Fixed Points on Asymmetric Riemann Surfaces

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Abstract. We study actions of cyclic groups on asymmetric Riemann surfaces for which all conformal automorphisms of prime orders have the same number of fixed points. We find the sharp upper bound on the number of points fixed by the square of an anticonformal automorphism of an asymmetric surface. Moreover, we determine the minimal genus of an asymmetric Riemann surface on which a given finite group G acts without fixed points.

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1. Introduction

In this paper, we study the so-called *asymmetric* (or *pseudo-real*, see [1]) Riemann surfaces. This term means that the surface admits orientation reversing automorphisms, but none of them is an involution. Such automorphisms are called *asymmetries* and they have orders divisible by 4. An asymmetry of order 4 is called a *pseudo-symmetry* and the underlying Riemann surface is called *pseudo-symmetric*. Let us also mention that an orientation reversing involution is called a *symmetry*, and the surface on which it acts is called *symmetric*.

The category of complex algebraic curves is equivalent to the category of compact Riemann surfaces. It is known that the moduli space $\mathcal{M}_g$ of complex algebraic curves of genus g is a quasi-projective variety which can be defined in $\mathbb{P}^n(\mathbb{C})$ by polynomials with rational coefficients. There is an antiholomorphic involution $\iota : \mathcal{M}_g \rightarrow \mathcal{M}_g$ which maps the class of a complex

curve to its conjugate. The fixed points of such a mapping are called *complex algebraic curves with real moduli*. The defining equations of symmetric Riemann surfaces are over reals and therefore they are fixed by ι . The other fixed points, which are not symmetric, correspond to the asymmetric surfaces.

There are many publications concerning asymmetric and pseudo-symmetric Riemann surfaces, let us only mention the most important, from our point of view, which were the motivation for this work. Many general results concerning asymmetric surfaces can be found in [1] and some asymmetric surfaces with cyclic automorphism groups were also considered in [4]. The hyperelliptic asymmetric surfaces were studied in [12] and in [2], where the defining equations for such surfaces are given and the special case of hyperelliptic pseudo-symmetric surfaces is also treated. The present paper can be seen as a continuation of the authors' papers [8, 9, 13], where asymmetric p -hyperelliptic and (q, n) -gonal surfaces, with cyclic automorphism group, were studied.

We shall also recall the two papers concerning fixed points on Riemann surfaces, which are related to the main theme of this work. In [5] the number of fixed points of the product of two symmetries of a Riemann surface is studied, while here we study the number of points fixed by the square of an asymmetry. Also, in [7] the first author of this paper considered the fixed point free actions on symmetric surfaces, while here in the last part of the paper we study similar problems in the asymmetric case.

In this paper, we study fixed points of conformal automorphisms of asymmetric Riemann surfaces of genus $g > 1$. We use the combinatorial approach to the problem based on the uniformization theorem and on the theory of non-euclidean crystallographic groups, called shortly NEC groups. The second chapter provides some information about NEC groups necessary for understanding the topic, and it presents two well-known theorems about fixed points. The first proved by Macbeath in [10], gives a formula on the total number of points fixed by a conformal automorphism of a Riemann surface. The second one proved by Izumi in [6] gives the upper bound on this number.

In the third chapter we use the necessary and sufficient conditions for the existence of an asymmetric Riemann surface of genus g with the cyclic automorphism group $\mathbb{Z}_{4k} = \langle \delta \rangle$, given in [13], in order to determine all possible integers which can be realized as the numbers of fixed points of δ^2 . In particular, we find the upper bound on this number and we study its coincidence with Izumi's bound.

For a given pair (g, k) , let $r = g \bmod 2k - 1$, $c = (g - r)/(2k - 1)$, $r_{(k)} = r \bmod k$ and $g_{(2)} = g \bmod 2$. We prove that if $g > 2k - 1$ and $g_{(2)} \leq r_{(k)} \leq c + 1$, then there is an asymmetric Riemann surface of genus g having an asymmetry δ of order $4k$, whose square has exactly $2(c + 1 - r_{(k)})$ fixed points. Later on, we find the minimal number of topologically non-equivalent actions of the group $\langle \delta \rangle$.

The fourth chapter is devoted to the so-called fixed point stable actions of the automorphism group G on an asymmetric Riemann surface. We say that such an action is β -fixed point stable if all conformal automorphisms of prime order in G fix exactly $2\beta + 2$ points for an integer $\beta > -1$. If this condition is satisfied by all such conformal automorphisms except involutions, then we say that the action is β -exceptional fixed point stable. We prove that for given integers β and k , there is an infinite sequence (g_t) corresponding to β -fixed point stable actions of the group $\mathbb{Z}_{4k}$ on asymmetric Riemann surfaces of genera g_t . We find the necessary and sufficient conditions for the existence of an asymmetric pseudo-symmetric Riemann surface with β -fixed point stable action of the group $\mathbb{Z}_{4k}$, and we determine the smallest genus of such a surface. In the fifth chapter we find the minimal number of β -exceptional fixed point stable actions of the group $\mathbb{Z}_{4k}$ on an asymmetric Riemann surface of genus g .

In the last part of the paper, we study fixed point free actions for which none automorphism of an asymmetric surface has fixed points. We construct such actions with a given finite group G standing as the group of conformal automorphisms of the surface and we find the lower bound on its genus. In particular, we prove that any nilpotent group of even order acts without fixed points on an asymmetric Riemann surface.

2. Preliminaries

We shall use the combinatorial approach, based on Riemann uniformization theorem and non-euclidean crystallographic groups. Recall that such a *NEC group* is just a discrete and cocompact subgroup of the group $\mathcal{G}$ of isometries of the hyperbolic plane $\mathcal{H}$, including those which reverse orientation, and if such a subgroup contains only orientation preserving isometries, then it is called a *Fuchsian* group. For every NEC group Λ , we have the associated *signature* $\sigma(\Lambda)$, which determines its algebraic structure. It has the form

$$\sigma(\Lambda) = (h; \pm; [m_1, \dots, m_t]; \{C_1, \dots, C_u, (-)^v\}). \quad (1)$$

The numbers $m_i \geq 2$ are called the *proper periods*, the non-empty brackets $C_i = (n_{i1}, \dots, n_{is_i})$, $i = 1, \dots, u$, or the empty ones $(-)$, are the *period cycles*, the numbers $n_{ij} \geq 2$ are the *link periods* and $h \geq 0$ is said to be the *orbit genus* of Λ . The orbit space $\mathcal{H}/\Lambda$ is a surface of topological genus h , having $u + v$ boundary components, and being orientable or not according to the sign being $+$ or $-$. The group with the signature (1) has the presentation given by the following generators and relations:

$$\left\{ \begin{array}{l} \text{generators:} \\ \text{(a) } x_i, i = 1, \dots, t, \\ \text{(b) } c_{ij}, i = 1, \dots, u + v, j = 0, \dots, s_i, \\ \text{(c) } e_i, i = 1, \dots, u + v, \\ \text{(d) } a_i, b_i, i = 1, \dots, h \text{ if the sign is } +, \\ \quad d_i, i = 1, \dots, h, \text{ if the sign is } -, \\ \text{relations:} \\ \text{(A) } x_i^{m_i} = 1, i = 1, \dots, t, \\ \text{(B) } c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1, i = 1, \dots, u + v, j = 1, \dots, s_i, \\ \text{(C) } c_{is_i} = e_i^{-1}c_{i0}e_i, i = 1, \dots, u + v, \\ \text{(D) } x_1 \dots x_re_1 \dots e_{u+v}[a_1, b_1] \dots [a_h, b_h] = 1, \text{ if the sign is } +, \\ \quad x_1 \dots x_re_1 \dots e_{u+v}d_1^2 \dots d_h^2 = 1, \text{ if the sign is } -. \end{array} \right. \quad (2)$$

A Fuchsian group can be regarded as a NEC group with the signature

$$(h; +; [m_1, \dots, m_t]; \{-\}), \quad (3)$$

usually shortened to $(h; m_1, \dots, m_t)$; a Fuchsian group without periods is called a *Fuchsian surface group*.

Any set of generators of a NEC group satisfying the above relations is called a *canonical set of generators*. Every NEC group has an associated fundamental region, whose hyperbolic area $\mu(\Lambda)$, for a NEC group Λ with signature (1), is given by

$$2\pi \left(\varepsilon h + u + v - 2 + \sum_{i=1}^t \left(1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^u \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}} \right) \right), \quad (4)$$

where $\varepsilon = 2$ if the sign is $+$ and $\varepsilon = 1$ otherwise. Since the above formula depends on the signature of the group, sometimes we shall write $\mu(\sigma)$ instead of $\mu(\Lambda)$ when we refer to the hyperbolic area of a fundamental region of a NEC group Λ with signature σ .

Now, if Γ is a finite index subgroup in a NEC group Λ , then it is a NEC group itself and there is a Hurwitz–Riemann formula, which says that

$$[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}.$$

Finally, it is known that an abstract group with the presentation (2) can be realized as a NEC group Λ with the signature (1) if and only if the quantity in (4) is positive.

Any NEC group Λ with the signature (1) has the so-called *canonical Fuchsian subgroup* Λ^+ which, by [11], has signature

$$(\varepsilon h + u + v - 1; m_1, m_1, \dots, m_t, m_t, n_{11}, \dots, n_{us_u}). \quad (5)$$

Every compact Riemann surface X of genus $g \geq 2$ is isomorphic to the orbit space $\mathcal{H}/\Gamma$ for some surface Fuchsian group Γ . A finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ containing Γ as a normal subgroup. In such a case we say that G acts on X with signature σ , where σ denotes the signature of Λ . If Λ is a *maximal NEC*

group, then there does not exist another NEC group containing it properly, and the group G is the full automorphism group of X . The detailed exposition of maximality can be found in [3]. A signature σ is called *maximal*, if for every NEC group Λ' with a signature σ' containing a NEC group Λ with the signature σ and having the same Teichmüller dimension, the equality $\Lambda = \Lambda'$ holds. For any maximal signature σ , there exists a maximal NEC group with the signature σ .

In the paper we will use the two well-known facts about the number of points fixed by a conformal automorphism of a Riemann surface. The first is the formula on the total number of such points given by Macbeath in [10].

Theorem 2.1. *If a cyclic group G of order n acts on a Riemann surface X with a signature $(\gamma; m_1, \dots, m_r)$, then an automorphism $h \in G$ of order d fixes exactly $f(h) = n \sum_{d|m_i} \frac{1}{m_i}$ points on X .*

The second fact, proved by Izumi in [6], gives the upper bound on the number m of fixed points.

Theorem 2.2. *An orientation preserving automorphism of order n of a Riemann surface of genus g has at most $\frac{2(g+n-1)}{n-1}$ fixed points.*

3. The Number of Points Fixed by the Square of an Asymmetry

Let δ be an orientation reversing automorphism of an asymmetric Riemann surface X of genus $g \geq 2$. We shall study the number of fixed points of the conformal automorphism δ^2 . Let us notice that δ has order divisible by 4 because if $d = \sharp(\delta)$ was odd, then δ would be conformal as a power of δ^2 while if $d \equiv 2 \pmod{4}$, then $\delta^{d/2}$ would be a symmetry. So assume that $d = 4k$ for some integer $k \geq 1$. Since X has no symmetries, it follows that $G = \langle \delta \rangle$ acts with a signature

$$(h + 1; -; [m_1, \dots, m_s, 2k, \dots, 2k]; \{-\}) \quad (6)$$

for some nonnegative integers l, h and some proper divisors $m_1, \dots, m_s$ of $2k$. In [13] the second author found the necessary conditions for the existence of a smooth epimorphism from a NEC group with the signature (6) onto $\mathbb{Z}_{4k}$ and justified that this signature must be maximal. As the result we have the following

Lemma 3.1. *There exists an anti-conformal automorphism δ of order $4k$ of an asymmetric Riemann surface of genus g if and only if*

$$g = 2k \left[h + \sum_{i=1}^s \left(1 - \frac{1}{m_i} \right) \right] + (l - 1)(2k - 1) \quad (7)$$

for some nonnegative integers h, l and some proper divisors $m_1, \dots, m_s$ of $2k$ such that

- (i) h has the same parity as g ,

- (ii) if $h = 0$, then $l \neq 0$ or $(m_{i_1} \cdot \dots \cdot m_{i_{s'}})/\text{g.c.d.}(m_{i_1}, \dots, m_{i_{s'}}) = 2k$ for some subset $\{i_1, \dots, i_{s'}\} \subset \{1, \dots, s\}$,
- (iii) $(h, s, l) \neq (1, 1, 0), (0, 1, 1)$ and $(0, 2, 0)$.

In order to simplify the notation, we will write $a_{(b)}$ instead of $a \bmod b$ for natural numbers a and b . In the case when a Riemann surface X has an automorphism of order $4k$, we will present the genus of X in the form $g = c(2k - 1) + r$ for $c \geq 0$ and $0 \leq r \leq 2k - 2$. Throughout the paper we shall use the following symbols:

$$r = g_{(2k-1)}, \quad c = (g - r)/(2k - 1) \text{ and } r_{(k)} = r \bmod k. \quad (8)$$

Moreover, for any group G , by $[g_1, g_2]$ we denote the commutator of two elements $g_1, g_2 \in G$ defined by the formula $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$.

Theorem 3.2. *The square of an anti-conformal automorphism δ of order $4k$ of an asymmetric Riemann surface of genus $g = c(2k - 1) + r$ has $f(\delta^2) = 2(c + 1 - R)$ fixed points for some integer R in the range $0 \leq R \leq c + 1$. Conversely, for any R satisfying the last inequalities, there exists such an automorphism δ if and only if*

$$r = 2k \left[h + \sum_{i=1}^s \left(1 - \frac{1}{m_i} \right) \right] - R(2k - 1) \quad (9)$$

for an integer $h \geq 0$ of the same parity as g and some proper divisors $m_1, \dots, m_s$ of $2k$ satisfying the conditions (ii) and (iii) of Lemma 3.1.

Proof. The cyclic group $G = \langle \delta \rangle$ of order $4k$ acts on an asymmetric Riemann surface of genus g with the signature (6) for some parameters $h, l, m_1, \dots, m_s$ satisfying the conditions of Lemma 3.1. According to (5), the subgroup $\langle \delta^2 \rangle$ acts with the signature

$$(h; +; [m_1, m_1 \dots, m_s, m_s, 2k, \dots, 2k]; \{-\}).$$

Thus by Theorem 2.1, δ^2 has $f(\delta^2) = 2l$ fixed points. By (7),

$$l = g/(2k - 1) + 1 - 2k \left[h + \sum_{i=1}^s \left(1 - \frac{1}{m_i} \right) \right] / (2k - 1) \leq \frac{g}{2k - 1} + 1. \quad (10)$$

Thus $f(\delta^2) \leq 2(\frac{g}{2k-1} + 1)$ which is consistent with Theorem 2.2. Using the notation (8), we can write l in the form $l = c + 1 - R$ for

$$R = \left[2k \left(h + \sum_{i=1}^s \left(1 - \frac{1}{m_i} \right) \right) - r \right] / (2k - 1). \quad (11)$$

By the last formula, r is equal to (9). Since $0 \leq l \leq \lfloor \frac{g}{2k-1} \rfloor + 1$, it follows that R must be a nonnegative integer not exceeding $c + 1$.

Conversely, let R be an integer in the range $0 \leq R \leq c + 1$. If there exist $h \geq 0$ of the same parity as g and some proper divisors $m_1, \dots, m_s$ of $2k$ satisfying the conditions (ii) and (iii) of Lemma 3.1 for which r can

be expressed by (9), then according to Lemma 3.1, there is an action of the group $G = \mathbb{Z}_{4k} = \langle \delta \rangle$ with the signature

$$(h+1; -; [m_1, \dots, m_s, 2k, {}^{c+1-R}_{-}, 2k]; \{-\})$$

on a Riemann surface X of genus $g = c(2k-1) + r$ and δ^2 has $2(c+1-R)$ fixed points. By the condition (iii), the signature is maximal and so we can assume that G is the full automorphism group. Thus X is asymmetric because it has an anti-conformal automorphism δ , but it has no symmetries.

We will say that an anti-conformal automorphism δ of an asymmetric Riemann surface X of genus g is a $[k, l, g]$ -asymmetry, if $\sharp(\delta) = 4k$ and δ^2 fixes exactly $2l$ points on X .

Corollary 3.3. *Suppose that $g = c(2k-1) + r > 2k-1$ and there is an integer γ which, depending on r , satisfies the following conditions:*

- (A) $r \geq k : g_{(2)} \leq \gamma \leq \min\{r_{(k)}, c+1\}$ and $2k-r+\gamma$ divides $2k$,
- (B) $r \leq k : k-r+g_{(2)} \leq \gamma \leq \min\{2k-2-r+g_{(2)}, c+1-g_{(2)}\}$ and $2k+g_{(2)}-r-\gamma$ divides $2k$, and $\gamma \neq c+1$ if g is even except the case when $\gamma = k-r$ for odd k .

Then there exists a $[k, l, g]$ -asymmetry for $l = c+1-R$, where $R = \gamma$ in the case (A) and $R = \gamma + g_{(2)}$ in the case (B).

Proof. Let $R = \gamma$ in the case (A). Then there exist parameters $h = g_{(2)}$, $s = 2(\gamma - g_{(2)}) + 1$, $m_1 = \dots = m_{s-1} = 2$ and $m_s = \frac{2k}{2k-r+\gamma}$ for which r can be expressed by formula (9). In the case (B) for $R = \gamma + g_{(2)}$ such parameters can be selected as follows $h = g_{(2)}$, $s = \gamma + 1$, $m_1 = \dots = m_{s-1} = k$ and $m_s = \frac{2k}{2k-r+g_{(2)}-\gamma}$. In both cases the period m_s is omitted when it is equal to 1.

According to Theorem 3.2, there exists a $[k, l, g]$ -asymmetry for $l = c+1-R$, if the selected above parameters satisfy the equality (9) and the conditions (ii) and (iii) of Lemma 3.1. It is easy to check that the condition (iii) fails only in few cases at which $g \leq 2k-1$. Suppose that the condition (ii) is not satisfied. Then g must be even and $\gamma = c+1$. However, in the case (A) for $g > 2k-1$, there are at least four periods equal to 2. Thus we can reduce the number of such periods by 4 and increase the number h by 2 at the same time. After this change the condition (ii) is satisfied. If k is odd and $\gamma = k-r$ in the case (B), then there are periods 2 and k for which the condition (ii) is also true.

Theorem 3.4. *Let δ be a $[k, l, g]$ -asymmetry for $g > 2k-1$. Then $l \leq c+1-g_{(2)}$, and the upper bound is sharp if and only if $2k+g_{(2)}-r$ divides $2k$.*

Proof. According to Theorem 3.2, $l = c+1-R$ for some nonnegative integer R for which there exist h of the same parity as g and some proper divisors $m_1, \dots, m_s$ of $2k$ such that r is given by (9). Thus an upper bound on l equals $c+1$. We will find the conditions on g for which this bound is sharp. If $R = 0$, then r given by (9) is in the range $0 \leq r \leq 2k-2$, if and only if $(h, s) = (0, 0)$ or $(h, s) = (0, 1)$. Thus $r = 0$ or $r = 2k - \frac{2k}{m_1}$. In both cases $2k-r$ divides $2k$, and g is even because it must have the same parity as h . Consequently, $R \geq 1$

for odd g . If $R = 1$, then r satisfies the inequalities $0 \leq r \leq 2k - 2$ for odd h if and only if $(h, s) = (1, 0)$ or $(h, s) = (1, 1)$. Thus $r = 1$ or $r = 1 + 2k - \frac{2k}{m_1}$. In both cases g is an odd integer for which $1 + 2k - r$ divides $2k$. Summing up, $l \leq c + 1 - g_{(2)}$ and if l attains this upper bound, then $2k - r + g_{(2)}$ divides $2k$. Conversely, if the last condition is satisfied, then applying Corollary 3.3 for $\gamma = g_{(2)}$ we get a $[k, l, g]$ -asymmetry for which l attains the upper bound $c + 1 - g_{(2)}$.

Corollary 3.5. *If $g = c(2k - 1) + r$ and $2k_{(2k-r)} = 0$, then the square of a $[k, l, g]$ -asymmetry has at most m fixed points, where*

$$m = \begin{cases} 2c + 2, & g_{(2)} = 0, \\ 2c, & g_{(2)} = 1 \text{ and } 2k_{(2k-r+1)} = 0, \\ 2c - 2, & g_{(2)} = 1 \text{ and } 2k_{(2k-r+1)} \neq 0, \end{cases} \quad (12)$$

and this bound is sharp.

Proof. We need only to consider the case at which $g_{(2)} = 1$ and $2k_{(2k-r+1)} \neq 0$. Then according to Theorem 3.4, $l \leq c - 1$. If $2k - r$ divides $2k$, then for $R = 2$, there are parameters $h = 1$, $s = 2$, $m_1 = 2k/(2k - r)$ and $m_2 = k$ satisfying the assumptions of Theorem 3.2. Thus there is a $[k, c - 1, g]$ -asymmetry.

There are some Riemann surfaces corresponding to complex algebraic curves with real moduli for which the Izumi's bound is sharp. In [5] G. Gromadzki and the first author proved that the maximal number $\frac{2(g+n-1)}{n-1}$ of fixed points is attained for the product of two symmetries, which generate a dihedral automorphism group D_n of a symmetric Riemann surface of genus g . In the asymmetric case, we have the following result.

Theorem 3.6. *Let δ be an anti-conformal automorphism of order $4k$ of an asymmetric Riemann surface of genus g . Then the number of fixed points of δ^2 attains the Izumi's bound if and only if $g \equiv 0 \pmod{4k - 2}$. In this case, δ^2 is the hyperelliptic involution if δ is a pseudo-symmetry.*

Proof. By the proof of Theorem 3.2, $f(\delta^2) = 2l$ for l given by (10). Thus $f(\delta^2) = 2(\frac{g}{2k-1} + 1)$ if and only if g is divisible by $2k - 1$ and $h = s = 0$. Hence the group $\langle \delta \rangle$ acts with the signature

$$(1; -, [2k, \cdot^l, 2k]; \{-\}), \quad l = \frac{g}{2k-1} + 1. \quad (13)$$

By Lemma 3.1, this action is possible only for even g because h must have the same parity as g . So $g \equiv 0 \pmod{4k - 2}$. If δ is a pseudo-symmetry, then the signature (13) has the form $(1; -, [2, {}^{g+1}, 2]; \{-\})$. Thus, by (5), the group $\mathbb{Z}_2 = \langle \delta^2 \rangle$ acts with the signature $(0; +; [2, {}^{2g+2}, 2]; \{-\})$. It means that δ^2 is a conformal involution with $2g + 2$ fixed points and $X/\langle \delta^2 \rangle$ has genus 0. Such an automorphism is called a *hyperelliptic involution*.

Corollary 3.7. *For any integer $g = c(2k - 1) + r$ such that $g > 2k - 1$ and $g_{(2)} \leq r_{(k)} \leq c + 1$, there exists a $[k, c + 1 - r_{(k)}, g]$ -asymmetry. In particular, there exists such an asymmetry for any $g \geq (k - 1)(2k - 1)$.*

Proof. Suppose that $r \geq k$. Then applying Corollary 3.3 for $\gamma = r_{(k)}$, we conclude that there exists a $[k, c + 1 - r_{(k)}, g]$ -asymmetry. If $r < k$, then for $R = r_{(k)}$, there are parameters $h = g_{(2)}$, $s = 2(r_{(k)} - g_{(2)})$ and $m_1 = \dots = m_s = 2$ satisfying the conditions of Theorem 3.2, and so such an asymmetry exists also in this case. In both cases the group generated by the asymmetry acts with the signature

$$(g_{(2)} + 1; -; [2, \frac{2(r_{(k)} - g_{(2)}) + \frac{r - r_{(k)}}{k}}{2}, 2k, c+1 - r_{(k)}, 2k]; \{-\}). \quad (14)$$

We say that a pair (g, k) of integers $g, k \geq 2$ satisfies the *cyclic action conditions*, if there exists a $[k, l, g]$ -asymmetry δ for some integer l in the range $0 \leq l \leq c + 1 - g_{(2)}$, where $c = \lfloor \frac{g}{2k-1} \rfloor$. In such a case there are parameters $h, m_1, \dots, m_s$ satisfying the conditions of Lemma 3.1, and δ^2 fixes $2l = 2(c + 1 - R)$ points for R given by (11). Among many choices of these parameters, there is one for which R has the smallest possible value. Then the number of points fixed by δ^2 attains the upper bound which is denoted by $f_{\max}^{k,g}$.

Proposition 3.8. *Suppose that a pair (g, k) satisfies the cyclic action conditions. Let $n_1, \dots, n_p$ be all proper divisors of $2k$, and let $e_1, \dots, e_p$ be non-negative integers such that*

- (i) $f = f_{\max}^{k,g} - \sum_{i=1}^p \frac{4ke_i}{n_i} \geq 0$,
- (ii) *if g is even and $f = 0$, then $(n_{i_1} \cdot \dots \cdot n_{i_{p'}})/\text{g.c.d.}(n_{i_1}, \dots, n_{i_{p'}}) = 2k$ for some subset $\{i_1, \dots, i_{p'}\} \subset \{1, \dots, p\}$ for which $(e_{i_1} \cdot \dots \cdot e_{i_{p'}}) \neq 0$. Then there exists a $[k, f/2, g]$ -asymmetry.*

Proof. For a given pair (g, k) , let us choose parameters $h, m_1, \dots, m_s$ for which the number of points fixed by a $[k, l, g]$ -asymmetry attains the upper bound $f_{\max}^{k,g}$. For a proper divisor n of $2k$, let e be a nonnegative integer such that $l_n = l - \frac{2ke}{n} \geq 0$. Then there is a $[k, l, g]$ -asymmetry acting with the signature

$$\sigma_n = (h + 1; -; [m_1, \dots, m_s, 2k, l_n, 2k, n, e, n, 2, \frac{2e(2k/n-1)}{2}, 2]; \{-\})$$

whose square has $f_{\max}^{k,l} - 4ek/n$ fixed points. Repeating this construction for all proper divisors of $2k$ in such a way that the conditions of Lemma 3.1 are satisfied, we obtain a $[k, l, g]$ -asymmetry whose square has $f = f_{\max}^{k,g} - \sum_{i=1}^p \frac{4ke_i}{n_i}$ fixed points.

4. Fixed Point Stable Actions on Asymmetric Surfaces

A Riemann surface X is called (q, n) -gonal if it admits a conformal automorphism ρ of prime order n such that $X/\langle \rho \rangle$ has genus q .

The cyclic group $G = \langle \delta \rangle = \mathbb{Z}_{4k}$ acts on an asymmetric Riemann surface with a signature (6). Thus by (5), the subgroup $\langle \delta^2 \rangle$ acts with the signature

$$(h; +; [m_1, m_1 \dots, m_s, m_s, 2k, \frac{2l}{2}, 2k]; \{-\}).$$

Hence by Theorem 2.1, any conformal automorphism in G has an even number $f(\rho)$ of fixed points.

We will say that an action of G is β -fixed point stable if all automorphisms of prime orders have the same positive number of fixed points equal to $2\beta + 2$ for some integer $\beta > -1$. When this condition is satisfied by all such automorphisms except involutions, we will say that an action is β -exceptional fixed point stable.

Lemma 4.1. *If an action of the group $G = \mathbb{Z}_{4k}$ on an asymmetric Riemann surface X of genus g is β -fixed point stable, then $g + \beta \equiv 0 \pmod{2k}$, and if it is β -exceptional fixed point stable, then $g + \beta \equiv 0 \pmod{k/2^t}$ for the greatest power 2^t dividing k . Moreover, the surface X is a n -sheeted covering of an orbifold of genus $\frac{g+\beta}{n} - \beta$, ramified over $2\beta + 2$ fixed points, for any prime n dividing $2k$, which is different from 2 in the exceptional case.*

Proof. Assume that $G = \mathbb{Z}_{4k}$ acts with signature (6) on an asymmetric Riemann surface X of genus g , and let $\rho \in G$ be a conformal automorphism of prime order n . Then ρ has an even number $f(\rho)$ of fixed points and by the Hurwitz–Riemann formula, the group $\langle \rho \rangle$ acts with the signature

$$\left(\frac{g + \beta}{n} - \beta; n, 2^{\beta+2}, n \right), \quad (15)$$

for $\beta = f(\rho)/2 - 1$. In particular, the surface X is $(\frac{g+\beta}{n} - \beta, n)$ -gonal and β satisfies the conditions:

$$-1 \leq \beta \leq g/(n-1) \text{ and } g + \beta \equiv 0 \pmod{n}. \quad (16)$$

Let D be the set of indices i in the signature (6) for which n divides m_i and let N be the set of indices for which this is not true. Then by Theorem 2.1, $\beta + 1 = \sum_{i \in D} \frac{2k}{m_i} + l$. By substituting the last into (7), we get

$$g + \beta = 2k \left(h - 1 + \sum_{i \in N} \left(1 - \frac{1}{m_i} \right) + |D| + l \right).$$

Thus $g + \beta \equiv 0 \pmod{n^a}$ for the greatest power n^a dividing $2k$. For a fixed point stable action, this congruence is true for all prime n dividing $2k$, which implies $g + \beta \equiv 0 \pmod{2k}$. For an exceptional fixed point stable action, we must exclude $n = 2$ and so $g + \beta \equiv 0 \pmod{k/2^t}$ for the greatest power 2^t dividing k .

Proposition 4.2. *For given $\beta \geq 0$ and $k \geq 2$, there is an infinite sequence (g_t) corresponding to β -fixed point stable actions of the group $\mathbb{Z}_{4k}$ on asymmetric Riemann surfaces of genera g_t .*

Proof. Let $g_t = 2k(\beta + \beta_{(2)}) - \beta + 4kt$ for any natural number t , which is different from 0 if $\beta = 0$. Then by Lemma 3.1, there is an action of the group $\mathbb{Z}_{4k}$ on an asymmetric Riemann surface of genus g_t with signature

$$(\beta_{(2)} + 2t + 1; -, [2k, \beta+1, 2k]; \{-\}).$$

By Theorem 2.1, this action is β -fixed point stable.

Lemma 4.3. *If there is a β -exceptional fixed point stable action of the cyclic group $G = \mathbb{Z}_{4k}$ on an asymmetric Riemann surface X of genus g , then $g + \beta \geq k(2g_{(2)} + \beta - \beta_{(2)})$.*

Proof. Suppose that an action of G with signature (6) is β -exceptional fixed point stable. Then any conformal automorphism $\rho \in G$ of odd prime order has $2(\beta + 1)$ fixed points. Let n be the order of one of such automorphisms. Without loss of generality, we can assume that periods $m_1, \dots, m_y$ are divisible by n , and periods $m_{y+1}, \dots, m_s$ are not. Then, according to Theorem 2.1,

$$\beta + 1 = \sum_{i=1}^y \frac{2k}{m_i} + l. \quad (17)$$

By substituting the last into (7), we get

$$g + \beta = 2k \left(h - 1 + \sum_{i=y+1}^s \left(1 - \frac{1}{m_i} \right) + y + l \right). \quad (18)$$

The sum $\sum_{i=y+1}^s (1 - \frac{1}{m_i})$ is the smallest if $m_i = 2$ for $i = y + 1, \dots, s$. In this case, $x = s - y = (g + \beta)/k - 2(h - 1 + y + l)$. Since the greater is $y + l$ then the smaller is x , we conclude by formula (17), that x has the smallest possible value in the case when $m_i = k$ for $i = 1, \dots, y$. Then $l + 2y = \beta + 1$ and $x = (g + \beta)/k - (2h - 1 + \beta + l)$. The last integer must be nonnegative and so $(g + \beta)/k \geq 2h - 1 + \beta + l$. Since $l \geq 1 - \beta_{(2)}$ and $h \geq g_{(2)}$, it follows that $(g + \beta)/k \geq 2g_{(2)} - \beta_{(2)} + \beta$.

Theorem 4.4. *There exists a β -exceptional fixed point stable action of the cyclic group $G = \mathbb{Z}_{4k}$ on an asymmetric pseudo-symmetric Riemann surface X of genus $g > 2k - 1$ if and only if*

$$g + \beta \equiv 0 \pmod{k} \quad \text{and} \quad (g + \beta)/k \geq 2g_{(2)} + \beta - \beta_{(2)}. \quad (19)$$

For even k , the conditions (19) are sufficient for the existence of such an action on an asymmetric Riemann surface, which is not pseudo-symmetric.

Proof. An asymmetric Riemann surface with the automorphism group $\mathbb{Z}_{4k}$ is pseudo-symmetric if and only if k is odd. If an action of the group $\mathbb{Z}_{4k}$ with odd k is β -exceptional fixed point stable, then by Lemmata 4.1 and 4.3, the conditions (19) are satisfied.

Conversely, assume that these conditions are satisfied by some integers $g, k \geq 2$ and $\beta > -1$. Let us consider the signature

$$(h + 1; -; [k, \dots, k, 2, \frac{g+\beta}{k} - 2(h+\beta-y), 2, 2k, 1+\beta-2y, 2k]; \{-\}), \quad (20)$$

for an integer y in the range $0 \leq y \leq \frac{\beta + \beta_{(2)}}{2}$ and an integer h of the same parity as g such that $0 \leq h \leq (g + \beta)/2k + (y - \beta)$. If $g > 2k - 1$, then for any pair (h, y) the condition (iii) of Lemma 3.1 is satisfied. The condition (ii) of this Lemma may not be satisfied for $(h, y) = (0, \frac{\beta+1}{2})$ in the case when g is even and β is odd. However, in this case we can choose $(h, y) = (0, 0)$. So there is at least one action of the group $\mathbb{Z}_{4k}$ with the signature (20) on an asymmetric Riemann surface of genus $g > 2k - 1$. By Theorem 2.1, this action is β -exceptional fixed point stable.

5. Classes of Asymmetric Surfaces with a Cyclic Automorphism Group

An action of a finite group G on a Riemann surface of genus $g \geq 2$ corresponds to a short exact sequence of homomorphisms

$$1 \rightarrow \Gamma \rightarrow \Lambda \xrightarrow{\theta} G \rightarrow 1,$$

where Λ is a NEC group and Γ is a surface Fuchsian group isomorphic to the fundamental group of the surface. Let us denote such an action by (Λ, θ, G) . An action (Λ, θ, G) is topologically equivalent to (Λ', θ', G') if $\varphi\theta = \theta'\psi$ for some isomorphisms $\psi: \Lambda \rightarrow \Lambda'$ and $\varphi: G \rightarrow G'$. Let $[\Lambda, \theta, G]$ be the equivalence class of (Λ, θ, G) . We refer to it as to the g -action class if the actions in the class take place on surfaces of genus g . Moreover, we say that the class is full, if it is topologically equivalent to an action of the full automorphism group of at least one of Riemann surface of genus g . Let $\mathcal{M}_g$ be the moduli space of conformal equivalence classes of Riemann surfaces of genus g . To each full g -action class $[\Lambda, \theta, G]$ there is an associated subvariety $V([\Lambda, \theta, G])$ of $\mathcal{M}_g$, where $X \in V([\Lambda, \theta, G])$ if and only if, up to topological equivalence, the action of $\text{Aut}(X)$ restricts to (Λ, θ, G) . The set of such subvarieties corresponding to all full g -action classes is called the *stratification* of the moduli space $\mathcal{M}_g$.

Theorem 5.1. *Assume that $g = c(2k - 1)$ for $c > 2$. Then there are at least N topologically non-equivalent $(c - 2g_{(2)})$ -exceptional fixed point stable actions of the group $G = \mathbb{Z}_{4k}$ on asymmetric Riemann surfaces of genus g , where*

$$N = \begin{cases} ((c+4)^2 - 4)/16, & c \equiv 2 \pmod{4}, \\ (c+4)^2/16, & c \equiv 0 \pmod{4}, \\ ((c+3)^2 - 4)/16, & c \equiv 3 \pmod{4}, \\ (c+3)^2/16, & c \equiv 1 \pmod{4}. \end{cases} \quad (21)$$

Proof. Let $g = c(2k - 1)$ for $c > 2$. Then for any integer t in the range $0 \leq t \leq \lfloor \frac{c+1}{2} \rfloor - g_{(2)}$ and an even nonnegative s not exceeding t the parameters in the signature

$$(1 + g_{(2)} + s; -; [k, \overset{t+g_{(2)}}{\underset{!}{!}{!}}, k, 2, \overset{2(t-s)}{\underset{!}{!}{!}}, 2, 2k, \overset{c+1-2g_{(2)}-2t}{\underset{!}{!}{!}}, 2k]) \quad (22)$$

satisfy all conditions of Theorem 3.1. Hence for any such pair (s, t) , there is an action of the group $\mathbb{Z}_{4k}$ on an asymmetric Riemann surface of genus g with signature (22), and according to Theorem 2.1, this action is $(c - 2g_{(2)})$ -exceptional fixed point stable. For a given t , the number $n(t)$ of all nonnegative even integers not exceeding t is equal to $\frac{t+1}{2}$ or $\frac{t}{2} + 1$ according to whether t is odd or even. Thus the total number N of all actions with signatures (22) is equal to

$$N = \sum_{t=0}^b n(t) = 2 \left(1 + 2 + \cdots + \frac{b+1}{2} \right) = \frac{(b+2)^2 - 1}{4}$$

or

$$N = \sum_{t=0}^b n(t) = 2 \left(1 + 2 + \cdots + \frac{b}{2} \right) + \frac{b}{2} + 1 = \frac{(b+2)^2}{4}$$

according to whether $b = \lfloor \frac{c+1}{2} \rfloor - g_{(2)}$ is odd or even. Hence we get (21).

Theorem 5.2. *Let $b = \frac{\beta+\beta_{(2)}}{2}$ and let $a = \lfloor \frac{g+\beta}{2k} \rfloor - (\beta + g_{(2)})$ for some integers $k \geq 2$, $\beta \geq 0$ and $g > 2k - 1$ such that*

$$g + \beta \equiv 0 \pmod{k} \quad \text{and} \quad (g + \beta)/k \geq 2(\beta + g_{(2)}).$$

Let $\varepsilon = 1$ in the case when g and k are even and β is odd, and let $\varepsilon = 0$ in the remaining cases. Then there are at least N topologically non-equivalent β -exceptional fixed point stable actions of the group $G = \mathbb{Z}_{4k}$ on asymmetric Riemann surfaces of genus g , where

$$N = \frac{b+1}{2} \left(1 + \frac{b+1}{2} + a \right) - \varepsilon \quad (23)$$

or

$$N = \frac{a + a_{(2)}}{2} (1 + b) + \left(1 + \frac{b}{2} \right) \left(1 + \frac{b}{2} - a_{(2)} \right) - \varepsilon, \quad (24)$$

according to whether b is odd or even, respectively.

Proof. By Lemma 3.1, there is an action of the group $G = \mathbb{Z}_{4k}$ on an asymmetric Riemann surface X of genus g with the signature

$$\sigma_0 = (g_{(2)} + 1; -; [2, \frac{g+\beta}{k} - 2(g_{(2)} + \beta), 2, 2k, 1+\beta, 2k]; \{-\}). \quad (25)$$

Let Λ and Γ be NEC groups such that $X = \mathcal{H}/\Gamma$ and $G = \Lambda/\Gamma$. Since σ_0 is a maximal signature, we can choose Λ as a maximal group, which is not contained properly in any other NEC group. Then G is the full automorphism group of X . By the Hurwitz–Riemann formula, $A = k\mu(\Lambda)/\pi = g - 1$, where $\mu(\Lambda)$ is given by (4). Actions of the group G with signatures which have different sets of proper periods or different orbit genera are not topologically equivalent. Thus all changes of parameters in σ_0 , for which the integer A remains unchanged, provide different classes of surfaces in $\mathcal{M}_g$ with automorphism group $\mathbb{Z}_{4k}$. We cannot reduce $l = \beta + 1$ by 1, because there is no change of other parameters in the signature, which could compensate for this. However, we can reduce l by $2t$ for any integer t in the range $0 \leq t \leq \lfloor \frac{l}{2} \rfloor$, by adding new periods $2, \frac{2t}{k}, 2, k, \frac{2t}{k}, k$ at the same time. The parameter $h = g_{(2)}$ in σ_0 has the smallest possible value. It can increase only by an even number because it must have the same parity as g . We can increase h by an even number s through reducing by $2s$ the number of periods equal to 2, under condition that there are at least $2s$ of such periods in the signature. As a result, we get topologically non-equivalent actions of the group $\mathbb{Z}_{4k}$ with signatures

$$(g_{(2)} + s + 1; -; [2, \frac{g+\beta}{k} - 2(g_{(2)} + \beta + s - t), 2, k, \frac{2t}{k}, k, 2k, 1+\beta-2t, 2k]; \{-\}), \quad (26)$$

where t is an integer in the range $0 \leq t \leq b$ for $b = \frac{\beta+\beta_{(2)}}{2}$, and s is an even nonnegative integer not exceeding $a + t$ for $a = \lfloor \frac{g+\beta}{2k} \rfloor - g_{(2)} - \beta$. We check that if $g > 2k - 1$, then the condition (iii) of Lemma 3.1 is satisfied for any pair (t, s) . The condition (ii) of Lemma 3.1 is not satisfied if and only if g and k are even and $(t, s) = (b, 0)$ for an odd β .

For a given t , let $n(t)$ be the number of all possible values of s . Then for $t \neq b$, $n(t) = \frac{a+t}{2} + 1$ or $n(t) = \frac{a+t+1}{2}$ depending on whether $a+t$ is even or odd. In the case when g and k are even and $t = b$ for odd β , we must reduce the number $n(t)$ by 1 in order to reject $s = 0$. The number N of all signatures of the form (26) is equal to $N = \sum_{t=0}^b n(t)$. By considering the four cases depending on the parity of a and b we calculate that N is equal to (23) or to (24) according to whether b is odd or even, respectively.

Corollary 5.3. *For any integer $g = c(2k - 1) + r$ such that $g > 2k - 1$ and $g_{(2)} \leq r_{(k)} \leq c + 1$ there are at least N topologically non-equivalent actions of a cyclic automorphism group $G = \mathbb{Z}_{4k}$ on asymmetric Riemann surfaces of genus g , for N given by (23) or (24) with $a = r_{(k)} - g_{(2)}$, according to whether $b = \lfloor \frac{c+1-r_{(k)}}{2} \rfloor$ is odd or even, respectively.*

Proof. This is a particular case of Theorem 5.2 with $\beta = c - r_{(k)}$. Now the signatures (26) have the form

$$(g_{(2)} + 1 + s; -; [2, {}^{2(r_{(k)}-g_{(2)}+s-t)+\alpha}, 2, k, \dots, k, 2k, {}^{c+1-r_{(k)}-2t}, 2k]; \{-\}) \quad (27)$$

for $\alpha = \frac{r-r_{(k)}}{k}$. Thus the number of non-equivalent actions cannot be smaller than the number of different choices of parameters t and s for which the conditions of Lemma 3.1 are satisfied. The last number is equal to (23) or (24), according to the parity of b , where $a = r_{(k)} - g_{(2)}$ and $b = \lfloor \frac{c+1-r_{(k)}}{2} \rfloor$. Let us notice that for $t = s = 0$ we get an action with the signature (14).

Corollary 5.4. *Let $\mathcal{M}_g^{(q,n)} \subset \mathcal{M}_g$ be the locus consisting of classes of surfaces admitting a (q, n) -gonal automorphism. For a given pair (g, k) such that $g \geq (k - 1)(2k - 1)$, let N be an integer given by (21) if $g_{(2k-1)} = 0$, and in the remaining cases let N be given by (23) or (24) for $a = r_{(k)} - g_{(2)}$ and $b = \lfloor \frac{c+1-r_{(k)}}{2} \rfloor$, according to whether b is odd or even. Then there are at least N topologically non-equivalent actions of the group $\mathbb{Z}_{4k}$ on surfaces in $\bigcap_n \mathcal{M}_g^{(q,n)}$, where n runs over all odd prime integers dividing k , and $q = \frac{g+\beta}{n} - \beta$ for $\beta = c - 2g_{(2)}$ if $g_{(2k-1)} = 0$, and $\beta = c - r_{(k)}$ in the remaining cases.*

Proof. Suppose that $G = \mathbb{Z}_{4k}$ acts with the signature (6) and let $\rho \in G$ be a conformal automorphism of a prime order n . Then the number of points fixed by ρ is equal to $f(\rho) = 2(l + \sum \frac{2k}{m_i})$, where the sum is taken over those i for which n divides m_i . In particular, if n is odd, then periods $m_i = 2$ in the signature of Λ do not influence the number $f(\rho)$, while each period $m_i = 2k$ provides 2 points fixed by ρ and period $m_i = k$ provides 4 such points. Thus the changes of the signatures, which we discussed in the proofs of Theorem 5.1 and Corollary 5.3, do not influence the number $f(\rho)$. The group $\langle \rho \rangle$ acts with the signature $(q; +; [n, {}^{2\beta+2}, n]; \{-\})$, where $\beta = f(\rho)/2 - 1$ and $q = \frac{g+\beta}{n} - \beta$. In particular, the surface X is $(\frac{g+\beta}{n} - \beta, n)$ -gonal. We can calculate β by Theorem 2.1.

6. Fixed Point Free Actions on Asymmetric Riemann Surfaces

Let $G^\pm$ be the group $\text{Aut}^\pm(X)$ of conformal and anti-conformal automorphisms of an asymmetric Riemann surface X . Let us notice that an automorphism of an asymmetric surface may have fixed points only if it is conformal. We will say that an action of the group $G^\pm$ is *fixed point free* if none of its nontrivial conformal automorphisms has fixed points.

Theorem 6.1. *Let X be an asymmetric Riemann surface of genus g with a fixed point free action of the group $G^\pm = \text{Aut}^\pm(X)$, and let s be the minimal number of generators of the subgroup $G^+ < G^\pm$ consisting of all conformal automorphisms. Then g is an odd integer such that $g \geq 1 + |G^+|$ if $s \leq 4$ and $g \geq 1 + \frac{|G^+|}{2}(s - 2)$ if $s > 4$.*

Proof. Let Λ and Γ be NEC groups for which $X \simeq \mathcal{H}/\Gamma$ and $G^\pm = \Lambda/\Gamma$. The group Γ is a surface Fuchsian group with the signature $(g; +; [-]; \{-\})$, where $g \geq 2$ is the genus of X . Since X has no symmetries, it follows that there can be no period cycles in the signature of Λ . Moreover, by Theorem 2.1, there can be no proper periods in it. Thus

$$\sigma(\Lambda) = (\gamma; -; [-]; \{-\}). \quad (28)$$

Applying the Hurwitz–Riemann formula for (Λ, Γ) , we get

$$g = 1 + \frac{|G^\pm|}{2}(\gamma - 2). \quad (29)$$

Thus $g \equiv 1 \pmod{2}$, because $G^\pm$ has an anticonformal automorphism whose order is a multiple of 4.

Let s denote the minimal number of generators of the subgroup $G^+ < G^\pm$ consisting of all conformal automorphisms. Then $G^+ = \Lambda^+/\Gamma$ for the canonical Fuchsian subgroup Λ^+ of Λ . By (5), Λ^+ has the signature $(\gamma - 1; +; [-]; \{-\})$. Thus G^+ is generated by $2\gamma - 2$ elements, being the images in G^+ of the $2\gamma - 2$ generators of the group Λ^+ . In particular, $2\gamma - 2 \geq s$ and by (29) we get $g \geq 1 + \frac{|G^\pm|}{4}(s - 2)$.

The group $G^\pm$ cannot act with signature (28) for $\gamma = 1, 2$, because then $\mu(\Lambda) \leq 0$. For $\gamma = 3$, this signature is non-maximal and there is a NEC group Λ' with the signature $(0; +; [2, 2, 2]; \{-\})$ which contains Λ as a subgroup of index 2. In this case, Λ' is generated by elements x_1, x_2, x_3, e, c which satisfy the relations

$$x_i^2 = 1 \text{ for } i = 1, 2, 3, \quad x_1 x_2 x_3 e = 1 \quad \text{and} \quad e^{-1} c e c = 1,$$

and an embedding $i : \Lambda \rightarrow \Lambda'$ is given by

$$i(d_1) = x_1 x_2 x_3 c x_3 x_2, \quad i(d_2) = x_2 x_3 c x_3 \quad \text{and} \quad i(d_3) = x_3 c,$$

where d_1, d_2, d_3 denote canonical glide reflections of Λ .

Let $\theta : \Lambda \rightarrow G^\pm$ be an epimorphism with kernel Γ . Then $G^\pm$ is generated by three anti-conformal elements h_1, h_2 and h_3 which satisfy the relation $h_1^2 h_2^2 h_3^2 = 1$. No composition of an odd number of these elements has order 2 because X has no symmetries. Suppose that there is a semidirect product $G' = G^\pm \rtimes \langle \rho : \rho^2 = 1 \rangle$ for which θ can be extended to an epimorphism

$\theta' : \Lambda' \rightarrow G'$ with kernel Γ . Since there is no anti-conformal automorphism of order 2 in $G^\pm$, it follows that $\theta'(c_0) = \rho$. By the equality $d_3 = x_3c$, we get $\theta'(x_3) = h_3\rho$. The surface Fuchsian group Γ has no elliptic elements and therefore the last element has order 2 which implies that $\rho h_3\rho = h_3^{-1}$. Next, by the equality $d_2 = x_2x_3cx_3$ we get $\theta'(x_2) = h_2h_3^2\rho$ and this element has order 2 if and only if $\rho h_2\rho = h_3^{-2}h_2^{-1}h_3^2$. Similarly, by the equality $d_1 = x_1x_2x_3cx_3x_2$ we get $\theta'(x_1) = h_1^{-1}\rho$ and $\rho h_1\rho = h_1^{-1}$. So the existence of an automorphism $\eta \in \text{Aut}(G^\pm)$ induced by

$$\eta(h_1) = h_1^{-1}, \quad \eta(h_2) = h_3^{-2}h_2^{-1}h_3^2 \quad \text{and} \quad \eta(h_3) = h_3^{-1}, \quad (30)$$

is the necessary and sufficient condition for the existence of an extension $G' = G^\pm \rtimes \langle \rho \rangle$ of $G^\pm$. Then θ can be extended to an epimorphism $\theta' : \Lambda' \rightarrow G'$ for which $\theta' \circ i = j \circ \theta$, where $j : G^\pm \rightarrow G'$ is the identity. This stands in contradiction to the assumption that $G^\pm$ is the full automorphism group of all conformal and anti-conformal automorphisms of the surface X . Moreover, in this case X is not asymmetric because G' has a symmetry ρ . However, if $G^\pm$ has not the epimorphism η , then an action of $G^\pm$ with the signature $(3; -; [-]; \{-\})$ on an asymmetric Riemann surface of genus $g = 1 + |G^\pm|/2 = 1 + |G^+|$ is still possible and this is the lower bound on g for $s \leq 4$.

Theorem 6.2. *The lower sharp bound on the genus g of an asymmetric Riemann surface, whose full automorphism group $G^\pm$ has no fixed points, is equal to $1 + |G^\pm|$ or $1 + \frac{|G^\pm|}{2}$, according to whether $G^\pm$ is abelian or non-abelian. This bound is attained for infinitely many values of g .*

Proof. The automorphism group $G^\pm$ of an asymmetric Riemann surface of genus $g \geq 2$ acts without fixed points with a signature $\sigma_\gamma = (\gamma; -; [-]; \{-\})$ for $\gamma \geq 3$. Thus according to the Hurwitz–Riemann formula, $g = 1 + \frac{|G^\pm|}{2}(\gamma - 2)$. By the proof of Theorem 6.1, an action with the signature σ_3 is possible under condition that $\eta : G^\pm \rightarrow G^\pm$, induced by (30), is not an automorphism, which implies that $G^\pm$ is not abelian. There are infinite families of non-abelian groups acting on asymmetric Riemann surfaces with the signature σ_3 . First, we will give one of them as an example. Let G be the 33-rd group of order 64 in the Small Groups Database. It is generated by two elements g_1 and g_2 of order 4, which satisfy the relations

$$[g_2, g_1^2] = [g_1, [g_1, g_2]], \quad [[g_1, g_2], g_1^2] = g_2^2 \quad \text{and} \quad [g_1, [g_1, [g_1, g_2]]] = g_2^2.$$

For a NEC group Λ with the signature σ_3 , there is a smooth epimorphism $\theta : \Lambda \rightarrow G$ given by

$$\theta(d_1) = g_1, \theta(d_2) = g_1 \cdot [g_1, [g_1, g_2]] \quad \text{and} \quad \theta(d_3) = g_2.$$

The kernel $\Gamma = \ker \theta$ is a surface Fuchsian group with the signature $(33; +; [-]; \{-\})$. Let $h_i = \theta(d_i)$ for $i = 1, 2, 3$. Using MAGMA we check that η is not an automorphism of G and no product of an odd number of h_i has order 2. Thus G is the full automorphism group $G^\pm$ of the asymmetric Riemann surface $X = \mathcal{H}/\Gamma$ of genus 33.

Now, for a positive integer m , let Γ_m be the characteristic subgroup of Γ generated by the commutator $[\Gamma, \Gamma]$ and all m -th powers of the generators

of Γ and their conjugates. Then Γ/Γ_m is an abelian group of order m^{66} and $\mu(\Gamma_m) = \mu(\Gamma) \cdot m^{66}$. The quotient group $G_m = \Lambda/\Gamma_m$ acts as the full automorphism group on the asymmetric Riemann surface $\mathcal{H}/\Gamma_m$ which, by the Hurwitz–Riemann formula, has genus

$$g = 1 + |G_m|/2 = 1 + |\mu(\Gamma_m)/\mu(\Lambda)|/2 = 32m^{66} + 1.$$

Hence the lower bound on g is attained for infinitely many values of g .

In the abelian case the lower bound $g \geq 1 + |G^\pm|$ is sharp because by Lemma 3.1, for any g satisfying the congruence $g \equiv 1 \pmod{4}$, there is an action of the cyclic group $Z_{g-1} = \langle \delta \rangle$ with the signature σ_4 on an asymmetric Riemann surface of genus g .

In order to prove that a given finite group G is the group of orientation preserving automorphisms of an asymmetric Riemann surface X and the action of the group $\text{Aut}^\pm(X)$ is fixed point free, we will construct some group $\tilde{G}$ containing G as a subgroup of index 2 and we will prove that there is an action of $\tilde{G}$ on X with a signature (28). In this case, we will use the following.

Remark 6.3. Let $\theta : \Lambda \rightarrow \tilde{G}$ be an epimorphism from a maximal NEC group Λ with a signature (28) onto a finite group $\tilde{G}$ such that $\Gamma = \ker \theta$ is a surface Fuchsian group. Then the group of all conformal and anti-conformal automorphisms of the Riemann surface $X = \mathcal{H}/\Gamma$ is isomorphic to $\tilde{G}$. The surface X is asymmetric if there is no symmetry in $\tilde{G}$. Moreover, a subgroup $G < \tilde{G}$ of index 2 whose generators can be written as products of even numbers of θ -images of the canonical glide reflections of Λ consists of all conformal automorphisms of X .

Proof. The group $\tilde{G}$ is isomorphic to the quotient group Λ/Γ acting on the Riemann surface $X = \mathcal{H}/\Gamma$. Since Λ is a maximal NEC group, it follows that $\tilde{G}$ cannot be contained properly in any other automorphism group of X , because otherwise there would be a NEC group Λ' containing Λ as a proper subgroup, what is impossible. Hence $\tilde{G}$ is the group $\text{Aut}^\pm(X)$ of all conformal and anticonformal automorphisms of the surface. It is generated by automorphisms $h_i = \theta(d_i)$ for $1 \leq i \leq \gamma$, where d_i are the canonical generating glide reflections of the group Λ . So the surface X has anticonformal automorphisms and it is asymmetric if none of them is a symmetry.

Let $G < \tilde{G}$ be a subgroup of index 2 whose generators can be written as products of even numbers of elements h_i . Then G is generated by conformal automorphisms and so it is contained in the group G^+ of all conformal automorphisms of X . Since $|G| = |G^+|$, it follows that $G = G^+$.

Theorem 6.4. *Let G be a finite group whose center $Z(G)$ has even order. Then there exists an asymmetric Riemann surface X , whose $\text{Aut}^\pm(X)$ acts without fixed points and G is the subgroup of all conformal automorphisms in $\text{Aut}^\pm(X)$.*

Proof. Let G be a finite group which has a central element a of even order $2k$ and let $C = \langle \delta \rangle$ be a cyclic group of order $4k$. Then the direct product $G \times C$ has a normal subgroup generated by $b = \delta^2 a^{-1}$ and $\tilde{G} = (G \times C)/\langle b \rangle$

has order $2|G|$. Let $\{g_1, \dots, g_s\}$ be a generating set of G with $g_i^2 \neq a^{-1}$ for all i . The signature (28) with $\gamma = 2s + 2$ is maximal. Let Λ be a maximal NEC group Λ with this signature, and let $\theta : \Lambda \rightarrow \tilde{G}$ be an epimorphism given by

$$\theta(d_{2i-1}) = \delta g_i, \theta(d_{2i}) = (\delta g_i)^{-1} \quad \text{for } 1 \leq i \leq s$$

$\theta(d_{2s+1}) = \delta$ and $\theta(d_{2s+2}) = \delta^{-1}$. Then $\theta(d_1)^2 \cdots \theta(d_\gamma)^2 = 1$ and so the kernel $\Gamma = \ker \theta$ is a surface Fuchsian group. By Remark 6.3, $\tilde{G}$ is isomorphic to the group $G^\pm = \Lambda/\Gamma$ of all conformal and anticonformal automorphisms of the Riemann surface $X = \mathcal{H}/\Gamma$, which by the Hurwitz–Riemann formula has genus $2|G|s + 1$. It is generated by reversing orientation automorphisms $\theta(d_i)$ for $1 \leq i \leq \gamma$ and its subgroup G^+ of all conformal automorphisms consists of elements which are products of an even number of some generators $\theta(d_i)$. In particular, the group G is contained in G^+ . Since $|G| = |G^+|$, it follows that these groups are equal. Finally, the surface X is asymmetric because it has anticonformal automorphisms but none of them is a symmetry.

Corollary 6.5. *For any nilpotent group G of even order there exists an asymmetric Riemann surface X on which G acts as the group of all conformal automorphisms and $\text{Aut}^\pm(X)$ acts without fixed points.*

Proof. Let P be the set of all prime integers dividing the order of a nilpotent group G , and let G_p be the unique Sylow p -group of G for $p \in P$. Then $G = \prod_{p \in P} G_p$ is the internal direct product of Sylow p -groups G_p . Each G_p has nontrivial center $Z(G_p)$ and $Z(G) = \prod_{p \in P} Z(G_p)$. Let z_p be a nontrivial element in $Z(G_p)$ for $p \in P$ and let $n_p = \sharp(z_p)$. Since for any distinct integers $p, q \in P$, $G_p \triangleleft G$, $G_q \triangleleft G$ and $G_p \cap G_q = 1$, it follows that $[z_p, z_q] = 1$ and so z_p and z_q commute. Let a be the product of z_p for all $p \in P$. Then $a \in Z(G)$ and $\sharp(a)$ is the product of n_p for all $p \in P$. Hence if $|G|$ is even, then a is a central element of even order. Thus by Theorem 6.4, there is a fixed point free action on an asymmetric Riemann surface such that G is the group of all conformal automorphisms of the surface.

Theorem 6.6. *Let G be a finite noncyclic group whose center has even order, and let $\{g_1, \dots, g_s\}$ be a generating set of G . If there is an automorphism of G given by the assignments $g_i \rightarrow g_i^{-1}$ for $i = 1, \dots, s$, then there is an asymmetric Riemann surface X of genus $g = 1 + |G|(s - s_{(2)})$ on which G acts as the full group of conformal automorphisms and the action of the group $\text{Aut}^\pm(X)$ is fixed point free.*

Proof. Let $a \in G$ be a central element of even order $2k$. For a cyclic group $C = \langle \delta \rangle$ of order 4, let $G \rtimes C$ be the semidirect product, where δ acts on generators of G by the formula: $\delta g_i \delta^{-1} = g_i^{-1}$ for $i = 1, \dots, s$. The subgroup $\langle \delta^2 a^{-k} \rangle$ is normal in $G \rtimes C$ and the quotient group $\tilde{G} = (G \rtimes C) / \langle \delta^2 a^{-k} \rangle$ has order $2|G|$. For even s , let Λ be a maximal NEC group with the signature $(s + 2; -, [-]; \{-\})$. Then there is a smooth epimorphism $\theta : \Lambda \rightarrow \tilde{G}$ given by

$$\theta(d_{2i-1}) = \delta g_{2i-1}, \quad \theta(d_{2i}) = \delta^{-1} g_{2i} \quad (31)$$

for $i = 1, \dots, \frac{s}{2}$, $\theta(d_{s+1}) = \delta$ and $\theta(d_{s+2}) = \delta^{-1}$. The group $\Gamma = \ker \theta$ is a surface Fuchsian group. If s is odd, then we can take a maximal NEC group Λ with the signature $(s+1; -; [-]; \{-\})$ and define θ by (31) for $i = 1, \dots, \frac{s-1}{2}$, $\theta(d_s) = \delta g_s$ and $\theta(d_{s+1}) = \delta^{-1}$. By Remark 6.3, $\tilde{G}$ is isomorphic to the group $\text{Aut}^\pm(X)$ of the Riemann surface $X = \mathcal{H}/\ker \theta$ which by the Hurwitz–Riemann formula has genus $g = 1 + |G|(s - s_{(2)})$.

Each generator g_i of the group G preserves orientation because it can be written as a product of two orientation reversing automorphisms being the θ -images of two canonical glide reflections. Thus G is contained in the subgroup G^+ of conformal automorphisms. Since $|G| = |G^+|$, it follows that $G = G^+$. The surface X is asymmetric because none of its anticonformal automorphisms has order 2.

Corollary 6.7. *Let G be a finite noncyclic abelian group of even order, and let $G = \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_s}$ be the unique decomposition of G into the product of cyclic subgroups generated by maximal elements of G , where n_{i+1} divides n_i for $i = 1, \dots, s-1$ and $n_1 \cdot \dots \cdot n_s = |G|$. Then there is an asymmetric Riemann surface of genus $g = 1 + |G|(s - s_{(2)})$ on which G acts as the group of conformal automorphisms and the action of $\text{Aut}^\pm(X)$ is fixed points free.*

Below we give examples of a finite group G which acts without fixed points as the group of conformal automorphisms on an asymmetric Riemann surface of genus $g = 1 + 2|G|$.

Example 1. Let G be a dihedral group of order $4k$, generated by two elements a and b of orders 2, such that their product has order $2k$. The group G has a central element $(ab)^k$, and there is an automorphism of G given by the assignments $a \rightarrow a^{-1}$ and $b \rightarrow b^{-1}$. Hence we can repeat the construction from the proof of Theorem 6.6. Let $C = \langle \delta \rangle$ be the cyclic group of order 4, and let $\tilde{G} = (G \times C)/\langle \delta^{-2}(ab)^k \rangle$. For a maximal NEC group Λ with signature $(4; -; [-]; \{-\})$, there is an epimorphism $\theta : \Lambda \rightarrow \tilde{G}$ with torsion-free kernel given by $\theta(d_1) = \delta a, \theta(d_2) = \delta^{-1}b, \theta(d_3) = \delta, \theta(d_4) = \delta^{-1}$. Thus by Theorem 6.6, $\tilde{G}$ is isomorphic to the group $\text{Aut}^\pm(X)$ of an asymmetric Riemann surface $X = \mathcal{H}/\ker \theta$ of genus $g = 1 + 8k = 1 + 2|G|$, and G is the subgroup of conformal automorphisms.

Corollary 6.8. *For any integer $g > 1$ such that $g \equiv 1 \pmod{8}$ and for any prime odd integer n dividing $g-1$, there is an asymmetric Riemann surface of genus g which is a n -sheeted unbranched covering of an orbifold of genus $\frac{g-1}{n} + 1$.*

Example 2. Let Q be the quaternion group consisting of 8 elements $\pm 1, \pm i, \pm j, \pm k$, which satisfy the relations:

$$i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j.$$

The set $\{i, j\}$ is a minimal set of generators of Q and -1 is a central element of order 2. Moreover, there is an automorphism of Q given by the assignments: $i \rightarrow -i$ and $j \rightarrow -j$. We can repeat the construction from the proof of Theorem 6.6. Let $C = \langle \delta \rangle$ be the cyclic group of order 4, and let $Q \rtimes C$ be the semidirect product, where δ acts on generators of Q by the formula:

$\delta i \delta^{-1} = -i$ and $\delta j \delta^{-1} = -j$. Then $\tilde{G} = (Q \rtimes C)/\langle \delta^2(-1) \rangle$ has order 16. Let Λ be a maximal NEC group with the signature $(4; -; [-]; \{-\})$, and let $\theta : \Lambda \rightarrow \tilde{G}$ be given by

$$\theta(d_1) = \delta i, \theta(d_2) = \delta^{-1} j, \theta(d_3) = \delta, \theta(d_4) = \delta^{-1}.$$

Then

$$\theta(d_1^2 d_2^2 d_3^2 d_4^2) = \delta i \delta i \delta^{-1} j \delta^{-1} j \delta^2 \delta^{-2} = \delta(-i^2) j \delta^{-1} j = \delta j \delta^{-1} j = -j^2 = 1$$

and so $\Gamma = \ker \theta$ is a surface Fuchsian group with signature $(17, -)$. According to Theorem 6.6, $\tilde{G}$ acts without fixed points on an asymmetric Riemann surface $X = \mathcal{H}/\Gamma$ of genus $17 = 1 + 2|Q|$, the quaternion group is the group of all conformal automorphisms of X and δ is a pseudo-symmetry of X .

Example 3. Let us consider two groups:

$$K = \langle x, y : x^4 = 1, y^3 = 1, (xy)^4 = 1, yx^2y^{-1}x^2 = 1 \rangle$$

and

$$H = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 = \langle z_1 \rangle \times \cdots \times \langle z_8 \rangle.$$

There is an action of the group K on H which we will write for simplicity as a permutation of the set $I = \{1, \dots, 8\}$ of indices of elements z_i . Assume that conjugation via y corresponds to the trivial permutation of the set I and that conjugation via x corresponds to the permutation

$$(1, 2, 3, 4)(5, 7, 6, 8).$$

Let G be the semidirect product $G = H \rtimes K$ with respect to this action, and let C be the cyclic group $C = \langle \delta : \delta^4 = 1 \rangle$. There is an automorphism of K given by the assignments $x \rightarrow x^{-1}$ and $y \rightarrow y^{-1}$ which allows us to define an action of the group C on G . Assume that

$$\delta x \delta^{-1} = x^{-1}, \quad \delta y \delta^{-1} = y^{-1}$$

and conjugation of z_i via δ corresponds to the permutation

$$(1, 5, 3, 6)(2, 7, 4, 8).$$

Let $G \rtimes C$ be the semidirect product with respect to this action. There is a normal subgroup $\langle \delta^2 x^2 \rangle$ of $G \rtimes C$ and the quotient group $\tilde{G} = (G \rtimes C)/\langle \delta^2 x^2 \rangle$ has order $2|G|$. Let Λ be a maximal NEC group with signature $(6; -; [-]; \{-\})$, and let $\theta : \Lambda \rightarrow \tilde{G}$ be the homomorphism given by

$$\theta(d_1) = \delta x, \theta(d_2) = \delta^{-1} y, \theta(d_3) = \delta z_1, \theta(d_4) = \delta^{-1} z_5, \theta(d_5) = \delta, \theta(d_6) = \delta^{-1}.$$

Then denoting $\theta(d_i)$ by h_i for $1 \leq i \leq 6$, we get

$$x = h_6 h_1, y = h_5 h_2, z_5 = h_5 h_4, z_1 = h_6 h_3$$

and

$$\begin{aligned} z_2 &= x z_1 x^{-1} = (h_6 h_1) h_6 h_3 (h_6 h_1)^{-1}, \\ z_3 &= \delta z_5 \delta^{-1} = h_5^2 h_4 h_6, \\ z_4 &= x z_3 x^{-1} = (h_6 h_1) h_5^2 h_4 h_6 (h_6 h_1)^{-1} \\ z_6 &= \delta^2 z_5 \delta^{-2} = h_5^3 h_4 h_6^2, \end{aligned}$$

$$\begin{aligned} z_7 &= xz_5x^{-1} = (h_6h_1)h_5h_4(h_6h_1)^{-1}, \\ z_8 &= xz_6x^{-1} = (h_6h_1)h_5^3h_4h_6^2(h_6h_1)^{-1}. \end{aligned}$$

So θ is an epimorphism whose kernel $\Gamma = \ker\theta$ is a surface Fuchsian group. By Remark 6.3, the group $\tilde{G}$ is isomorphic to the group $G^\pm$ of all conformal and anticonformal automorphisms of an asymmetric Riemann surface $X = \mathcal{H}/\Gamma$ which by the Hurwitz–Riemann formula has genus $g = 1 + 4|G|$. Let us notice that $g = 1 + |G|(s-2)/2$, where $s = 10$ is the number of generators of the group G . The group $\tilde{G}$ is generated by anticonformal automorphisms h_i for $1 \leq i \leq 6$. Since G is generated by elements $x, y, z_1, \dots, z_8$ which can be written as products of even numbers of elements h_i , it follows that G is contained in the subgroup $G^+ < G^\pm$ consisting of all conformal automorphisms. Moreover, G has the same order as G^+ , what implies that $G = G^+$.

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