ \sum_i \pi_i' = \rho\right\} \\
&= \inf\left\{\sum_i \Phi(\rho)\Phi(\pi_i')T\pi_i' : \ \sum_i \pi_i' = \rho\right\}.
\end{aligned}$$

Thus, $W\rho = \Phi(\rho)W$ for every $\rho \in \mathfrak{B}(E)$. By the equivalence (3) which is established above, we obtain $W = R(W)$. □

We remark that a similar theorem for orthomorphisms was proved in [20].

3. Atomic Operators in Spaces of Measurable Functions

In this section, we investigate atomic operators in spaces of real-valued, measurable functions and get an analytic representation for this class of operators.

Let (B, Ξ, ν) be a σ-finite measure space. Choose an *equivalent* finite measure λ on Ξ such that ν and λ have the same sets of measure 0. As in

Example 2.3 above, we denote by $L_0(B, \Xi, \nu)$ (or $L_0(\nu)$ for brevity) the set of all real-valued, measurable functions on B. More precisely, $L_0(\nu)$ consists of equivalence classes of such functions, where as usual two functions f and g are said to be equivalent if they coincide almost everywhere on B; note that $L_0(\nu)$ and $L_0(\lambda)$ coincide. The vector space $L_0(\nu)$ with the metric ρ_{L_0}, defined by

$$\rho_{L_0}(f, g) := \int_B \frac{|f(s) - g(s)|}{1 + |f(s) - g(s)|} \,\mathrm{d}\lambda \quad (f,\, g \in L_0(\nu)),$$

becomes a complete metric space, and the convergence with respect to the metric ρ_{L_0} is equivalent to the convergence in measure, meaning here the measure λ. Recall that (f_n) converges to f in measure (notation $f_n \xrightarrow{\lambda} f$; see [3, Theorem 1.82]), if, for every $\delta > 0$, $\lim_{n\to\infty} \lambda(\{|f_n - f| > \delta\}) = 0$. More precisely, the convergence in measure is characterised by the following statement.

Lemma 3.1 [3, Theorem 1.82]. *Let (B, Ξ, ν) be a σ-finite measure space. Choose an equivalent finite measure λ on Ξ such that ν and λ have the same sets of measure* 0. *For a sequence $(f_n) \subseteq L_0(\nu)$ and element $f \in L_0(\nu)$, the following equivalent:*

(1) $f_n \xrightarrow{\lambda} f$;
(2) *every subsequence of (f_n) has a subsequence that converges pointwise ν-almost everywhere to f;*
(3) *for every $D \in \Xi$ with $\nu(D) < \infty$,*

$$\lim_{n\to\infty} \int_D \frac{|f_n(s) - f(s)|}{1 + |f_n(s) - f(s)|} \,\mathrm{d}\nu = 0.$$

We say that a function $N : B \times \mathbb{R} \to \mathbb{R}$ is a *superpositionally measurable function*, or briefly that it is *sup-measurable*, if

(C_1') $N(\cdot, f(\cdot))$ is measurable for every $f \in L_0(\nu)$.

We call the function N an $\mathfrak{S}$-*function*, if it is sup-measurable and if

(C_0) $N(s, 0) = 0$ for ν-almost all $s \in B$.

This normalisation condition already appeared in Example 2.3, where we also defined $\mathfrak{K}$-functions and Caratheodory functions. Note that every sup-measurable function N satisfies automatically the condition (C_1) from Example 2.3, that is, $N(\cdot, r)$ is measurable for all $r \in \mathbb{R}$. Indeed, it suffices to identify $r \in \mathbb{R}$ with the corresponding constant function $r 1_B$. It is well known that every Carathéodory function N is sup-measurable; see for instance [5, Chapter 1.4]. Sup-measurability of a function $N : B \times \mathbb{R} \to \mathbb{R}$ is the weakest condition under which the superposition operator T_N given by

$$T_N f := N(\cdot, f(\cdot)) \quad (f \in L_0(\nu))$$

is well defined on $L_0(\nu)$.

Given two sup-measurable functions N, $K : B \times \mathbb{R} \to \mathbb{R}$, we write $N \preceq K$ if, for every $f \in L_0(\nu)$, $N(\cdot, f(\cdot)) \leq K(\cdot, f(\cdot))$ ν-almost everywhere on B. We say that N and K are *sup-equivalent* (notation $N \simeq K$) if both $N \preceq K$ and $K \preceq N$.

Let $N : B \times \mathbb{R} \to \mathbb{R}$ be an $\mathfrak{K}$-function, and let T_N be the associated superposition operator on $L_0(\nu)$. Then, by Example 2.3, T_N is atomic with respect to the identity Boolean homomorphism. In addition, since $L_0(\nu)$ is Dedekind complete and by Lemma 2.5 and Theorem 1.1, T_N is regular orthogonally additive and disjointness preserving, but actually these two properties can easily be verified directly for a superposition operator. The main result of this section shows that in $L_0(\nu)$, and *up to Boolean homomorphisms*, all sequentially order continuous, atomic operators are superposition operators. Before stating the precise statement, let us recall a definition and introduce an operator.

We recall that an orthogonally additive operator $T : E \to F$ is *sequentially order continuous*, if for every order convergent sequence $(x_n)_n \subseteq E$ with $x_n \xrightarrow{(\mathrm{o})} x$ the sequence $(Tx_n)_n \subseteq F$ is order convergent to Tx.

Now, let (A, Σ, μ) be a second σ-finite measure space. Recall that for every measurable set $A' \in \Sigma$ the multiplication operator $\pi_{A'}$ associated with the multiplication by the characteristic function $1_{A'}$ is an order projection on $L_0(\mu)$. In fact, every order projection is of this form, and when we consider the factor Boolean algebra $\Sigma' = \Sigma/\Sigma_0$ as in Example 2.3 (factorization by the sets of μ-measure zero), then we obtain a one-to-one correspondence, that is, the Boolean algebras Σ' and $\mathfrak{B}(L_0(\mu))$ are isomorphic.

Now let $\Phi : \mathfrak{B}(L_0(\mu)) \to \mathfrak{B}(L_0(\nu))$ be a Boolean homomorphism. We identify it with a Boolean homomorphism $\Phi : \Sigma' \to \Xi'$, and we define an associated, linear *shift operator* $S_\Phi : L_0(\mu) \to L_0(\nu)$ in the following way. First, for every simple function $f = \sum_{i=1}^n r_i 1_{A_i} \geq 0$ (where $r_i \in \mathbb{R}$ and the $A_i \in \Sigma$ are mutually disjoint), we set

$$S_\Phi f := \sum_{i=1}^n r_i 1_{\Phi(A_i)}.$$

The function $S_\Phi f$ is a simple function and, therefore, measurable, and its definition does not depend on the representation of f. Note in this context that since Φ is a Boolean homomorphism, then the $\Phi(A_i)$ are mutually disjoint, too. Second, for every positive, measurable function $f \in L_0(\mu)^+$ there exists an increasing sequence (f_n) of positive, simple functions such that $f = \sup_n f_n$. One can easily show that the sequence (Sf_n) is order bounded in $L_0(\nu)$. We then put

$$S_\Phi f := \sup_n S_\Phi f_n \in L_0(\nu)^+.$$

This definition of $S_\Phi f$ does not depend on the choice of the approximating sequence (f_n). Finally, for arbitrary $f \in L_0(\mu)$, we set

$$S_\Phi f := S_\Phi f_+ - S_\Phi f_-.$$

The operator S_Φ thus defined is a linear, positive operator from $L_0(\mu)$ into $L_0(\nu)$.

Now let Φ be, in addition, a Boolean isomorphism. Then $\Phi^{-1} : \Xi' \to \Sigma'$ is a Boolean isomorphism, too, and it follows from the definition that S_Φ is invertible and $S_\Phi^{-1} = S_{\Phi^{-1}}$. In particular, S_Φ^{-1} is linear and positive, too.

We show that S_Φ is a sequentially order continuous operator. Assume, on the contrary, that S_Φ is not sequentially order continuous. Then there exists a sequence (f_n) in $L_0(\mu)$ such that $f_n \downarrow 0$ and $S_\Phi f_n \not\downarrow 0$. Passing to an appropriate subsequence, we can find $g > 0$ such that $S_\Phi f_n \geq g$ for every $n \in \mathbb{N}$. Applying S_Φ^{-1} to this inequality yields $f_n \geq S_\Phi^{-1} g$ for every $n \in \mathbb{N}$. Since $S_\Phi^{-1} g > 0$, this yields a contradiction.

The next theorem is the main result of the section.

Theorem 3.2. *Let (A, Σ, μ) and (B, Ξ, ν) be σ-finite measure spaces, $\Phi : \Sigma' \to \Xi'$ be a Boolean isomorphism, and $T : L_0(\mu) \to L_0(\nu)$ be a regular orthogonally additive operator. Then the following statements are equivalent:*

1. *T is a continuous (with respect to the metric ϱ_{L_0}), atomic operator subordinate to Φ;*
2. *there exists a $\mathfrak{K}$-function $N : B \times \mathbb{R} \to \mathbb{R}$ such that $T = T_N \circ S_\Phi$, where T_N is the superposition operator associated with N and S_Φ is the shift operator associated with Φ, that is,*

$$Tf = N(\cdot, S_\Phi f(\cdot)) \quad (f \in L_0(\mu)). \tag{3.1}$$

Proof. (1) $\Rightarrow$ (2). Let $T : L_0(\mu) \to L_0(\nu)$ be a continuous, atomic operator subordinate to Φ. Then we define a function $\hat{N} : B \times \mathbb{R} \to \mathbb{R}$ by

$$\hat{N}(\cdot, r) := T(r 1_A)(\cdot) \quad (r \in \mathbb{R}).$$

We note that $\hat{N}(\cdot, 0) = T(0) = 0$ and, therefore, $\hat{N}(\cdot, 0) = 0$ ν-almost everywhere. Moreover, $\hat{N}(\cdot, r)$ is Ξ-measurable for every $r \in \mathbb{R}$. Now, take a simple function $f = \sum_{i=1}^n r_i 1_{A_i}$, where the A_i are mutually disjoint measurable subsets of A and $r_i \in \mathbb{R}$, $1 \leq i \leq n$. Then

$$\begin{aligned} Tf &= T\left(\sum_{i=1}^n r_i 1_{A_i}\right) \\ &= \sum_{i=1}^n T(r_i 1_{A_i}) \\ &= \sum_{i=1}^n T\pi_{A_i}(r_i 1_A) \\ &= \sum_{i=1}^n \Phi(\pi_{A_i}) T(r_i 1_A) \\ &= \sum_{i=1}^n \hat{N}(\cdot, r_i)\, 1_{\Phi(A_i)} \\ &= \sum_{i=1}^n \hat{N}(\cdot, r_i 1_{\Phi(A_i)}) \\ &= \hat{N}\left(\cdot, \sum_{i=1}^n r_i 1_{\Phi(A_i)}\right) \end{aligned}$$

$$= \hat{N}\left(\cdot, S_\Phi\left(\sum_{i=1}^n r_i 1_{A_i}\right)\right)$$
$$= \hat{N}(\cdot, S_\Phi f).$$

In other words, when we define the operator $T_{\hat{N}} : L_0(\nu) \to L_0(\nu)$, $T_{\hat{N}} := T \circ S_\Phi^{-1}$, then $T_{\hat{N}} f = \hat{N}(\cdot, f(\cdot))$ for every finite step function f, so that on the space of finite step functions, the operator $T_{\hat{N}}$ acts like a superposition operator. At the same time, $T_{\hat{N}}$ is defined everywhere on $L_0(\nu)$ and it is continuous with respect to the metric ϱ_{L_0} by assumption on T and by sequential order continuity of S_Φ^{-1} which implies continuity with respect to ϱ_{L_0}.

It is, however, not clear whether $\hat{N}$ is sup-measurable. If it was sup-measurable, then we could invoke [5, Theorem 1.4] to show that $\hat{N}$ is sup-equivalent to a Caratheodory function N. For the construction of a Caratheodory function associated with $T_{\hat{N}}$, we proceed as in the proof of [5, Lemma 1.7], that is, by regularisation and approximation.

We may for our purposes without loss of generality assume that (B, Ξ, ν) is a finite measure space. In fact, if this measure space was only σ-finite, then we could replace the measure ν by an equivalent finite measure λ, as in the definition of the metric ϱ_{L_0}. Define, for every $k \in \mathbb{N}$, the function $\mathfrak{t}_k : \Xi \times L_0(\nu) \to \mathbb{R}$ by

$$\mathfrak{t}_k(D, f) := \int_D ((-k) \vee (T_{\hat{N}} f)(x) \wedge k) \, \mathrm{d}\nu(x) \quad (D \in \Xi, f \in L_0(\nu)),$$

and then for every $k \in \mathbb{N}$ and every $\lambda > 0$ the regularized function $\mathfrak{t}_{k,\lambda} : \Xi \times L_0(\nu) \to \mathbb{R}$ by

$$\mathfrak{t}_{k,\lambda}(D, f) := \inf_{g \in L_0} [\mathfrak{t}_k(D, g) + \lambda \int_D (|g(x) - f(x)| \wedge k) \, \mathrm{d}\mu(x)].$$

Then, for every $k \in \mathbb{N}$, $D \in \Xi$, $f \in L_0(\nu)$, r, $\hat{r} \in \mathbb{R}$,

$$\mathfrak{t}_{k,\lambda}(D, f) \leq \mathfrak{t}_{k,\lambda'}(D, f) \leq \mathfrak{t}_k(D, f) \text{ for every } 0 < \lambda \leq \lambda', \tag{3.2}$$

$$\lim_{\lambda \to \infty} \mathfrak{t}_{k,\lambda}(D, f) = \mathfrak{t}_k(D, f), \text{ and} \tag{3.3}$$

$$|\mathfrak{t}_{k,\lambda}(D, r) - \mathfrak{t}_{k,\lambda}(D, \hat{r})| \leq \lambda \, \mu(D) \, |r - \hat{r}|, \tag{3.4}$$

where in the third line, we identify a real number r with the corresponding constant function $r \, 1_B$. To see that $\mathfrak{t}_{k,\lambda}(D, f) \leq \mathfrak{t}_k(D, f)$ (see (3.2)), it suffices simply to take $g = f$ in the definition of $\mathfrak{t}_{k,\lambda}$. Similarly, from the definition, one sees that $\mathfrak{t}_{k,\lambda}$ increasing in $\lambda > 0$ (see (3.2)). The property (3.3) follows from the order continuity of $T_{\hat{N}}$. Finally, to prove (3.4), fix f, $\hat{f} \in L_0(\nu)$. By definition, for every $g \in L_0(\nu)$,

$$\mathfrak{t}_{k,\lambda}(D, f) \leq \mathfrak{t}_k(D, g) + \lambda \int_D (|g(x) - f(x)| \wedge k) \, \mathrm{d}\nu(x).$$

Moreover, for every $\varepsilon > 0$, there exists $g_\varepsilon \in L_0(\nu)$ such that

$$\mathfrak{t}_{k,\lambda}(D, \hat{f}) \geq \mathfrak{t}_k(D, g_\varepsilon) + \lambda \int_D (|g_\varepsilon(x) - \hat{f}(x)| \wedge k) \, \mathrm{d}\nu(x) - \varepsilon.$$

When we subtract both inequalities and take $g = g_\varepsilon$ in the first inequality, then we obtain

$$\mathfrak{t}_{k,\lambda}(D,f) - \mathfrak{t}_{k,\lambda}(D,\hat{f}) \le \lambda \int_D (|f(x) - \hat{f}(x)| \wedge (2k)) \,\mathrm{d}\nu(x) + \varepsilon,$$

or, when $f = r$ and $\hat{f} = \hat{r}$ are constant functions,

$$\mathfrak{t}_{k,\lambda}(D,r) - \mathfrak{t}_{k,\lambda}(D,\hat{r}) \le \lambda\, \mu(D)\, |r - \hat{r}| + \varepsilon.$$

Since this inequality holds for arbitrary $\varepsilon > 0$, and by changing the roles of r and $\hat{r}$, one obtains (3.4).

One easily shows that for every $k \in \mathbb{N}$, $\lambda > 0$ and $r \in \mathbb{R}$ the function $\mathfrak{t}_{k,\lambda}(\cdot, r)$ is a measure on (B, Ξ). By the inequality (3.4), this measure is absolutely continuous with respect to ν. By the Radon–Nikodym theorem, for every $k \in \mathbb{N}$, $\lambda \in \mathbb{N}$ and $r \in \mathbb{Q}$, there exist densities $N_{k,\lambda}(\cdot, r)$ such that

$$\mathfrak{t}_{k,\lambda}(D,r) = \int_D N_{k,\lambda}(x,r) \,\mathrm{d}\nu(x).$$

Since we are only dealing with a countable set of parameters k, λ and r, by the definition of $\mathfrak{t}_{k,\lambda}$, and by the properties (3.2) and (3.4), there exists a set $D_0 \in \Xi$ of ν-measure zero such that, for every $k \in \mathbb{N}$, $\lambda \in \mathbb{N}$, r, $\hat{r} \in \mathbb{Q}$ and $x \in B \setminus D_0$,

$$-k \le N_{k,\lambda}(x,r) \le k, \tag{3.5}$$

$$N_{k,\lambda}(x,r) \le N_{k,\lambda+1}(x,r), \quad \text{and} \tag{3.6}$$

$$|N_{k,\lambda}(x,r) - N_{k,\lambda}(x,\hat{r})| \le \lambda\, |r - \hat{r}|. \tag{3.7}$$

From the last inequality it follows that, for every $k \in \mathbb{N}$, $\lambda \in \mathbb{N}$ and every $x \in B \setminus D_0$, the function $N_{k,\lambda}(x,\cdot)$ uniquely extends to a Lipschitz continuous function on $\mathbb{R}$, which we still denote by $N_{k,\lambda}(x,\cdot)$. In particular, the functions $N_{k,\lambda}$ are Caratheodory functions (more precisely, they are $\mathfrak{K}$-functions), and the associated superposition operators are continuous on $L_0(\nu)$ by Example 2.3. Set

$$N_k(x,r) := \sup_{\lambda \in \mathbb{N}} N_{k,\lambda}(x,r).$$

As a pointwise supremum of (Lipschitz) continuous functions, for every $x \in B \setminus D_0$, the function $N_k(x,\cdot)$ is lower semicontinuous. By (3.6), for every $x \in B \setminus D_0$, $r \in \mathbb{R}$,

$$N_k(x,r) = \lim_{\lambda \to \infty} N_{k,\lambda}(x,r).$$

By [5, Theorem 1.1], N_k is a so-called Shragin function.

By Lebesgue's dominated convergence theorem, for every $k \in \mathbb{N}$, every $D \in \Xi$ and every $r \in \mathbb{R}$,

$$\begin{aligned}\int_D N_k(x,r) \,\mathrm{d}\nu(x) &= \lim_{\lambda \to \infty} \int_D N_{k,\lambda}(x,r) \,\mathrm{d}\nu(x) \\ &= \lim_{\lambda \to \infty} \mathfrak{t}_{k,\lambda}(D,r)\end{aligned}$$

$$= \mathfrak{t}_k(D, r)$$
$$= \int_D ((-k) \vee \hat{N}(x, r) \wedge k) \, \mathrm{d}\nu(x).$$

As a consequence, there exists a set $D_1 \in \Xi$ of ν-measure zero, such that $D_1 \supseteq D_0$ and, for every $x \in B \backslash D_1$ and every $r \in \mathbb{Q}$,

$$N_k(x, r) = (-k) \vee \hat{N}(x, r) \wedge k. \tag{3.8}$$

In particular, the superposition operator associated with the Shragin function N_k and the superposition operator associated with the function $(-k) \vee \hat{N} \wedge k$ coincide on the space of rational step functions (step functions taking values in $\mathbb{Q}$). The latter operator, however, uniquely extends to a continuous operator on $L_0(\nu)$. By [5, Theorem 1.3], the function N_k already is a Caratheodory function. It remains now to let $k \to \infty$, and to note that $N_k(x, \cdot)$ is uniquely determined on $[-k, k] \cap \mathbb{Q}$ by (3.8), independently of $k \in \mathbb{N}$, to obtain a Caratheodory function $N : B \times \mathbb{R} \to \mathbb{R}$ such that the associated superposition operator T_N coincides with $T_{\hat{N}}$ on the space of rational step functions. Hence, by continuity, $T_N = T_{\hat{N}}$ everywhere on $L_0(\nu)$. Since $T_N 0 = 0$, N is in fact a $\mathfrak{K}$-function, and we have proved one implication.

(2) $\Rightarrow$ (1). Assume that there exists a $\mathfrak{K}$-function $N : B \times \mathbb{R} \to \mathbb{R}$ such that for any $f \in L_0(\nu)$

$$Tf = N(\cdot, S_\Phi f(\cdot)),$$

that is, $T = T_N \circ S_\Phi$. Since any superposition operator associated with a Caratheodory function is order continuous, and since S_Φ is order continuous, then $T : L_0(\mu) \to L_0(\nu)$ is order continuous.

By Lemma 2.4, the superposition operator T_N is an atomic operator subordinate to the identity homomorphism. By construction, the shift operator S_Φ is an atomic operator subordinate to Φ. It follows easily that the composition $T = T_N \circ S_\Phi$ is an atomic operator subordinate to Φ. The proof is finished. $\square$

4. An Extension of Positive Atomic Operators and Laterally Continuous Orthogonally Additive Operators

In this section, we show that any atomic operator is laterally-to-order continuous. We also prove that an atomic, orthogonally additive map defined on a lateral ideal can be extended to an atomic orthogonally additive operator defined on the whole space.

Let E, F be vector lattices. A net $(x_\alpha)_{\alpha \in \Lambda} \subseteq E$ is said to be *laterally convergent to* $x \in E$ if $x_\alpha \xrightarrow{(o)} x$ and $(x_\beta - x_\gamma) \perp x_\gamma$ for all $\beta, \gamma \in \Lambda$, $\beta \geq \gamma$. In this case, we write $x_\alpha \xrightarrow{\text{lat}} x$. An orthogonally additive operator $T : E \to F$ is said to be *laterally-to-order continuous*, if for every laterally convergent net $(x_\alpha) \subseteq E$ with $x_\alpha \xrightarrow{\text{lat}} x$ the net (Tx_α) order converges to Tx.

The following lemma is a variant of Lemma 2.5.

Lemma 4.1. *Let E be a vector lattice with the principal projection property, F be a vector lattice, $\Phi : \mathfrak{B}(E) \to \mathfrak{B}(F)$ be an order continuous homomorphism of Boolean algebras and $T \in \Phi(E,F)$. Then T is orthogonally additive, laterally-to-order continuous and disjointness preserving.*

Proof. Take a laterally convergent net $(x_\lambda)_{\lambda\in\Lambda}$ with $x_\lambda \xrightarrow{\text{lat}} x$. Denote by π_λ, ρ_λ and π the order projections onto the bands $\{x_\lambda\}^{\perp\perp}$, $\{x - x_\lambda\}^{\perp\perp}$ and $\{x\}^{\perp\perp}$, respectively. Since the elements $x - x_\lambda$ and x_λ are disjoint for any $\lambda \in \Lambda$ it follows that

$$T(x) = T(x - x_\lambda + x_\lambda) = T(x - x_\lambda) + T(x_\lambda).$$

Moreover, the net (π_λ) order converges to π and the net (ρ_λ) order converges to $\mathbf{0}_{\mathfrak{B}(E)}$ in the Boolean algebra $\mathfrak{B}(E)$. Taking into account that Φ is an order continuous homomorphism of Boolean algebras we deduce that the net $\Phi(\rho_\lambda)$ converges to $\mathbf{0}_{\mathfrak{B}(F)}$ in the Boolean algebra $\mathfrak{B}(F)$. Hence,

$$\begin{aligned}|T(x) - T(x_\lambda)| &= |T\pi(x) - T\pi_\lambda(x)|\\ &= |T(\pi - \pi_\lambda)(x)|\\ &= |T\rho_\lambda(x)|\\ &= |\Phi(\rho_\lambda)Tx| \xrightarrow{(\mathrm{o})} 0,\end{aligned}$$

and this completes the proof. $\square$

A subset D of a vector lattice E is said to be a *lateral ideal* if the following conditions hold:
(1) if $x \in D$ and $y \in \mathcal{F}_x$, then $y \in D$;
(2) if x, $y \in D$ and $x \perp y$, then $x + y \in D$.

Example 4.2. Let E be a vector lattice. Then any order ideal in E is a lateral ideal.

Example 4.3. Let E be a vector lattice and $x \in E$. Then $\mathcal{F}_x$ is a lateral ideal.

Example 4.4. Let E, F be vector lattices and $T : E \to F$ a positive, orthogonally additive operator. Then the kernel

$$\ker(T) = \{y \in E : T(y) = 0\}$$

is a lateral ideal.

Let E, F be vector lattices, D be a lateral ideal in E. A map $T : D \to F$ is said to be

- *orthogonally additive*, if $T(x + y) = Tx + Ty$ for every disjoint elements x, $y \in D$;
- *positive*, if $Tx \geq 0$ for every $x \in D$;
- *atomic*, if $T\pi = \Phi(\pi)T$ for every order projection $\pi \in \mathfrak{B}(E)$.

Theorem 4.5 [15, Theorem 4.4]. *Let E, F be vector lattices with F Dedekind complete, $D \subseteq E$ be a lateral ideal, and $T : D \to F$ be a positive, orthogonally additive operator. Then the operator $\widetilde{T} : E \to F$ defined by*

$$\widetilde{T}_D x = \sup\{Ty : y \in \mathcal{F}_x \cap D\} \quad (x \in E),$$

with the interpretation $\sup \emptyset = 0$*, is positive, orthogonally additive and laterally-to-order continuous, that is,* $\widetilde{T}_D \in \mathcal{P}_+(E,F)$*. Moreover,* $\widetilde{T}_D x = Tx$ *for every* $x \in D$*.*

The operator $\widetilde{T}_D \in \mathcal{P}_+(E,F)$ is called the *minimal extension* of the positive, orthogonally additive operator $T : D \to F$.

We recall the following auxiliary result.

Lemma 4.6 [17, Lemma 2]. *Let* E *be a vector lattice. Then the relation* $\sqsubseteq$ *is a partial order on* E*. Moreover, for every* $x \in E$ *the set* $\mathcal{F}_x$*, partially ordered by* $\sqsubseteq$*, is a Boolean algebra with the least element* 0*, maximal element* x*, and the Boolean operations*

$$\begin{aligned} z \cup y &:= (z^+ \vee y^+) - (z^- \vee y^-), \\ z \cap y &:= (z^+ \wedge y^+) - (z^- \wedge y^-), \\ \overline{z} &:= x - z \quad (y,\, z \in \mathcal{F}_x). \end{aligned}$$

The next theorem is the main result of this section. It shows that the minimal extension of an atomic orthogonally additive map is an atomic operator as well.

Theorem 4.7. *Let* E *be vector lattice with the principal projection property and* F *be a Dedekind complete vector lattice,* D *be a lateral ideal in* E*, and* $T : D \to F$ *be an atomic, positive, orthogonally additive map. Then the minimal extension* $\widetilde{T}_D$ *of* T *is an atomic, positive, orthogonally additive operator from* E *to* F *as well.*

Proof. By Theorem 4.5, the operator $\widetilde{T}_D$ is well defined and $\widetilde{T}_D \in \mathcal{OA}_+(E,F)$. We show that $\widetilde{T}_D$ is an atomic operator. Take an order projection $\pi \in \mathfrak{B}(E)$ and $x \in E$. First, we show that $\mathcal{D} := \{Ty : \, y \in \mathcal{F}_x \cap D\}$ is an upward directed set. Indeed, take $y,\, z \in \mathcal{D}$. By Lemma 4.6 there exists $u \in \mathcal{F}_x$ such that $u = z \cap y$. Then $y' := y - u \in \mathcal{F}_x \cap D$ and $y' \perp z$. Hence, $v := y' + z \in \mathcal{F}_x \cap D$, $y \sqsubseteq v$ and $z \sqsubseteq v$. Thus, for every $y,\, z \in \mathcal{F}_x \cap D$ there exists v. Taking into account that the relation $z \sqsubseteq x$ implies that $Tz \le Tx$, we deduce that for every $y,\, z \in \mathcal{F}_x \cap D$ there exists $v \in \mathcal{F}_x \cap D$ such that $Ty \le Tv$ and $Tz \le Tv$. Now,

$$\begin{aligned} \widetilde{T}_D \pi(x) &= \sup\{T\pi(y) : \, y \in \mathcal{F}_x \cap D\} \\ &= \sup\{\Phi(\pi)T(y) : \, y \in \mathcal{F}_x \cap D\}. \end{aligned}$$

Taking into account that the set $\{T(y) : \, y \in \mathcal{F}_x \cap D\}$ is upward directed and that $\Phi(\pi)$ is an order continuous positive linear operator for any order projection $\pi \in \mathfrak{B}(E)$, we get

$$\begin{aligned}
\widetilde{T}_D\pi(x) &= \sup\{\Phi(\pi)T(y) :\ y \in \mathcal{F}_x \cap D\} \\
&= o\text{-}\lim_{\lambda}\{\Phi(\pi)T(y_\lambda) :\ y_\lambda \in \mathcal{F}_x \cap D\} \\
&= \Phi(\pi)\left(o\text{-}\lim_{\lambda}\{T(y_\lambda) :\ y_\lambda \in \mathcal{F}_x \cap D\}\right) \\
&= \Phi(\pi)\sup\{T(y) :\ y \in \mathcal{F}_x \cap D\} \\
&= \Phi(\pi)\widetilde{T}_D(x),
\end{aligned}$$

and the proof is finished. □

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Ralph Chill
Fakultät für Mathematik
Institut für Analysis, TU Dresden
Zellescher Weg 12-14
01062 Dresden
Germany
e-mail: `ralph.chill@tu-dresden.de`

Marat Pliev
Southern Mathematical Institute of the Russian Academy of Sciences
362027 Vladikavkaz
Russia

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